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Another positivity proof and gravitational energy localizations

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Two locally positive expressions for the gravitational Hamiltonian, one using four-spinors and the other special orthonormal frames, are reviewed. A new quadratic three-spinor-curvature identity is used to obtain another positive expression for the Hamiltonian and thereby a localization of gravitational energy and positive energy proof. These new results provide a link between the other two methods. Localization and prospects for quasilocalization are discussed.

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I. INTRODUCTION

For asymptotically flat gravitating systems *total* energy is well defined and must be non-negative. Each new positive total energy proof (e.g., [1]) offers some more insight. Concerning the localization of the total energy, although the equivalence principle forbids a true local gravitational energy density, a suitable "quasilocalization" is desirable [2]. A good candidate for a gravitational energy density is the Hamiltonian density. For asymptotically flat Einstein gravity the Hamiltonian density has the general form [3]

$$H(N) = \int d^3x\, 2N^\mu G^0_{\ \mu} + \oint B$$
$$= \int d^3x\ N\mathcal{H} + N^k\mathcal{H}_k + \oint B\ , \tag{1.1}$$

which includes a boundary term at spatial infinity. On a solution the spatial integral vanishes; the *value* of the Hamiltonian $-16\pi G N^\mu p_\mu$ comes from the integral of the boundary term over the two-sphere at spatial infinity and determines the total four-energy-momentum p_μ. The integrand of the boundary term B is only well defined up to $O(r^{-2})$; moreover, we have the freedom to choose the *lapse* N and *shift* N^k. Together these allow a certain latitude which can be exploited to obtain a locally non-negative Hamiltonian density. Indeed, such a form can be achieved in more than one way.

II. THE FOUR-COVARIANT QUADRATIC SPINOR HAMILTONIAN

The first constructions of this type [4] were done in the wake of the Witten positive-energy proof. It was shown that the *Hamiltonian density* for Einstein gravity could be expressed as a four-covariant quadratic spinor three-form:

$$\mathcal{H}(\psi) := 2\{D(\overline{\psi}\gamma_5\gamma)\wedge D\psi - D\overline{\psi}\wedge D(\gamma_5\gamma\psi)\}\ . \tag{2.1}$$

This remarkable result follows from (i) the identity

$$\mathcal{H}(\psi) \equiv 2N^\mu G^\nu_{\ \mu}\eta_\nu + d\{\overline{\psi}\gamma_5\gamma\wedge D\psi + D(\overline{\psi}\gamma_5\gamma)\psi$$
$$-\overline{\psi}D(\gamma_5\gamma\psi) + D\overline{\psi}\wedge(\gamma_5\gamma\psi)\}\ , \tag{2.2}$$

where $N^\mu = \overline{\psi}\gamma^\mu\psi$ (for conventions see the appendix) revealing that $\mathcal{H}(\psi)$ contains the appropriate projected components of the Einstein tensor (needed to generate the equations of motion) up to an exact differential (which does not change the variational derivatives), and (ii) the observation that $\delta\int\mathcal{H}(\psi)$ has an asymptotically vanishing boundary integral since $\mathcal{H}(\psi)$ is asymptotically of order $O(r^{-4})$.

The Hamiltonian density (2.1) can be decomposed, with respect to the normal to any spacelike hypersurface, into positive and negative definite parts,

$$\mathcal{H}(\psi) \simeq 4(g^{ab}D_a\psi^\dagger D_b\psi - |\gamma^a D_a\psi|^2)\,\eta_0\ , \tag{2.3}$$

and is locally non-negative if ψ satisfies the Witten equation (or certain modifications thereof)

$$\gamma^a D_a\psi = 0\ , \tag{2.4}$$

thereby permitting a non-negative "localization" of gravitational energy.

This mathematically elegant form of the gravitational Hamiltonian has several virtues: in particular it (i) is manifestly four-covariant, (ii) shows that total four-momentum is future timelike, and (iii) can be evaluated on a spacelike surface extending to future null infinity thereby showing that the Bondi four-momentum also is future timelike. However it also has some liabilities: in particular (a) the spinor field is physically mysterious, (b) there is no direct relation to the customary variables, and (c) it yields an unintuitive energy localization. [For the Schwarzschild solution in isotropic Cartesian frames $\psi = (1+m/2r)^{-2}\psi_{\rm flat}$ solves the Witten equation; using this result in the boundary integral yields 1/8 of the total mass-energy inside the horizon and 7/8 outside.] Consequently other Hamiltonian-based positivity proofs and/or localizations were sought and found.

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III. THE SPECIAL ORTHONORMAL FRAME APPROACH

Another approach [5] used orthonormal frames and exploited their rotational gauge freedom. The Arnowitt-Deser-Misner (ADM) Hamiltonian (1.1) in an asymptotically Cartesian frame has the form

$$H(N) = \int d^3x \; N(g^{-\frac{1}{2}}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2) - g^{\frac{1}{2}}R) + 2\pi^m{}_k\nabla_m N^k + \oint dS_k \; N\,\delta^{kc}_{ab}\,\Gamma^{ab}{}_c \; . \quad (3.1)$$

We choose $N^k = 0$, use the divergence theorem to eliminate the boundary term, parametrize the metric with orthonormal frames, split the connection coefficients algebraically into a symmetric tensor q_{ab}, a vector $q_c := -\Gamma^a{}_{ca}$ and a scalar $q := \epsilon^{abc}\Gamma_{abc}$, and use the *special orthonormal frame* (SOF) [6] rotational gauge conditions

$$q_k = 4\partial_k \ln\Phi, \qquad q = \text{const}, \quad (3.2)$$

to obtain the Einstein Hamiltonian (i.e., energy) density in the form

$$\mathcal{H}(N) = 8g^{\frac{1}{2}}g^{nm}\partial_n(N\Phi^{-1})\partial_m\Phi + N\{g^{-\frac{1}{2}}(\pi^{ab}\pi_{ab} - \tfrac{1}{2}\pi^2) + g^{\frac{1}{2}}(q^{ab}q_{ab} - \tfrac{1}{2}q^2)\} \; . \quad (3.3)$$

This expression is good for both compact spatial surfaces (in which case q is a nonvanishing constant) and for asymptotically flat spatial surfaces (in which case q vanishes). For the latter case total energy is well defined; a suitable choice of the lapse gives a positive total energy proof.

Many choices for the lapse give a positive local energy density, in particular $N = \Phi^a, (a \geq 1)$. An especially attractive choice is $N = \Phi$, which leads to the *gravitational energy density*

$$\mathcal{H}(\Phi) = \Phi\{g^{-\frac{1}{2}}(\pi^{ab}\pi_{ab} - \tfrac{1}{2}\pi^2) + g^{\frac{1}{2}}(q^{ab}q_{ab} - \tfrac{1}{2}q^2)\} \; , \quad (3.4)$$

and the value

$$E = (16\pi G)^{-1}\int_V \mathcal{H}(\Phi)\,d^3x = (2\pi G)^{-1}\oint_S g^{\frac{1}{2}}\nabla^k\Phi dS_k \; , \quad (3.5)$$

for the amount of energy localized within a volume V bounded by a surface S. The value is non-negative for asymptotically flat maximal spatial hypersurfaces. The *gravitational potential* Φ (generalized Newtonian potential) satisfies the generalized Poisson equation

$$8g^{\frac{1}{2}}\Delta\Phi = \mathcal{H}(\Phi) + 16\pi\Phi g^{\frac{1}{2}}G\rho \; . \quad (3.6)$$

(This is just the Hamiltonian constraint and is essentially the scale equation of the usual conformal approach to the Einstein initial value constraints, see, e.g., Choquet-Bruhat and York [7]).

This SOF Hamiltonian has certain virtues, especially (i) the gauge conditions are conformally invariant so SOF's are closely related to the usual variables of the standard initial value constraints, (ii) the oscillating physical modes are apparent in the SOF gravitational energy density, and (iii) the energy localization is physically reasonable; in particular, all of the mass of the Schwarzschild solution [note: $\Phi = (1 + m/2r)^{-1}$ for the isotropic cartesian frame] is within the horizon. (Moreover there is some freedom here; the choice $N = \Phi^4$ produces the same 1/8 inside the horizon as the four-covariant spinor Hamiltonian for the Schwarzschild solution.) However the SOF approach also has certain liabilities: in particular (a) the expression concerns only energy, it gives no restraint on the momentum, (b) the energy is guaranteed to be *locally* non-negative only for maximal spacelike hypersurfaces, and (c) a maximal spacelike hypersurface cannot be extended to future null infinity so this approach cannot give the Bondi mass energy.

IV. A NEW THREE-SPINOR PROOF AND LOCALIZATION

A link between the special orthonormal frame approach and the four-covariant quadratic spinor form of the Hamiltonian has now been found in terms of a new Hamiltonian-based gravitational energy positivity proof and localization which uses three-dimensional spinors.

The key is a new spinor identity (see Appendix)

$$2[\nabla(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \nabla\varphi^\dagger\wedge\nabla(i\sigma\varphi)] \equiv dB - (\varphi^\dagger\varphi)\Omega^{ab}\wedge\zeta_{ab} \; , \quad (4.1)$$

where

$$B := \varphi^\dagger i\sigma\wedge\nabla\varphi - \varphi^\dagger\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\varphi + (\nabla\varphi^\dagger)\wedge i\sigma\varphi \; . \quad (4.2)$$

Using this identity we replace the scalar curvature term $NRg^{1/2}d^3x = N\Omega^{ab}\wedge\zeta_{ab}$ and the boundary term in the ADM Hamiltonian (3.1) with the left-hand side of (4.1). The Einstein Hamiltonian (with $N = \varphi^\dagger\varphi$, $N^k = 0$) can then be written as

$$H(N) = \int d^3x\,(\varphi^\dagger\varphi)g^{-\frac{1}{2}}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2) + 2[\nabla(\varphi^\dagger i\sigma)\wedge\varphi - \nabla\varphi^\dagger\wedge\nabla(i\sigma\varphi)] \; . \quad (4.3)$$

An important property of the three-spinor Hamiltonian (4.3) is that no additional boundary term at infinity is needed. As Regge and Teitelboim [8] have nicely explained, it is necessary that the boundary terms in the variation of the Hamiltonian vanish asymptotically. To verify this property for (4.3) we need only check the variation of the quadratic spinor terms (see Appendix):

$$\delta[\nabla(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \nabla\varphi^\dagger\wedge\nabla(i\sigma\varphi)] \equiv -\Omega^{ab}\wedge\delta[(\varphi^\dagger\varphi)\zeta_{ab}] - \tfrac{1}{2}\delta\omega^{ab}\wedge\nabla[(\varphi^\dagger\varphi)\zeta_{ab}] + d\{\delta(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \delta\varphi^\dagger\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\delta\varphi + (\nabla\varphi^\dagger)\wedge\delta(i\sigma\varphi)\}. \quad (4.4)$$

Since $\varphi = \text{const} + O(1/r)$ and $\delta\varphi = O(1/r)$ the boundary term falls off as $O(r^{-3})$, therefore vanishing at spatial infinity for aymptotically flat initial data.

On a maximal hypersurface $\pi = 0$, the kinetic terms in the Hamiltonian (4.3) are non-negative. The quadratic spinor terms can also be made non-negative. Since the torsion vanishes the spinor terms reduce to

$$\begin{aligned} 2[\nabla(\varphi^\dagger i\sigma) \wedge \nabla\varphi - \nabla\varphi^\dagger \wedge \nabla(i\sigma\varphi)] \\ &= -4[\nabla_a\varphi^\dagger i\zeta^{abc}\sigma_c\nabla_b\varphi]\zeta \\ &= 4[g^{ab}\nabla_a\varphi^\dagger\nabla_b\varphi - \nabla_a\varphi^\dagger\sigma^a\sigma^b\nabla_b\varphi]\zeta . \end{aligned} \tag{4.5}$$

Hence the spinor terms are non-negative for any asymptotically constant φ satisfying the three-dimensional Dirac equation

$$\sigma^a\nabla_a\varphi = 0 \ . \tag{4.6}$$

This is a linear elliptic equation similar to the Witten equation; essentially the same arguments show that unique solutions exist. Hence the local energy density is non-negative on asymptotically spacelike maximal slices.

This three-spinor Hamiltonian approach by itself has most of the liabilities of the other two approaches: (a) it yields the same sort of unintuitive energy localization as the four-covariant spinor expression (indeed they have identical values for the Schwarzschild solution, but differ when $K_{ab} \neq 0 \neq \overline{\psi}\gamma^a\psi$); (b) the expression concerns only energy, it gives no restraint on the momentum; (c) the energy is guaranteed to be *locally* non-negative only for *maximal* spacelike hypersurfaces; (d) the maximal spacelike hypersurface cannot be extended to future null infinity so it cannot give the Bondi mass-energy. However, for the three-spinor Hamiltonian some of the other four-spinor liabilities are not so severe since in this case (i) the spinor field is not so mysterious, for (ii) there is a relation to the customary variables via the SOF variables as we shall show below. Indeed the principal virtue of this approach is that it relates the other two methods we have discussed.

V. RELATIONS BETWEEN THE METHODS

The three-spinor Hamiltonian expression (4.3) is intermediate between the four-covariant spinor Hamiltonian (2.1) discussed above and the SOF Hamiltonian (3.3).

On the one hand it can be extracted as a piece of the 3+1 decomposition of the four-covariant spinor Hamiltonian. The orthonormal frame components of the metric compatible four-connection project into the components of the three-connection and the extrinsic curvature $K^a{}_b = -\Gamma^{0a}{}_b$; hence,

$$\begin{aligned} D_c\psi &= \partial_c\psi - \tfrac{1}{4}\Gamma^{\alpha\beta}{}_c\gamma_{[\alpha}\gamma_{\beta]}\psi \\ &= \partial_c\psi - \tfrac{1}{4}\Gamma^{ab}{}_c\gamma_{[a}\gamma_{b]}\psi - \tfrac{1}{2}\Gamma^{0b}{}_c\gamma_{[0}\gamma_{b]}\psi \\ &= \nabla_c\psi + \tfrac{1}{2}K^b{}_c\gamma_{[0}\gamma_{b]}\psi . \end{aligned} \tag{5.1}$$

Consequently the quadratic in $D\psi$ Hamiltonian decomposes into (a) quadratic terms in $K^a{}_b$ along with quadratic terms in $\nabla\psi$, which are essentially the three-spinor Hamiltonian density (4.3), and (b) linear terms in $K^a{}_b$ (they are of the form $2\pi^m{}_c\nabla_m N^c$, where $N^c = \overline{\psi}\gamma^c\psi$), which represent the momentum constraint. (Note that the three-spinor method decouples the spinor field from N^k; this has both advantages and disadvantages.)

On the other hand the three-spinor Hamiltonian not only resembles the SOF approach in (i) using a vanishing shift, (ii) considering the kinetic terms separately, (iii) relying on maximal slices, and (iv) replacing the potential terms by an expression using different variables, but, moreover, there is a close relation between the SOF variables and spinor fields via solutions to the three-dimensional Dirac equation (4.6).

Indeed the three-dimensional Dirac equation explicitly depends only on the parts of the connection which appear in the gauge conditions (3.2):

$$\begin{aligned} \sigma^c\nabla_c\varphi &= \sigma^c(\varphi_{,c} + \tfrac{1}{4}\Gamma^{ab}{}_c\sigma_{[a}\sigma_{b]}\varphi) \\ &= \sigma^c\varphi_{,c} - \tfrac{1}{2}q_b\sigma^b\varphi + \tfrac{1}{4}iq\varphi \ . \end{aligned} \tag{5.2}$$

An asymptotically constant solution to $\sigma^a\nabla_a\varphi = 0$ can be factored into a magnitude and a unitary transformation which determines an SOF [9] . Conversely, expressed in terms of an SOF, the Dirac equation reduces to $\sigma^a\partial_a\Phi^{-2}\varphi = 0$, hence $\varphi = \Phi^2\varphi_{\text{const}}$.

VI. LOCALIZATION AND QUASILOCALIZATION

Our considerations have been concerned with obtaining a positive *localization* of the *total energy* by finding a good expression for the Hamiltonian density. Each localization depends on the solution to an elliptic equation, which, in turn, depends on the values on the boundary of the region. Since the boundary is at spatial infinity we can simply choose suitable constant values as the physically appropriate boundary conditions.

Beyond distributing the total gravitational energy, there is considerable interest in "quasilocalization," i.e., determining the amount of energy in a finite region *without* reference to what is outside. The expressions we have discussed could also be used for a finite region. Then each of the (locally positive) Hamiltonian densities provides a quasilocal energy. The value of the positive quasilocal energy can be obtained from the associated boundary integral. The quasilocalization will depend on the choice of *boundary values* on the finite two-surface bounding the region. We, however, do not know how to decide which values on a finite boundary are a physically good choice. Several "quasilocalizations" based upon four-covariant spinor expressions such as (2.1) and (2.2) have been investigated by others [10]. Their methods of choosing boundary values for the four-spinor field could also be adapted to our orthonormal frame or three-spinor fields. Canonical investigations associated with a *finite* region, with particular attention to the possible boundary terms and their relation to what is held fixed on the boundary, have been done for the standard variables [11]. Such a study of the spinor or SOF parametrized Hamiltonian should provide some guidance for the choice of appropriate boundary values for finite regions.

Nontrivial examples of the localizations produced by the four-spinor, SOF, and three-spinor techniques, e.g., for the Kerr solution, would be instructive. However, as noted, the localizations depend on solving an elliptic system of equations, essentially the Dirac equation. Unfortunately, aside from the aforementioned spherically symmetric case, there are hardly any known exact solutions for the Dirac equation in curved spacetime [12].

Forgoing direct comparisons for actual solutions we can compare the expressions by considering desirable properties. We know of no gravitational energy localization method which is satisfactory. One list [13], for example, requires (i) zero for flat spacetime, (ii) the standard value for spherical solutions, (iii) the ADM value for an asymptotically flat slice, (iv) the Bondi value for an asymptotically null slice, (v) the irreducible mass for the apparent horizon, and (vi) positivity and monotonicity. Of the methods considered here, the quadratic four-spinor expression certainly fails (ii) and (v). The SOF Hamiltonian satisfies the positivity requirement (vi) only on maximal slices, while the maximal slice restriction precludes satisfying (iv). The new three-spinor technique has all of these failings.

VII. CONCLUSION

We have presented a new positive total energy proof for asymptotically flat Einstein gravity. The proof uses a three-dimensional spinor parametrization of the Hamiltonian and a new three-spinor-curvature identity. What insight has it yielded so far?

As a proof, considered on its own, this method has no advantages and indeed is less general than some other known proofs. More interesting is the fact that it provides *another* independent method for obtaining a *positive localization* of gravitational energy; yet again, as a localization method, it has no apparent advantages.

Probably the most interesting thing is that it provides a *link* between two other Hamiltonian-based proofs and their associated localizations. At the very least this link connects the somewhat mysterious Witten spinor field proof and localization to the more usual type variables.

Perhaps this link will play an essential role in finding a modification of our expressions into a better Hamiltonian density—one which permits a positive energy proof and gravitational energy localization combining the virtues of the four-covariant spinor Hamiltonian and special orthonormal frame approaches without the liabilities. Such an expression, complimented by a good choice of boundary values for finite regions, would provide a physically reasonable quasilocalization of gravitational energy.

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APPENDIX: CONVENTIONS AND IDENTITIES

Our four-dimensional conventions are metric signature $(-1,+1,+1,+1)$, orthonormal coframe θ^μ, unit volume element η, and unit three-form basis $\eta_\mu = *\theta_\mu$. The spinor conventions are $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2g_{\mu\nu} = 2\mathrm{diag}(+1,-1,-1,-1),$$

$\gamma = \gamma_\mu\theta^\mu$ and $D\psi = d\psi - \frac{1}{4}\omega^{\mu\nu}\gamma_{[\mu}\gamma_{\nu]}\psi$ is the covariant differential.

The three-dimensional spinor conventions used are $\sigma = \sigma_c\theta^c$, where

$$\sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab},$$

with $\sigma_{ab} = \frac{1}{4}[\sigma_a,\sigma_b]$, so $\sigma_c\sigma_{ab} + \sigma_{ab}\sigma_c = i\zeta_{abc}$, and $\zeta_{ab} = \zeta_{abc}\theta^c$, where ζ_{abc} is the three-dimensional Levi-Civita tensor with $\zeta_{123} = +1$.

The identity connecting the three-dimensional scalar curvature to the spinor expression in the Hamiltonian can be verified as follows:

$$\begin{aligned}
2[\nabla(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \nabla\varphi^\dagger\wedge\nabla(i\sigma\varphi)] &\equiv dB - [-\varphi^\dagger i\sigma\nabla^2\varphi + \nabla^2(\varphi^\dagger i\sigma)\varphi - \varphi^\dagger\nabla^2(i\sigma\varphi) + (\nabla^2\varphi^\dagger)i\sigma\varphi]\\
&\equiv dB - \tfrac{1}{2}\Omega^{ab}\wedge[-\varphi^\dagger i\sigma\sigma_{ab}\varphi - \varphi^\dagger i\sigma\sigma_{ab}\varphi - \varphi^\dagger i\sigma_{ab}\sigma\varphi - \varphi^\dagger i\sigma_{ab}\sigma\varphi]\\
&\equiv dB + \Omega^{ab}\wedge[\varphi^\dagger(i\sigma\sigma_{ab} + i\sigma_{ab}\sigma)\varphi] \equiv dB - (\varphi^\dagger\varphi)\Omega^{ab}\wedge\zeta_{ab}, \qquad \text{(A1)}
\end{aligned}$$

where $B := \varphi^\dagger i\sigma\wedge\nabla\varphi - \varphi^\dagger\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\varphi + (\nabla\varphi^\dagger)\wedge i\sigma\varphi$.

Similarly, we calculate the variation of the quadratic spinor terms:

$$\begin{aligned}
\delta[\nabla(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \nabla\varphi^\dagger\wedge\nabla(i\sigma\varphi)] &\equiv [\nabla\delta(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \nabla\delta\varphi^\dagger\wedge\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\wedge\nabla\delta\varphi - (\nabla\varphi^\dagger)\wedge\nabla\delta(i\sigma\varphi)]\\
&\quad + [(\delta\nabla)(\varphi^\dagger i\sigma)\wedge\nabla\varphi - (\delta\nabla)\varphi^\dagger\wedge\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\wedge(\delta\nabla)\varphi - \nabla\varphi^\dagger\wedge(\delta\nabla)(i\sigma\varphi)]\\
&\equiv d[\delta(\varphi^\dagger i\sigma)\wedge\nabla\varphi - \delta\varphi^\dagger\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\delta\varphi + (\nabla\varphi^\dagger)\wedge\delta(i\sigma\varphi)]\\
&\quad - [-\delta(\varphi^\dagger i\sigma)\wedge\nabla^2\varphi - \delta\varphi^\dagger\nabla^2(i\sigma\varphi) + \nabla^2(\varphi^\dagger i\sigma)\delta\varphi + \nabla^2\varphi^\dagger\wedge\delta(i\sigma\varphi)]\\
&\quad + \tfrac{1}{2}\delta\omega^{ab}\wedge[-(\varphi^\dagger i\sigma)\sigma_{ab}\wedge\nabla\varphi + \varphi^\dagger\sigma_{ab}\nabla(i\sigma\varphi) + \nabla(\varphi^\dagger i\sigma)\sigma_{ab}\varphi + \nabla\varphi^\dagger\sigma_{ab}\wedge i\sigma\varphi]\\
&\equiv d[\cdots] - \tfrac{1}{2}\Omega^{ab}\wedge[-\delta(\varphi^\dagger i\sigma)\sigma_{ab}\varphi - \delta\varphi^\dagger\sigma_{ab}(i\sigma\varphi) - (\varphi^\dagger i\sigma)\sigma_{ab}\delta\varphi - \varphi^\dagger\sigma_{ab}\delta(i\sigma\varphi)]\\
&\quad + \tfrac{1}{2}\delta\omega^{ab}\wedge\nabla[\varphi^\dagger(i\sigma\sigma_{ab} + \sigma_{ab}i\sigma)\varphi]
\end{aligned}$$

$$\equiv d[\cdots] + \Omega^{ab} \wedge \delta[\varphi^\dagger(i\sigma\sigma_{ab} + \sigma_{ab}i\sigma)\varphi] + \tfrac{1}{2}\delta\omega^{ab} \wedge \nabla[\varphi^\dagger(i\sigma\sigma_{ab} + \sigma_{ab}i\sigma)\varphi]$$
$$\equiv d[\cdots] - \Omega^{ab} \wedge \delta[(\varphi^\dagger\varphi)\zeta_{ab}] - \tfrac{1}{2}\delta\omega^{ab} \wedge \nabla[(\varphi^\dagger\varphi)\zeta_{ab}]. \qquad \text{(A2)}$$

We have recently discovered [14] that there are many identities such as (2.2) and (4.1) in Riemannian and Riemann-Cartan spaces of any dimension.

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