Gravitational waves from a point particle in circular orbit around a black hole: Logarithmic terms in the post-Newtonian expansion

Hideyuki Tagoshi
Department of Physics, Kyoto University, Kyoto 606-01, Japan

Takashi Nakamura
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan
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We find logarithmic terms in a post-Newtonian expansion of gravitational radiation induced by a particle traveling a circular orbit of radius r_0 around a Schwarzschild black hole of mass M. We calculate the gravitational wave luminosity using the Teukolsky equation to high accuracy (~ 20 figures) and determine the coefficients of the post-Newtonian expansion by means of least squares fitting. We find that there are terms proportional to $x^6 \ln x$ and $x^8 \ln x$ where $x = (M/r_0)^{1/2}$. We also examine the accumulated phase of coalescing compact star binaries by means of the post-Newtonian expansion as it sweeps through the bandwidth at which the future laser interferometric detectors have good sensitivity.

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One of the promising sources of gravitational waves for the planned laser interferometer detectors [1] is the last stage of inspiral of compact binaries. Accurate theoretical models of waveforms are needed to extract physical information from gravitational waves [2]. Much theoretical effort has already been directed towards this purpose [3–11]. In the case when one of the stars in a binary is much more massive than the other, a linear perturbation theory is available. To investigate the effectiveness of a post-Newtonian method, Cutler *et al.* [4] numerically integrated the Teukolsky equation to high accuracy and fitted their numerical data for the gravitational wave luminosity to the post-Newtonian expansion form by means of least squares fitting up to sixth order.

In this paper, we try to extend the post-Newtonian expansion to the eighth order by using the fitting method. To do this it is essential to calculate the gravitational wave luminosity in higher accuracy than Cutler *et al.*'s work in which numerical integrations were used to obtain a solution of the Teukolsky equation. In this paper we adopt a method which was originally proposed by Leaver [12] to obtain higher accuracy. In this method, solutions of the Teukolsky equations are represented in an analytic manner.

We consider the case when a test particle of mass μ travels a circular orbit around a Schwarzschild black hole of mass $M \gg \mu$. Then the specific energy $\tilde{E}$ and the angular momentum $\tilde{L}$ of the particle is given by

$$\tilde{E} = (r_0 - 2M)/\sqrt{r_0(r_0 - 3M)}$$

and

$$\tilde{L} = \sqrt{Mr_0}/\sqrt{1 - 3M/r_0} ,$$

where r_0 is orbital radius. The angular frequency is given by $\Omega = (M/r_0^3)^{1/2}$. To calculate the gravitational wave luminosity we use the Teukolsky equation [13,14]

$$\left[\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{d}{dr}\right) - U(r)\right] R_{lm\omega} = T_{lm\omega}(r) , \tag{1}$$

where

$$U(r) = \frac{r}{r-2M}[\omega^2 r^2 - 4i\omega(r-3M)] - (l-1)(l+2) , \tag{2}$$

and $\Delta = r(r-2M)$. $T_{lm\omega}$ is given by

$$\begin{aligned}(T_{lm\omega}/\pi)/\mu = &-2\,{}_0b_{lm}(r_0-2M)^2\delta(r-r_0) - {}_{-1}b_{lm}2ir_0[(r_0-2M)^2\delta'(r-r_0) - (r_0-2M)(2-i\omega r_0)\delta(r-r_0)] \\ &+ {}_{-2}b_{lm}[r_0^2(r_0-2M)^2\delta''(r-r_0) + \{2i\omega r_0^3(r_0-2M) - 2r_0(3r_0^2 - 8Mr_0 + 4M^2)\}\delta'(r-r_0) \\ &+ \{4r_0^2 - 8M^2 - \omega^2 r_0^4 - 6i\omega r_0^2(r_0-M)\}\delta(r-r_0)] ,\end{aligned} \tag{3}$$

where $\delta(r)$ is the usual δ function and $\omega = m\Omega$. ${}_sb_{lm}$ are given by

$${}_0b_{lm} = \tfrac{1}{2}[(l-1)l(l+1)(l+2)]^{1/2}\,{}_0Y_{lm}\left(\frac{\pi}{2},0\right)\tilde{E}r_0/(r_0-2M) ,$$

$${}_{-1}b_{lm} = [(l-1)(l+2)]^{1/2}\,{}_{-1}Y_{lm}\left(\frac{\pi}{2},0\right)\tilde{L}/r_0 ,$$

and

$$ {}_{-2}b_{lm} = {}_{-1}Y_{lm}\left[\frac{\pi}{2},0\right]\tilde{L}\Omega , $$

where ${}_sY_{lm}$ are the spin-weighted spherical harmonics [14].

We solve Eq. (1) under the outgoing boundary condition at infinity and the ingoing boundary condition at the horizon. Then gravitational wave luminosity is given by

$$ \frac{dE}{dt} = \sum_{l=2}^{\infty}\sum_{m=1}^{l} |Z_{lm}^{\text{out}}|^2/2\pi\omega^2 , \tag{4} $$

$$ Z_{lm}^{\text{out}} = \frac{1}{2i\omega B_{\text{in}}}\int_{2M}^{\infty} dr\, R_{\text{in}}(r) T_{lm\omega}(r)/\Delta^2 , \tag{5} $$

where l and m are the spherical harmonic degree and order. $R_{\text{in}}(r)$ is a solution of the homogeneous Teukolsky equation and has a boundary condition

$$ R_{\text{in}}(r) = \begin{cases} \Delta^2 e^{-i\omega r^*}, & r^* \to -\infty , \\ r^3 B_{\text{out}} e^{i\omega r^*} + r^{-1} B_{\text{in}} e^{-i\omega r^*}, & r^* \to +\infty , \end{cases} \tag{6} $$

where $r^* = r + 2M\ln(r/2M-1)$. In the case of the circular orbit, $T_{lm\omega}$ contains only the terms which are proportional to a δ function and its derivatives. So Z_{lm}^{out} can be evaluated using $R_{\text{in}}(r_0)$ and its derivatives.

It is often convenient to use a Regge-Wheeler (RW) equation [15] instead of the Teukolsky equation. If we transform R_{in} as [16–18]

$$ R_{\text{in}}(r) = \Delta\left[\frac{d}{dr^*}+i\omega\right]\frac{r^2}{\Delta}\left[\frac{d}{dr^*}+i\omega\right] rX_{\text{in}} , \tag{7} $$

the homogeneous Teukolsky equation becomes

$$ \left[\frac{d^2}{dr^{*2}}+\omega^2-V(r)\right]X_{\text{in}}(r)=0 \tag{8} $$

and

$$ V(r) = \left[1-\frac{2M}{r}\right]\left[\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right] . $$

The asymptotic form of X_{in}, after being normalized appropriately, is given by

$$ X_{\text{in}}(r) = \begin{cases} e^{-i\omega r^*}, & r^* \to -\infty , \\ A_{\text{out}} e^{i\omega r^*} + A_{\text{in}} e^{-i\omega r^*}, & r^* \to +\infty . \end{cases} \tag{9} $$

To solve X_{in} we put Leaver's procedure [12] into practice. The Teukolsky equation is classified into generalized spheroidal wave equations which can be computed by means of a power-series expansion around the horizon or a series of Coulomb wave functions.

First, X_{in} can be represented as

$$ X_{\text{in}} = \exp(-i\omega r^*)\sum_{n=0}^{\infty} a_n\left[1-\frac{2M}{r}\right]^n . \tag{10} $$

This expansion is a power-series expansion in variable $u=1-2M/r$ around a regular singular point $u=0$. a_n's are determined by a three-term recurrence relation

$$ (n+1-4iM\omega)a_{n+1} = \left[2n+\frac{l(l+1)-3}{n+1}\right]a_n - \frac{n^2-4}{n+1}a_{n-1} \quad (n\ge 1) , \tag{11} $$

$$ a_1 = [l(l+1)-3]a_0/(1-4iM\omega) . $$

From the boundary condition (9), we must set $a_0=1$.

For convergence of (10), we need a summation only up to $n\sim 10^5$ for $r_0<10^4M$. As R_{in} is expressed by X_{in} and their derivatives, it also converges for $r_0<10^4M$. Thus, Z_{lm}^{out} can be calculated almost analytically for $r_0 \lesssim 10^4M$ except for the term B_{in}. To obtain B_{in}, we need to determine A_{in}. However, since $r=\infty$ is an irregular singular point, a power-series expansion with variable r^{-1} converges only asymptotically. So it is hard to obtain an accurate value of A_{in} using (10). To overcome this, we expand X_{in} using a series of Coulomb wave functions discussed by Leaver [12]. That is

$$ X_{\text{in}} = \left[A_{\text{out}}\sum_{L=-\infty}^{\infty} b_L u_{L+\nu}^{(+)}(z) + A_{\text{in}}\sum_{L=-\infty}^{\infty} b_L u_{L+\nu}^{(-)}(z)\right]\left[\frac{r}{r-2M}\right]^{2iM\omega} , \tag{12} $$

where $u^{(\pm)}$ are a certain kind of Coulomb wave function. In Eq. (12), ν is real and is determined so that a series of Coulomb wave functions converges. We show the details in the Appendix.

Numerical results of dE/dt are shown in Table I. These data contain multipoles from $l=2$ to 6. All numerical calculations are done with precision of 32 figures. Checking the convergence, we confirmed that the accuracy of the summation in Eqs. (10), (12), and (5) has at least 20 figures. We compared our numerical data with those by Cutler *et. al.* [4] and found that their data agree with ours up to 8–9 figures which seems to be consistent with their error estimates.

Using these data, we perform the post-Newtonian expansion. First we assume that dE/dt can be expanded in powers of x $[=(M/r_0)^{1/2}]$, that is, we assume that

$$ \frac{dE}{dt} = \left[\frac{dE}{dt}\right]_N \sum_{k=0} a_k x^k , \tag{13} $$

where $(dE/dt)_N$ is derived from a quadrupole formula:

$$ \left[\frac{dE}{dt}\right]_N = \frac{32}{5}\frac{\mu^2M^3}{r_0^5} . \tag{14} $$

The first four terms are known analytically [3,20]:

$$ a_0=1,\quad a_1=0,\quad a_2=-\tfrac{1247}{336},\quad a_3=4\pi . \tag{15} $$

The next three terms are known approximately [4]:

$$ a_4\simeq 4.89(2\%),\quad a_5\simeq -38(10\%),\quad a_6\simeq 135(50\%) , \tag{16} $$

TABLE I. Numerical value of dE/dt including multipoles up to $l=6$. This table is not intended to indicate the accuracy of these values over 30 figures. However accuracy is at least 20 figures.

r_0/M	$(M/\mu)^2 dE/dt$
100.000 00	$6.238\,203\,398\,978\,509\,412\,124\,176\,192\,704\,334\times10^{-10}$
300.000 00	$2.607\,338\,426\,816\,527\,765\,774\,966\,086\,335\,620\times10^{-12}$
400.000 00	$6.201\,576\,481\,329\,383\,382\,603\,167\,326\,567\,270\times10^{-13}$
500.000 00	$2.035\,048\,609\,173\,431\,022\,305\,459\,014\,285\,948\times10^{-13}$
600.000 00	$8.186\,438\,196\,960\,483\,600\,132\,340\,003\,588\,864\times10^{-14}$
700.000 00	$3.790\,284\,023\,911\,361\,535\,919\,647\,135\,042\,982\times10^{-14}$
800.000 00	$1.945\,130\,336\,339\,855\,005\,289\,216\,166\,030\,152\times10^{-14}$
900.000 00	$1.079\,872\,581\,297\,294\,600\,591\,999\,349\,334\,023\times10^{-14}$
1 000.000 0	$6.378\,752\,660\,479\,986\,315\,456\,777\,531\,270\,354\times10^{-15}$
1 200.000 0	$2.564\,828\,833\,240\,345\,107\,029\,159\,654\,956\,498\times10^{-15}$
1 400.000 0	$1.187\,107\,733\,351\,135\,240\,943\,577\,415\,669\,260\times10^{-15}$
1 600.000 0	$6.090\,542\,750\,440\,027\,869\,703\,840\,948\,937\,458\times10^{-16}$
1 800.000 0	$3.380\,585\,421\,529\,206\,602\,594\,378\,236\,237\,482\times10^{-16}$
2 000.000 0	$1.996\,566\,834\,529\,090\,273\,039\,578\,847\,096\,921\times10^{-16}$
2 200.000 0	$1.239\,897\,486\,733\,535\,981\,491\,638\,714\,136\,889\times10^{-16}$
2 400.000 0	$8.025\,973\,521\,482\,039\,884\,440\,516\,887\,789\,591\times10^{-17}$
2 600.000 0	$5.379\,398\,644\,465\,723\,875\,822\,126\,316\,487\,929\times10^{-17}$
2 800.000 0	$3.714\,072\,389\,983\,293\,585\,415\,708\,907\,420\,717\times10^{-17}$
3 000.000 0	$2.630\,686\,432\,040\,147\,798\,074\,361\,413\,233\,360\times10^{-17}$
3 200.000 0	$1.905\,267\,894\,555\,250\,203\,290\,888\,243\,178\,670\times10^{-17}$
3 400.000 0	$1.407\,143\,600\,435\,670\,282\,337\,691\,474\,289\,615\times10^{-17}$
3 600.000 0	$1.057\,412\,945\,896\,035\,827\,263\,146\,787\,254\,939\times10^{-17}$
3 800.000 0	$8.069\,762\,995\,913\,967\,648\,059\,791\,767\,590\,368\times10^{-18}$
4 000.000 0	$6.244\,509\,390\,803\,978\,517\,372\,128\,153\,303\,494\times10^{-18}$
4 200.000 0	$4.892\,935\,859\,814\,708\,021\,886\,566\,563\,356\,111\times10^{-18}$
4 000.000 0	$3.877\,650\,922\,832\,691\,958\,929\,296\,005\,872\,690\times10^{-18}$

TABLE II. The results of the least squares fitting. The model functions of (top) are power in x, (middle) contains an $x^6\ln x$ term, and (bottom) contains both $x^6\ln x$ and $x^8\ln x$ terms.

n	a_n
0	0.999 999 999 999 567 370 32
1	$2.554\,010\,726\,891\,945\,0071\times10^{-10}$
2	−3.711 309 596 750 765 212 0
3	12.566 384 350 926 089 206
4	−4.930 481 640 323 767 992 9
5	−37.999 982 610 332 799 195
6	179.273 234 265 923 125 902
7	−817.510 801 390 776 742 99
0	0.999 999 999 999 999 718 56
1	$1.926\,112\,189\,023\,945\,531\,28\times10^{-13}$
2	−3.711 309 523 874 391 092 1
3	12.566 370 629 097 184 200 5
4	−4.928 463 902 822 523 863 3
5	−38.292 319 775 967 105 066
6	115.913 907 630 116 282 263
$x^6\ln x$	−16.262 941 934 716 851 792
7	−104.972 507 345 727 096 16
8	−259.250 929 833 168 860 42
0	1.000 000 000 000 000 005 303 0
1	$-4.115\,960\,526\,368\,818\,532\,705\times10^{-15}$
2	−3.711 309 523 807 923 530 949
3	12.566 370 613 929 664 016 796
4	−4.928 461 103 278 450 962 933
5	−38.292 858 801 057 043 217 67
6	115.719 471 267 127 908 783 61
$x^6\ln x$	−16.307 354 957 551 625 440 55
7	−101.173 170 530 364 872 684 5
8	−72.608 762 809 630 517 670 49
$x^8\ln x$	64.218 530 396 595 549 400 97

where numbers in the parentheses denote the error. We calculated multipoles from $l=2$ to 6. Then we can determine the coefficients up to the order x^9 because the $l=7$ multipole contributes from order x^{10}.

We determine the coefficients a_n by means of least squares fitting implemented with the method of singular value decomposition [19]. We used 120 data of dE/dt in the fitting. The results are shown in Table II (top). Using these results we plot in Fig. 1 the residual ζ_n of dE/dt which is defined by

$$\frac{dE}{dt}\zeta_n=\frac{dE}{dt}-\left(\frac{dE}{dt}\right)_N\sum_{k=0}^{n}a_k x^k \,. \tag{17}$$

We see that, at $n=6$, $|\zeta_n|$ shows a strange behavior. In general, it is argued that the post-Newtonian expansion may contain terms proportional to $x^k(\ln x)^p$ ($k,p=$ nonnegative integers) [3,4,21]. Cutler *et al.* said in their paper [4] that Ori suggested in a private communication that there is a term proportional to $x^6\ln x$. To take this into account, we add the $b_6x^6\ln x$ term to the model functions. That is,

$$\frac{dE}{dt}=\left(\frac{dE}{dt}\right)_N\left(\sum_{k=0}a_k x^k+b_6x^6\ln x\right) \,. \tag{18}$$

The result is shown in Table II (middle). It is clear that the accuracy of the determination of first four values is improved about 3 orders. The residual of dE/dt including the $x^6\ln x$ term also becomes better. From the value in Table II (middle), we can understand the behavior of $|\zeta_6|$ in Fig. 1. Since $b_6x^6\ln x+a_7x^7$ becomes zero near $r\sim20$, ζ_6 change sign near this point. All of these facts are consistent with the existence of the term proportional

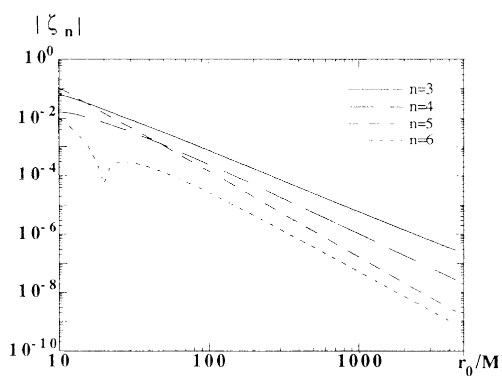

FIG. 1. The plot of the residual $|\zeta_n|$ of dE/dt using the coefficients in Table II (top).

to $x^6 \ln x$ in the post-Newtonian expansion. From the analysis of the data dE/dt of each modes (l,m), we find that this term comes from the $l=m=2$ mode. We also tried model functions with $x^6(\ln x)^n$ $(n\geq 2)$ and found that the coefficients of these terms are less than 10^{-2}. This fact suggests that the terms proportional to $x^6(\ln x)^n$ $(n\geq 2)$ do not exist.

In a similar way, we examine the coefficients at the order x^7 and x^8. We find that there seem to exist no $x^7(\ln x)^n$ $(n\geq 1)$ terms but there is a $b_8 x^8 \ln x$ term. This term comes from both $l=m=2$ and $l=m=3$ modes. Fitting results are shown in Table II (bottom). We see that the accuracy of the determination of the first four values is improved about 2 orders compared with Table II (middle).

We also performed the fitting to the data in which the first four terms are subtracted analytically and found that the post-Newtonian expansion has the form

$$\frac{dE}{dt}=\left(\frac{dE}{dt}\right)_N\left[1-\frac{1247}{336}x^2+4\pi x^3-4.928\,461\,199\,0x^4-38.292\,835x^5 +115.730x^6-16.3049x^6\ln x-101.450x^7-93x^8+57x^8\ln x\right]. \tag{19}$$

To investigate the accuracy of these coefficients, we compare these results to the value in Table II (bottom) and count the number of figures at which the value is not changed. We adopt these numbers as significant figures of the coefficients and the accuracy of these values are estimated as a_4, $10^{-5}\%$; a_5, $10^{-3}\%$; a_6, 0.1%; b_6, 0.1%; a_7, 1%; a_8, 30%; and b_8, 20%.

Using these data and log terms, we plot in Fig. 2 the residual ζ'_n which is defined by

$$\frac{dE}{dt}\zeta'_n=\frac{dE}{dt}-\left(\frac{dE}{dt}\right)_N\sum_{k=0}^{n}(a_k x^k+b_k x^k\ln x)\,, \tag{20}$$

where $b_k\neq 0$ for $k=6$ and 8. We see all ζ'_n except ζ'_7 show a power-law behavior. ζ'_7 changes sign around $r_0\sim 15M$. This may suggest that the coefficients of x^9 may be very large (about 650). However, for $n\geq 9$, more detailed analysis is needed.

Finally, we estimate the accuracy of the post-Newtonian expansion to construct the template waveform from an inspiraling compact binary. Consider that a binary starts to inspiral from orbital separation r_0 and that there are two template waveforms. One is constructed using correct the $\dot{E}$ and the other is constructed using an approximate form for $\dot{E}$ (the overdot denotes a time derivative). Then at the end of the inspiraling, the accumulated phase difference $\Delta\Phi$ between two template waveforms is given by [4]

$$\Delta\Phi/\Phi\sim\Delta\dot{E}(r_0)/\dot{E}(r_0)\,, \tag{21}$$

where Φ is the total phase and the right-hand side refers to the relative error in $\dot{E}$.

The planned ground-based laser interferometer detectors [1] will observe inspiraling compact binaries in the wave frequency $f\sim 10$–1000 Hz. In the case of a $1.4M_\odot$ neutron star binary, when the wave frequency is 10 Hz, r_0 is about 723 km $(175M)$ and the total accumulated phase from 10 to 10^3 Hz is $\Phi\simeq 10^5$. We assume that the expansions for non-negligible mass μ/M will be subject to the same convergence problems as for negligible μ/M. Then, from Fig. 2, we can estimate the accumulated phase error $(\Delta\Phi)_n$ when we include the post-Newtonian expansion of order x^n: $(\Delta\Phi)_3\sim 20$, $(\Delta\Phi)_4\sim 7$, $(\Delta\Phi)_5\sim 3$, $(\Delta\Phi)_6\sim 0.2$, $(\Delta\Phi)_7\sim 0.02$. This suggests that if we construct the template waveform which includes the post-Newtonian effect up to $n=7$, we will obtain an accuracy $\sim 3\times 10^{-3}$ cycle after $\sim 1.5\times 10^4$ cycles. Note that the $b_6 x^6\ln x$ term contributes to $\Delta\dot{E}(r_0)/\dot{E}(r_0)$ one-third of the $a_6 x^6$ term at $r_0\simeq 175M$. This shows the importance of the $b_6 x^6 \ln x$ term in constructing the template wave form.

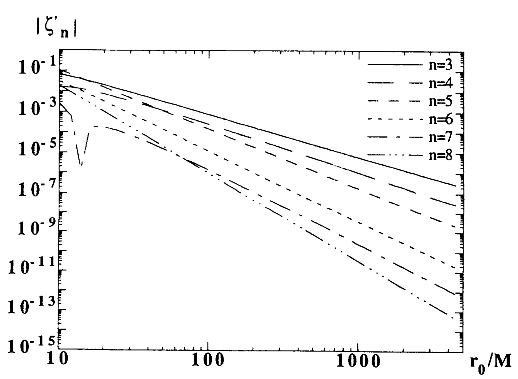

FIG. 2. The plot of the residual $|\zeta'_n|$ of dE/dt using the coefficients in Eq. (19).

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APPENDIX

In this appendix, we explain the details of the expansion in a series of Coulomb wave functions and the numerical method to calculate $A_{\rm in}$. We set $M=1$. The Teukolsky equation is classified into generalized spheroidal wave equations. Leaver [12] showed that generalized spheroidal wave equations can be calculated by means of a power-series expansion in variable $u=1-2/r$ around a black hole horizon $r=2$ and a series of the Coulomb wave functions. If we convert $X_{\rm in}(r)$ as $X_{\rm in}=r^3(r-2)^{-2i\omega}y(r)$, the Regge-Wheeler (RW) equation becomes a general form of the generalized spheroidal wave equations:

$$r(r-2)\frac{d^2y}{dr^2}+(B_1+B_2r)\frac{dy}{dr}+[\omega^2r(r-2)-2\eta\omega(r-2)+B_3]=0\ , \quad \text{(A1)}$$

where

$$\eta=-2\omega\ ,$$
$$B_1=-10\ ,$$
$$B_2=6-4i\omega\ ,$$
$$B_3=8\omega^2+6-l(l+1)-10i\omega\ .$$

With the substitution $y=r^{-B_2/2}h(r)$ and $z=r\omega$, this equation becomes

$$z(z-2\omega)[h_{,zz}+(1-2\eta/z)h]+C_1\omega h_{,z}+(C_2+C_3\omega/z)h=0\ , \quad \text{(A2)}$$

where

$$C_1=B_1+2B_2=2-8i\omega\ ,$$
$$C_2=B_3-\tfrac{1}{2}B_2(\tfrac{1}{2}B_2-1)=12\omega^2-l(l+1)\ , \quad \text{(A3)}$$
$$C_3=-\tfrac{1}{2}B_2(B_1+B_2+2)=6+8\omega^2+8i\omega\ .$$

The relation between $X_{\rm in}$ and $h(z)$ is

$$X_{\rm in}=\left[\frac{r}{r-2}\right]^{2i\omega}h(z)\ . \quad \text{(A4)}$$

According to Leaver, $h(z)$ can be expanded in a series of Coulomb wave functions:

$$h(z)=\left[A_{\rm out}\sum_{L=-\infty}^{\infty}b_Lu_{L+\nu}^{(+)}(z)+A_{\rm in}\sum_{L=-\infty}^{\infty}b_Lu_{L+\nu}^{(-)}(z)\right]\left[\frac{r}{r-2}\right]^{2i\omega}\ , \quad \text{(A5)}$$

$$u_{L+\nu}^{(\pm)}=\left\{\sum_{L=-\infty}^{\infty}b_L\exp\left[\pm i\left[2\omega\ln 4\omega-\frac{\pi}{2}(L+\nu)+\sigma_L\right]\right]\right\}^{-1}[G_{L+\nu}(\eta,z)\pm iF_{L+\nu}(\eta,z)]\ , \quad \text{(A6)}$$

where $F_{L+\nu}(\eta,z)$ and $G_{L+\nu}(\eta,z)$ are the regular and irregular Coulomb wave functions, respectively [22], and ν is defined later. The Coulomb wave functions are solutions of the differential equations

$$\frac{d^2}{dz^2}u_{L+\nu}+\left[1-\frac{2\eta}{z}-\frac{(L+\nu)(L+\nu+1)}{z^2}\right]u_{L+\nu}=0 \quad \text{(A7)}$$

and

$$\frac{d}{dz}u_{L+\nu}=-\frac{L+\nu}{2L+2\nu+1}R_Lu_{L+\nu+1}-Q_Lu_{L+\nu}+\frac{L+\nu+1}{2L+2\nu+1}R_Lu_{L+\nu-1}\ , \quad \text{(A8)}$$

where

$$Q_L=\eta/[(L+\nu)(L+\nu+1)]\ ,$$
$$R_L=[(L+\nu)^2+\eta^2]^{1/2}/(L+\nu)\ .$$

They satisfy the recurrence relation

$$\frac{1}{2L+2\nu+1}R_{L+1}u_{L+\nu+1}-\left[\frac{1}{z}+Q_L\right]u_{L+\nu}+\frac{1}{2L+2\nu+1}R_Lu_{L+\nu-1}=0\ . \quad \text{(A9)}$$

$F_{L+\nu}$ and $G_{L+\nu}$ are normalized such that the Wronskian becomes

$$F_{L+\nu,z}G_{L+\nu}-F_{L+\nu}G_{L+\nu,z}=1\ .$$

They have asymptotic forms such as

$$G_{L+\nu}\pm iF_{L+\nu}\rightarrow\exp\left[\pm i\left[z-\eta\ln 2z-\frac{\pi}{2}(L+\nu)+\sigma_L\right]\right]\ , \quad \text{(A10)}$$

where

$$\sigma_L=\arg\Gamma(L+\nu+1+i\eta)\ . \quad \text{(A11)}$$

From this equation, we can see that $h(z)$ has the desired asymptotic property. The coefficients b_L in the expansion (A5) are defined by the recurrence relation

$$\alpha_Lb_{L+1}+\beta_Lb_L+\gamma_Lb_{L-1}=0\ , \quad \text{(A12)}$$

where

$$\alpha_L=-\frac{\omega R_{L+1}}{2L+2\nu+3}[2(L+\nu+1)(L+\nu+2)-(L+\nu+2)C_1-C_3]\ ,$$
$$\beta_L=(L+\nu)(L+\nu+1)+C_2+\omega Q_L[2(L+\nu)(L+\nu+1)-C_1-C_3]\ ,$$
$$\gamma_L=-\frac{\omega R_L}{2L+2\nu-1}[2(L+\nu)(L+\nu-1)+(L+\nu-1)C_1-C_3]\ .$$

It is necessary that expansion (A5) converges. The recurrence relation (A12) has two independent solutions which have the asymptotic property

$$b_{L+1}/b_L \sim L/\omega \text{ or } \omega/L \text{ as } L\to\infty \ .$$

It is also known that $F_{L+\nu}$ and $G_{L+\nu}$ have asymptotic properties such as

$$F_{L+\nu}\sim z/(2L),\ \ G_{L+\nu}\sim 2L/z \ \text{ as } L\to\infty \ .$$

If we chose expansion coefficients which have the asymptotic property $b_{L+1}/b_L \sim w/L$, the series $\sum_{L=0}^{\infty} b_L u_{L+\nu}$ is absolutely convergent for $r>2$. In this case coefficients b_L are said to be the minimal solution of the three-term recurrence relation (A12) [23]. The proof of the convergent property as $L\to-\infty$ can be shown in the same way.

From recurrence relation (A12), we can formally derive continued fractions:

$$b_{L+1}/b_L = \frac{-\gamma_{L+1}}{\beta_{L+1}-}\frac{\alpha_{L+1}\gamma_{L+2}}{\beta_{L+2}-}\frac{\alpha_{L+2}\gamma_{L+3}}{\beta_{L+3}-} \ , \tag{A13}$$

$$b_{L-1}/b_L = \frac{-\alpha_{L-1}}{\beta_{L-1}-}\frac{\alpha_{L-2}\gamma_{L-1}}{\beta_{L-2}-}\frac{\alpha_{L-3}\gamma_{L-2}}{\beta_{L-3}-} \ . \tag{A14}$$

From the theory of the three-term recurrence relation, we can show that the continued fraction (A13) and (A14) converges if and only if $\{b_L\}$ is a minimal solution of the recurrence relation (A12) as $L\to\pm\infty$. When $\{b_L\}$ is minimal, from (A12), (A13), and (A14) and by setting $L=0$, we obtain

$$\beta_0 = \frac{\alpha_{-1}\gamma_0}{\beta_{-1}-}\frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-}\frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-}\cdots + \frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\frac{\alpha_2\gamma_3}{\beta_3-}\cdots \ . \tag{A15}$$

This equation can be satisfied by appropriately choosing ν. Conversely $\{b_L\}$ is minimal as $L\to\pm\infty$ if ν is a solution of an implicit equation (A15).

If we have a solution ν of Eq. (A15), $\nu+n$ (n = integer) are also solutions. When $\omega\to 0$, the differential equation (A2) has the limiting form

$$\frac{d^2h}{dz^2}+\left[1-\frac{2\eta}{z}+\frac{C_2}{z^2}\right]h=0 \ .$$

This equation has solutions $h(z)=G_{\nu 0}(0,z)$ and $F_{\nu 0}(0,z)$, where

$$\nu_0 = -\tfrac{1}{2}(1\pm\sqrt{1-4C_2}) = l \ \text{ or } \ -(l+1) \ .$$

We must determine ν and b_L such that they reproduce the above solutions when ω approaches 0.

In this way, we obtain the solutions which behave as purely ingoing and purely outgoing solutions at infinity and we can calculate $A_{\rm in}$ and $A_{\rm out}$ from

$$X_{\rm in} = \left[A_{\rm out}\sum_{L=-\infty}^{\infty} b_L u_{L+\nu}^{(+)}(z) + A_{\rm in}\sum_{L=-\infty}^{\infty} b_L u_{L+\nu}^{(-)}(z)\right]\left[\frac{r}{r-2}\right]^{2i\omega} , \tag{A16}$$

and its derivatives at some point $z=\omega r$.

Next we explain the details of numerical calculations. First we calculate a homogeneous solution of the RW equation $X_{\rm in}$ from Eqs. (11) and (10). Since a_n is not the minimal solution of the recurrence relation, there are no difficulties in calculating a_n. When the orbital radius r_0 is large, the convergence of expansion (10) is slow.

To evaluate the series of Coulomb wave functions, we first determine a parameter ν as a solution of (A15) using Brent's algorithm [19]. In our case, (A15) is a real function. We did not have any trouble in finding this solution. ν must approach ν_0 when ω approaches zero and this property can be checked directly. Second, the minimal solution $\{b_L\}$ must be determined. It is known that a minimal solution cannot be calculated numerically by the forward recursion of the recurrence relation from $L=0$ to infinity. So we use the property (A13) and (A14) to evaluate b_L from $L=0$ to infinity. We always set $b_0=1$. The continued fractions are determined by Steed's algorithm [24]. The Coulomb wave functions are also determined by the method shown in Barnett *et al.* [24]. The convergence of the series of Coulomb wave functions are usually very rapid and we have to add only the first several ten terms. To determine $A_{\rm in}$ and $A_{\rm out}$, we must evaluate (A16) and its derivative at some point r. We must choose appropriately the point r where we evaluate Eq. (A16) because when r is small, irregular Coulomb wave functions $G_{L+\nu}(\eta,z)$ diverge. This property is enhanced when ω is small because $G_{L+\nu}(\eta,z)$ have an asymptotic property such as z^{-L} as $L\to\infty$. We choose r around $\sim 1/\omega$.

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