Relativistic effects in systems of three constituent quarks

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We investigate the most general form of the three-body mass operator in the instant form of relativistic dynamics. It is shown that this operator is defined not only by the two-body mass operators and the three-body interaction operator but also by some three extra unitary operators. The latter are nontrivial already on the one-gluon-exchange level and, therefore, must be present in any relativistic constituent quark model. The restrictions imposed by the relativistic invariance on the form of the two-body energy operator in first order in $1/m^2$ are discussed in detail. We write down the explicit expression for the three-quark mass operator in this order and discuss the dependence of this operator on the total momenta of each pair of quarks in the c.m. frame of three quarks.

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I. INTRODUCTION

The present status of constituent quark models (CQM's) is such that they by no means can be considered "naive" and "nonrelativistic" [1-3]. Relativistic effects, especially in systems of light quarks, must be important since, as follows from the results of calculations in the CQM's framework, the quark velocities are comparable with the velocity of light.

It is generally accepted that the relativistic description of any bound system must be based on the Bethe-Salpeter equation (BSE) or its modifications. Then one should take into account that, according to the notion of constituent quarks, all degrees of freedom different from those related to constituent quarks must be integrated out. This corresponds to some reduction of the BSE (for example, the equal time or null plane reduction). However, in practice it is difficult to control the validity of approximations used in reducing the BSE, and, as we shall see in Sec. IV, even the crucial property of relativistic invariance may be lost in the reduction procedure.

Of course, the description of hadrons in terms of constituent quarks can be only approximate, but if we assume this approximation then, in our opinion, the most suitable approach consistent with the notion of relativistic constituent quarks is relativistic quantum mechanics (RQM), i.e., the relativistic theory of systems with a given number of degrees of freedom.

The main notions of RQM are those of relativistic invariance and cluster separability. The relativistic invariance implies that the system under consideration is described by a unitary representation of the Poincaré group, and, therefore, the generators of this representation satisfy the commutation relations

$$[P^\mu, P^\nu] = 0, \quad [M^{\mu\nu}, P^\rho] = -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu),$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\, g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}), \quad (1)$$

where $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$; P^μ are the four-momentum operators, $M^{\mu\nu}$ are the four-dimensional angular momentum operators, the metric tensor in Minkowski space has the nonzero components $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, and we choose the system of units with $\hbar = c = 1$.

The notion of cluster separability can be formulated in different ways [4-7]. In the context of CQM's, taking into account the confinement, it seems natural to use the cluster separability in its weakest form, i.e., only algebraically. This implies that if all interactions between any subsystems comprising our system are turned off, then the generators for the whole system become sums of the corresponding generators for these subsystems.

In the literature the term "relativistic invariance" means often not only the conditions (1), but also some "dynamical" conditions which can be formulated only in the given relativistic model. In this study, speaking about relativistic invariance, we shall mean only the "kinematical" restrictions imposed by (1).

Let us consider, for example, a system, consisting of a quark and antiquark. Equations (1) impose considerable restrictions on the form of corresponding generators (see Secs. II–IV), and this is important, for example, in calculating the matrix elements of the electromagnetic or weak current for our system. However, if we are interested only in finding the spectrum, we can restrict ourselves to the consideration of the mass operator in the c.m. frame of the quark and antiquark. Then Eqs. (1) reduce to the restriction that the mass operator must commute with the system spin operator (see Sec. III), and, therefore, in this instance we have the same restriction as in the nonrelativistic theory (though in concrete relativistic models the form of the mass operator differs from that in the nonrelativistic theory). However, in the case of three quarks, the relativistic invariance and clus-

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ter separability impose considerable restrictions already on the mass operator level (see Sec. V).

Let us make a few remarks about the role of relativity in three-quark systems.

In the nonrelativistic approach the mass operator of a three-quark system is written as

$$\hat{M}^{\rm nr} = \tilde{T}^{\rm nr} + \sum_{i<j} \tilde{v}_{ij} + v_{123}, \tag{2}$$

where $T^{\rm nr}$ is the nonrelativistic kinetic-energy operator, $v_{ij}(i,j=1,2,3)$ is the interaction operator for quarks i and j, v_{123} is the three-quark-interaction operator, and the tilde over $T^{\rm nr}$ and v_{ij} shows that we consider the reduction of these operators on the states describing the three quarks in their c.m. frame (see Sec. V). The operator $\tilde{v}_{ij}$ contains the dependence on the relative distance between particles i and j but it does not depend on the momentum of the pair ij in the c.m. frame of the three quarks. In order to reproduce the experimental data, the operators $\tilde{v}_{ij}$ must contain the dependence on the spin-orbit, spin-spin, and, possibly other interactions. This dependence is usually considered only phenomenologically and the corresponding models are still called "nonrelativistic." Meanwhile, it is easy to see that the spin-orbit and spin-spin terms in $\tilde{v}_{ij}$ have the relativistic origin since they contain the factor $1/c^2$ in comparison with the Coulomb-like interaction. Therefore the theory becomes partially relativistic, and, as we shall see in Sec. VII, such a simple composition of interactions as in (2) contradicts the relativistic invariance already in first order in $1/c^2$. This is clear also from the fact that the order-$(1/c^2)$ corrections to the Coulomb-like potential contain not only the spin-orbit and spin-spin terms, but also the terms depending on the total momentum of the given pair in the c.m. frame of three quarks.

Some authors consider relativistic effects in three-quark systems in the framework of the so-called semirelativistic approach [8]. This implies that the kinetic-energy operator is chosen in the fully relativistic form, while the pair potentials remain as in Eq. (2). Such an approach is also unjustified since there are no reasons for the relativistic corrections (RC's) to the kinetic energy to be more considerable than to the potential energy. For example, calculating the RC's to the triton binding energy (when the triton is considered as the bound state of three nucleons), different authors observed that the RC's to the kinetic and potential energies were separately substantial, but, owing to strong cancellation of these contributions the resulting effect appeared to be small [9]. In addition, as in the above case of the "nonrelativistic" approach, there is no reason to neglect the terms containing the dependence of the interaction on the total momentum of each pair in comparison with the spin-orbit and spin-spin terms depending on the interquark distances.

To the best of our knowledge, the most detailed calculations of the baryonic spectra have been carried out in Ref. [2] where a good agreement with the experimental data was achieved by using almost the same parameters as in the calculation of mesonic spectra in Ref. [10]. The kinetic-energy operator in Ref. [2] was taken in the fully relativistic form and the importance of the relativization of interaction operators was demonstrated. However, some terms in the interaction operators depending on the total momentum of the corresponding pair of quarks were not taken into account. Therefore the problem arises whether the remaining discrepancy with the data can be explained by the contribution of these terms.

Since the quark velocities in systems of light quarks are comparable with the velocity of light, then one of the main difficulties in solving the relativistic three-quark problem is that the results of calculations are expected to be reliable only if relativity is taken into account *exactly*, i.e., an expansion in powers of $1/c$ is not used. Therefore not only the kinetic-energy operator but the interaction operators as well should be taken in the fully relativistic form. In view of the above discussion, this can be naturally done in RQM.

The first results in this theory were obtained by Dirac [11], and since that time major efforts of physicists working on RQM were devoted to the solution of the problem whether it is possible to introduce interactions into a system of N particles (where N is arbitrary but fixed), preserving the relativistic invariance and cluster separability. For this purpose Sokolov has developed the so-called method of packing operators and partially solved the problem [5,12]. The complete solution with the use of this method was first given by Coester and Polyzou [6] and Mutze [7] (see also Ref. [13]). It has turned out that the solution not only exists but is not unique. A detailed review of RQM can be found in Refs. [14,15].

In the case $N=3$ the explicit form of the mass operators was given by several authors in different forms of relativistic dynamics [16–20,5–7], and some authors applied the results to CQM's [21,22]. The most general investigation has been carried out in Ref. [6], and the results of this work, along with the results of Refs. [5,17–20] and [7] give grounds to state that in principle the three-body problem in RQM is solved. However, since the solution is not unique, the problem remains what is the physical choice of the three-body mass operator in three-quark systems.

Several authors argue that relativistic CQM's can be consistent only in the front form (or null plane) dynamics, since, in their opinion, only in this form it is possible to ensure the cluster separability, to separate the motion of the system as a whole and the internal motion, and to truncate the Fock space leaving only the states with a given N. However, as shown by Sokolov and Shatny [23], the front, instant, and point forms of RQM are physically equivalent for any N.

In the present work we use the instant form, since, in our opinion, it is technically more convenient, and the known corrections to the Coulomb-like potentials can easily be obtained in this form. We suppose to discuss elsewhere the most general form of the three-body mass operator in the front form and to show that the results are fully equivalent to those in the instant form. Therefore, the choice of the form is only the matter of convenience but not the matter of principle.

The main result of the present work is that the most

general form of the relativistic three-body mass operator is derived explicitly. Namely, we show that it depends not only on the two-body interaction operators v_{ij} and the three-body interaction operator v_{123} (as usual), but also on three extra unitary operators A_{ij}, and in QCD inspired models the latter dependence must be necessarily taken into account. We also discuss the choice of the solution for the relativistic three-body mass operator. For the convenience of readers we briefly summarize our results in Sec. VI, where we give an algorithm of constructing the three-body mass operator when the v_{ij}, v_{123}, and A_{ij} are taken as input. Then in Sec. VII we discuss what additional difficulties arise in the relativistic three-body problem in comparison with the nonrelativistic one and roughly estimate the size of the relativistic correction to the nucleon mass in the $1/m^2$ approximation.

II. REALIZATION OF THE REPRESENTATION OF THE POINCARÉ GROUP FOR A SYSTEM OF TWO FREE PARTICLES

To describe a relativistic system of interacting particles it is necessary to choose first the explicit form of the unitary irreducible representation (UIR) of the Poincaré group describing an elementary particle of mass m and spin $1/2$. There are many equivalent ways to construct an explicit realization of such a representation [24]. For our purposes it is convenient to choose the realization in the space of functions $\varphi(\mathbf{p},\sigma)$ such that

$$\sum_{\sigma}\int |\varphi(\mathbf{p},\sigma)|^2\, d^3\mathbf{p} < \infty, \tag{3}$$

where $\mathbf{p}$ is the particle momentum and σ is the spin projection ($\sigma = \pm 1/2$). Then the generators of the UIR, which satisfy the commutation relations (1) can be realized in the form (see, for example, Ref. [24])

$$\mathbf{P} = \mathbf{p}, \quad E = (m^2+\mathbf{p}^2)^{1/2}, \quad \mathbf{M} = \boldsymbol{l}(\mathbf{p}) + \mathbf{s},$$

$$\mathbf{N} = -i(m^2+\mathbf{p}^2)^{1/4}\frac{\partial}{\partial \mathbf{p}}(m^2+\mathbf{p}^2)^{1/4} + \frac{\mathbf{s}\times\mathbf{p}}{m+(m^2+\mathbf{p}^2)^{1/2}}, \tag{4}$$

where $\mathbf{P}$ is the momentum operator, $\mathbf{p}$ is the operator of multiplication by $\mathbf{p}$, E is the energy operator,

$$\boldsymbol{l}(\mathbf{p}) = -i\mathbf{p}\times\frac{\partial}{\partial \mathbf{p}}$$

is the orbital angular-momentum operator, $\mathbf{s}$ is the spin operator and $\mathbf{M} = (M^{23}, M^{31}, M^{12})$, $\mathbf{N} = (M^{01}, M^{02}, M^{03})$.

Let us now consider a system of two free particles of spin $1/2$ and the masses m_1 and m_2. The representation describing this system is realized in the space of functions $\varphi(\mathbf{p}_1,\sigma_1,\mathbf{p}_2,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}\int |\varphi(\mathbf{p}_1,\sigma_1,\mathbf{p}_2,\sigma_2)|^2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2 < \infty \tag{5}$$

and each of the representation generator is equal to the sum of the corresponding one-particle generators (4).

Instead of the variables $\mathbf{p}_1$ and $\mathbf{p}_2$ describing the separate particle momenta, we introduce the variables $\mathbf{P}$ and $\mathbf{k}$:

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{k} = \mathbf{p}_1 - \frac{\mathbf{P}\omega_1(\mathbf{p}_1)}{M} + \frac{(\mathbf{P}\cdot\mathbf{p}_1)\mathbf{P}}{M(M+E)},$$
$$\omega_j(\mathbf{p}) = (m_j^2+\mathbf{p}^2)^{1/2},\ E = \omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_2),\ M = (E^2-\mathbf{P}^2)^{1/2}. \tag{6}$$

Conversely, $\mathbf{p}_1$ and $\mathbf{p}_2$ are expressed in terms of $\mathbf{P}$ and $\mathbf{k}$ as

$$\mathbf{p}_1 = \mathbf{k} + \frac{\mathbf{P}\omega_1(\mathbf{k})}{M} + \frac{(\mathbf{P}\cdot\mathbf{k})\mathbf{P}}{M(M+E)}, \quad \mathbf{p}_2 = -\mathbf{k} + \frac{\mathbf{P}\omega_2(\mathbf{k})}{M} - \frac{(\mathbf{P}\cdot\mathbf{k})\mathbf{P}}{M(M+E)},$$
$$M = M(\mathbf{k}) = \omega_1(\mathbf{k})+\omega_2(\mathbf{k}),\ E = E(\mathbf{P},\mathbf{k}) = [M(\mathbf{k})^2+\mathbf{P}^2]^{1/2}. \tag{7}$$

Then, instead of the realization of the representation in the space of functions satisfying (5), we obtain the realization in the space H, in the space of functions $\varphi(\mathbf{P},\mathbf{k},\sigma_1,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}\int |\varphi(\mathbf{P},\mathbf{k},\sigma_1,\sigma_2)|^2\, \frac{M\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)}{E\omega_1(\mathbf{k})\omega_2(\mathbf{k})} d^3\mathbf{k} d^3\mathbf{P} < \infty, \tag{8}$$

where M, E, $\mathbf{p}_1$, and $\mathbf{p}_2$ are expressed in terms of $\mathbf{P}$ and $\mathbf{k}$ according to (7).

We introduce the operator

$$U = \left[\frac{E\omega_1(\mathbf{k})\omega_2(\mathbf{k})}{M\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)}\right]^{1/2} U_{12},$$
$$U_{12} = U_{12}(\mathbf{P},\mathbf{k}) = \gamma(\mathbf{P}/M,\ \boldsymbol{\sigma}_1, \mathbf{k}/m_1)\,\gamma(\mathbf{P}/M,\ \boldsymbol{\sigma}_2, -\mathbf{k}/m_2), \tag{9}$$

where

$$\gamma(\mathbf{P}/M,\ \boldsymbol{\sigma}_j, \mathbf{k}/m_j) = \frac{(E+M)(\omega_j(\mathbf{k})+m_j) + (\mathbf{P}\cdot\mathbf{k}) + i\,\boldsymbol{\sigma}_j\cdot(\mathbf{P}\times\mathbf{k})}{\{2(E+M)[\omega_j(\mathbf{k})+m_j][E\omega_j(\mathbf{k})+Mm_j+\mathbf{P}\cdot\mathbf{k}]\}^{1/2}}, \tag{10}$$

$j = 1, 2$ and the Pauli matrices $\boldsymbol{\sigma}_j$ act only on the corresponding variable σ_j.

Let $\tilde{H}$ be the space of functions $\tilde{\varphi}(\mathbf{P}, \mathbf{k}, \sigma_1, \sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2} \int |\tilde{\varphi}(\mathbf{P}, \mathbf{k}, \sigma_1, \sigma_2)|^2 \, d^3\mathbf{k} d^3\mathbf{P} < \infty. \tag{11}$$

Then it is easy to see that $\mathcal{U}^{-1}$ is a unitary operator from H to $\tilde{H}$, and by direct calculation it can be verified that the action of the generators of this two-particle representation in $\tilde{H}$ has the form

$$\mathbf{P} = \mathbf{P}, \qquad E = (M^2 + \mathbf{P}^2)^{1/2}, \qquad \mathbf{M} = \boldsymbol{l}(\mathbf{P}) + \mathbf{S},$$
$$\mathbf{N} = -i(M^2+\mathbf{P}^2)^{1/4}\frac{\partial}{\partial \mathbf{P}}(M^2+\mathbf{P}^2)^{1/4} + \frac{\mathbf{S}\times\mathbf{P}}{M + (M^2+\mathbf{P}^2)^{1/2}}, \tag{12}$$

where $\mathbf{P} = \mathbf{P}$ implies that the momentum operator is equal to the operator of multiplication by the variable $\mathbf{P}$ defined in (6), $\mathbf{S} = \boldsymbol{l}(\mathbf{k}) + \mathbf{s}_1 + \mathbf{s}_2$, and $\mathbf{s}_j$ is the spin operator of the corresponding particle. This calculation is rather complicated, and a simpler (and more general) method of deriving (12) is to use the direct-integral formalism (see, for example, Ref. [13]).

Comparing Eqs. (3) and (4) on the one hand, and Eqs. (11) and (12) on the other, we can conclude that the transformation to the space $\tilde{H}$ in the two-particle case can be interpreted as a splitting of the variables into external and internal ones: the role of the external variables is played by $\mathbf{P}$, and that of the internal variables is played by $\mathbf{k}, \sigma_1, \sigma_2$. Here M is the mass operator of the compound system, and $\mathbf{S}$ is its spin operator. These operators act only in H_{int}, in the space of functions $\chi(\mathbf{k}, \sigma_1, \sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2} \int |\chi(\mathbf{k}, \sigma_1, \sigma_2)|^2 \, d^3\mathbf{k} < \infty. \tag{13}$$

Returning to the space H we find that the action of the representation generators Γ^i, $(i = 1, 2, \ldots, 10)$ in this space has the form

$$\Gamma^i = \mathcal{U}\tilde{\Gamma}^i\mathcal{U}^{-1}, \tag{14}$$

where $\tilde{\Gamma}^i$ are the generators (12) in $\tilde{H}$.

In the nonrelativistic limit the operator $\mathcal{U}$ obviously becomes unity, and the relativistic relative momentum $\mathbf{k}$ becomes the vector $\mathbf{q}$, which is defined by the usual expression

$$\mathbf{q} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{M_0} \quad (M_0 = m_1 + m_2) \tag{15}$$

When the particle velocities are small compared with the velocity of light, we can expand physical quantities in powers of $(v/c)^2$ where v means the particle velocities. In our system of units the leading terms in such an expansion contain the quantities $|\mathbf{p}_1/m_1|^2$, $|\mathbf{p}_2/m_2|^2$, or $|\mathbf{p}_1|\,|\mathbf{p}_2|/m_1m_2$. For brevity, speaking below about the expansion in powers of $1/m^2$ we shall always mean the expansion of such a type. In particular, in first order in $1/m^2$,

$$\mathbf{k} = \mathbf{q} - \frac{\mathbf{P}\mathbf{q}^2}{2M_0}\left(\frac{1}{m_1} - \frac{1}{m_2}\right) - \frac{\mathbf{P}(\mathbf{P}\mathbf{q})}{2M_0^2}. \tag{16}$$

Let us note however that the $1/m^2$ expansion is used in the present work only with auxiliary purposes while, as noted above, the physical three-quark problem should be solved not using such an expansion.

III. RELATIVISTICALLY INVARIANT INTRODUCTION OF THE INTERACTION INTO THE TWO-PARTICLE SYSTEM

Let us now suppose that we wish to describe the interaction between the two particles only in terms of variables relating to these particles. Then we have to introduce the interaction into the operators Γ^i and the relativistic invariance implies that such an obtained system of generators $\hat{\Gamma}^i$ $(i = 1, 2, \ldots, 10)$, as well as the system Γ^i satisfies the commutation relations (1). In principle, all ten generators $\hat{\Gamma}^i$ may contain the interaction but in practice it is convenient to introduce the interaction into the least possible number of generators. It is clear that the generators which do not contain the interaction (i.e., the generators for which $\hat{\Gamma}^i = \Gamma^i$) form a subalgebra of the Poincaré group Lie algebra. In this study we shall work in the so-called instant form when $\hat{\mathbf{P}} = \mathbf{P}$, $\hat{\mathbf{M}} = \mathbf{M}$ and thus, the interaction can be present only in $\hat{E}$ and $\hat{\mathbf{N}}$.

As follows from (12) and (14), the simplest way to introduce the interaction into $(\hat{\Gamma}^i)$ is to replace $(\tilde{\Gamma}^i)$ by $(\hat{\tilde{\Gamma}}^i)$ where the set $(\hat{\tilde{\Gamma}}^i)$ is obtained from (12) by replacing M by some operator $\hat{M}$ which depends on the interaction and acts only in H_{int}. In order to preserve the commutation relations (1) it is necessary and sufficient that $\hat{M}$ commutes with $\mathbf{S}$. As it is clear from (12) and (14), in such a way we obtain the system of generators in the instant form. However, in the general case, the interaction can also be introduced into the operator $\mathcal{U}$, i.e., the operators $(\hat{\Gamma}^i)$ can be written as

$$\hat{\Gamma}^i = \hat{\mathcal{U}}\hat{\tilde{\Gamma}}^i\hat{\mathcal{U}}^{-1}. \tag{17}$$

We can write $\hat{\mathcal{U}} = A\mathcal{U}$ and if we wish to stay in the instant

form, we must obviously require that A commutes with $\mathbf{P}$ and $\mathbf{M}$.

Let us assume that $\hat{M} = M + v$ where the operator v in $H_{\rm int}$ is the integral operator, such that [see (13)]

$$v\,\chi(\mathbf{k}) = \int v(\mathbf{k},\mathbf{k}')\chi(\mathbf{k}')d^3\mathbf{k}'. \tag{18}$$

It is assumed in this expression that $\chi(\mathbf{k})$ is the spinor in the variables σ_1, σ_2 and the kernel $v(\mathbf{k},\mathbf{k}')$ is the operator in these variables. Then the condition $[\hat{M}, \mathbf{S}] = 0$ implies that, as in the usual case, the kernel $v(\mathbf{k},\mathbf{k}')$ can contain the dependence only on the scalar and mixed products of the quantities $\mathbf{k}, \mathbf{k}'$ and the operators $\mathbf{s}_1, \mathbf{s}_2$.

As follows from (9), (12), (14), and (17), the energy operator in the space H can be written as

$$\hat{P}^0 = A[(M+W)^2 + \mathbf{P}^2]^{1/2}A^{-1}, \tag{19}$$

where the operator W has the kernel

$$W(\mathbf{P},\mathbf{k};\mathbf{P}',\mathbf{k}') = \delta^{(3)}(\mathbf{P}-\mathbf{P}')\left[\frac{M'E\omega_1(\mathbf{k})\omega_2(\mathbf{k})\omega_1(\mathbf{p}_1')\omega_2(\mathbf{p}_2')}{ME'\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)\omega_1(\mathbf{k}')\omega_2(\mathbf{k}')}\right]^{1/2} U_{12}(\mathbf{P},\mathbf{k})v(\mathbf{k},\mathbf{k}')U_{12}(\mathbf{P},\mathbf{k}')^{-1}. \tag{20}$$

Here M, E, $\mathbf{p}_1$, and $\mathbf{p}_2$ are expressed in terms of $\mathbf{P}$ and $\mathbf{k}$, while M', E', $\mathbf{p}_1'$, and $\mathbf{p}_2'$ are expressed analogously in terms of $\mathbf{P}$ and $\mathbf{k}'$.

If two particles are considered in their c.m. frame then we can set $\mathbf{P} = 0$ in (19) and (20). Let A_0 be the action of A when $\mathbf{P} = 0$ (the rigorous definition of A_0 can be given in the direct-integral formalism [13]). Then $\hat{P}^0 = A_0(M+v)A_0^{-1}$. When $\mathbf{P} = 0$ the energy operator becomes the mass operator and (since $\hat{M} = M + v$), not losing generality, we can set $A_0 = 1$ since otherwise this condition can be achieved by a redefinition of v. Therefore, taking into account the commutation of A with $\mathbf{P}$ and $\mathbf{M}$, we can conclude that the most general form of A is

$$A = \exp(\mathbf{P}\cdot\mathbf{B}), \tag{21}$$

where $\mathbf{B}$ is some anti-Hermitian vector operator commuting with $\mathbf{P}$. Therefore, the action of $\mathbf{B}$ can be written as

$$\mathbf{B}\varphi(\mathbf{P},\mathbf{k}) = \int \mathbf{B}(\mathbf{P};\mathbf{k},\mathbf{k}')\varphi(\mathbf{P},\mathbf{k}')d^3\mathbf{k}', \tag{22}$$

where $\varphi(\mathbf{P},\mathbf{k})$ is a spinor over the variables σ_1 and σ_2 and $\mathbf{B}(\mathbf{P};\mathbf{k},\mathbf{k}')$ is an operator over these variables.

IV. THE SYSTEM OF TWO CONSTITUENT QUARKS IN QCD INSPIRED MODELS

Above we used only relativistic invariance and did not use the fact that the particles under consideration are constituent quarks, and, therefore, the interaction between them must be described in the QCD framework.

In local quantum field theory a system with a fixed number of interacting particles cannot be described only in terms of variables relating to these particles since one must also take into account the degrees of freedom relating to the creation of other particles. However, in realistic theories a system of a fixed number of particles can be described in a self-consistent way not only in zero, but also in first order in $1/m^2$ (in general relativity this can be done even in order $1/m^4$ since the gravitational dipole bremsstrahlung is absent). This fact is well known in QED where in first order in $1/m^2$ the Coulomb potential must be replaced by the Breit-Fermi potential. In QCD the similar statement is correct at least for the one-gluon-exchange potential. Therefore, to make contact with QCD, we must expand the Hamiltonian in powers of $1/m^2$ and find the first term in such an expansion. However, it is clear, that in order to compare the result with the nonrelativistic Hamiltonian we must work in the same space in which the nonrelativistic Hamiltonian is defined, i.e., $H^{\rm nr}$, in the space of functions $\varphi(\mathbf{P},\mathbf{q},\sigma_1,\sigma_2)$ such that

$$\sum_{\sigma_1\sigma_2}\int |\varphi(\mathbf{P},\mathbf{q},\sigma_1,\sigma_2)|^2\, d^3\mathbf{P}\, d^3\mathbf{q} < \infty. \tag{23}$$

As follows from (16), the Jacobian of the transformation from the variables $\mathbf{k}$ to the variables $\mathbf{q}$ in order $1/m^2$ is equal to

$$J = \left|\frac{\partial^3\mathbf{k}}{\partial^3\mathbf{q}}\right| = 1 - \frac{\mathbf{P}\cdot\mathbf{q}}{M_0}\left(\frac{1}{m_1}-\frac{1}{m_2}\right) - \frac{\mathbf{P}^2}{2M_0^2}. \tag{24}$$

In this approximation we have from (9) and (10) that

$$U_{12} = 1 + \frac{1}{2M_0}(\mathbf{P}\times\mathbf{q})\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right). \tag{25}$$

From the definition of the operator A it is clear that it describes the contribution of the interaction into $\hat{U}$, and in the nonrelativistic limit $A = 1$. Therefore, as follows from (21), in the $1/m^2$ approximation,

$$A = 1 + \mathbf{P}\cdot\mathbf{B}, \tag{26}$$

where the operator $\mathbf{B}$ is of order $1/m^2$. We now use v to denote the operator, the action of which in $H^{\rm nr}$ is defined by the kernel $v(\mathbf{q},\mathbf{q}')$ (this operator does not act on the variable $\mathbf{P}$). Then from (16)–(26) we find that in first order in $1/m^2$ the action of the Hamiltonian in $H^{\rm nr}$ can be written as

$$\hat{P}^0 = E + V, \tag{27}$$

where E is the free Hamiltonian and the kernel of the operator V in the $1/m^2$ approximation has the form

$$V(\mathbf{P},\mathbf{q};\mathbf{P}',\mathbf{q}') = \delta^{(3)}(\mathbf{P}-\mathbf{P}')V(\mathbf{P},\mathbf{q},\mathbf{q}'),$$

$$\begin{aligned} V(\mathbf{P},\mathbf{q},\mathbf{q}') = {} & v(\mathbf{q},\mathbf{q}')\left[1-\frac{\mathbf{P}^2}{M_0^2}-\frac{(\mathbf{P},\mathbf{q}+\mathbf{q}')}{2M_0}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\right]-\frac{\mathbf{P}}{2M_0}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\left[\mathbf{q}^2\frac{\partial v(\mathbf{q},\mathbf{q}')}{\partial \mathbf{q}}+\mathbf{q}'^2\frac{\partial v(\mathbf{q},\mathbf{q}')}{\partial \mathbf{q}'}\right] \\ & -\frac{\mathbf{P}}{2M_0^2}\left[(\mathbf{P}\cdot\mathbf{q})\frac{\partial v(\mathbf{q},\mathbf{q}')}{\partial \mathbf{q}}+(\mathbf{P}\cdot\mathbf{q}')\frac{\partial v(\mathbf{q},\mathbf{q}')}{\partial \mathbf{q}'}\right]+\frac{i}{2M_0}(\mathbf{P}\times\mathbf{q})\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)v(\mathbf{q},\mathbf{q}') \\ & -\frac{i}{2M_0}v(\mathbf{q},\mathbf{q}')(\mathbf{P}\times\mathbf{q}')\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)-\frac{M_0(\mathbf{q}^2-\mathbf{q}'^2)}{2m_1m_2}\mathbf{P}\cdot\mathbf{B}(\mathbf{P},\mathbf{q},\mathbf{q}')+\mathbf{P}\cdot\mathbf{B}_1(\mathbf{P},\mathbf{q},\mathbf{q}'), \end{aligned} \tag{28}$$

where $\mathbf{B}_1(\mathbf{P},\mathbf{q},\mathbf{q}')$ is the kernel of the operator $\mathbf{B}_1 = [\mathbf{B},v]$.

This expression can also be obtained directly (not using the vector $\mathbf{k}$ in intermediate calculations) in the framework of approach developed by Foldy and Krajcik [25]. However, their approach is convenient only in calculations to some order in $1/m^2$, while if we are going to perform calculations exactly (not using the expansion in powers of $1/m^2$) then we must use as the internal momentum variable the quantity $\mathbf{k}$ but not $\mathbf{q}$. In this case the latter quantity is needed only to make contact between the exact expression for the interaction operator and its expression in the $1/m^2$ approximation.

If the system of two particles is considered in their c.m. frame, then, as follows from (28), $V = v$ in this reference frame. The operator v contains not only the pure nonrelativistic contribution $v^{\rm nr}$, but also the corrections of order $1/m^2$ to $v^{\rm nr}$: spin-orbit, spin-spin, and other relativistic corrections. In particular, the contribution to $v(\mathbf{q},\mathbf{q}')$ is given by the annihilation graphs if they exist.

It is clear that in the $1/m^2$ approximation we can replace $v(\mathbf{q},\mathbf{q}')$ by $v^{\rm nr}(\mathbf{q},\mathbf{q}')$ in all terms with $\mathbf{P}$ in (28). It is usually supposed that the kernel of $v^{\rm nr}$ does not depend on spins and depends only on $Q = |\mathbf{Q}|$ where $\mathbf{Q} = \mathbf{q}-\mathbf{q}'$. With this assumption we can replace $v^{\rm nr}(\mathbf{q},\mathbf{q}')$ by $v^{\rm nr}(Q)$ and then Eq. (28) can be written as

$$\begin{aligned} V(\mathbf{P},\mathbf{q},\mathbf{q}') = {} & v(\mathbf{q},\mathbf{q}')-\left[\frac{\mathbf{P}^2}{M_0^2}+\frac{(\mathbf{P},\mathbf{q}+\mathbf{q}')}{2M_0}(\frac{1}{m_1}-\frac{1}{m_2})\right]v^{\rm nr}(Q)-\frac{\mathbf{P}\cdot\mathbf{Q}(\mathbf{q}^2-\mathbf{q}'^2)}{2M_0Q}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\frac{dv^{\rm nr}(Q)}{dQ} \\ & -\frac{(\mathbf{P}\cdot\mathbf{Q})^2}{2M_0^2Q}\frac{dv^{\rm nr}(Q)}{dQ}+\frac{i}{2M_0}(\mathbf{P}\times\mathbf{Q})\cdot\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)v^{\rm nr}(Q) \\ & -\frac{M_0(\mathbf{q}^2-\mathbf{q}'^2)}{2m_1m_2}\mathbf{P}\cdot\mathbf{B}(\mathbf{P},\mathbf{q},\mathbf{q}')+\mathbf{P}\cdot\mathbf{B}_1(\mathbf{P},\mathbf{q},\mathbf{q}'). \end{aligned} \tag{29}$$

Our nearest aim is to compare Eq. (29) with the one-gluon-exchange potential in the $1/m^2$ approximation. In turn, this potential can easily be derived from the well-known result in QED: if the particles have the electric charges e_1 and e_2 then (see, for example, Eq. (83.9) in Ref. [26])

$$\begin{aligned} V(\mathbf{P},\mathbf{q},\mathbf{q}') = v(\mathbf{q},\mathbf{q}')+\frac{4\pi e_1e_2}{Q^2}\Bigg\{ & \frac{(\mathbf{P}\cdot\mathbf{Q})^2}{M_0^2Q^2}-\frac{\mathbf{P}^2}{M_0^2}+\frac{m_2-m_1}{2M_0m_1m_2Q^2}[\mathbf{P}\cdot\mathbf{Q}(\mathbf{q}^2-\mathbf{q}'^2)-Q^2(\mathbf{P},\mathbf{q}+\mathbf{q}')] \\ & +\frac{i}{2M_0}(\mathbf{P}\times\mathbf{Q})\cdot\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)\Bigg\}. \end{aligned} \tag{30}$$

It is important that this expression is valid for the particle-particle interaction as well as for the particle-antiparticle interaction [the annihilation graphs contribute only to $v(\mathbf{q},\mathbf{q}')$ but not to the terms with $\mathbf{P}$]. For the case of the one-gluon exchange Eq. (30) must be modified as follows. First, e_1e_2 must be replaced by the strong interaction running coupling constant $\alpha_s(Q)$. Then we must take into account the retardation in α_s, i.e., the fact that α_s depends actually on $Q^2-\omega^2$ where ω is the energy transfer. It is easily seen that this results in the additional term $-\omega^2\alpha_s'(Q)/2Q\alpha_s(Q)$ in the figure brackets in (30), where the prime stands for the differentiation in Q. Finally, as is well known [27], if we consider mesons and baryons as color singlets then we must introduce the multiplier β which is equal to $-4/3$ for the quark-antiquark and $-2/3$ for the quark-quark interaction. Therefore, the one-gluon-exchange potential in the $1/m^2$ approximation can be written as

$$\begin{aligned} V(\mathbf{P},\mathbf{q},\mathbf{q}') = v(\mathbf{q},\mathbf{q}')+v^{\rm nr}(Q)\Bigg\{ & -\frac{\omega^2\alpha_s'(Q)}{2Q\alpha_s(Q)}+\frac{(\mathbf{P}\cdot\mathbf{Q})^2}{M_0^2Q^2}-\frac{\mathbf{P}^2}{M_0^2}+\frac{m_2-m_1}{2M_0m_1m_2Q^2}[\mathbf{P}\cdot\mathbf{Q}(\mathbf{q}^2-\mathbf{q}'^2)-Q^2(\mathbf{P},\mathbf{q}+\mathbf{q}')] \\ & +\frac{i}{2M_0}(\mathbf{P}\times\mathbf{Q})\cdot\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)\Bigg\}, \end{aligned} \tag{31}$$

where we use $v^{\rm nr}(Q)$ to denote $4\pi\beta\alpha_s(Q)/Q^2$.

Now we are faced with the problem which for a long time is discussed in the literature: what expression for ω^2 as the function of $\mathbf{P},\mathbf{q},\mathbf{q}'$ should be used in deriving the potential to order $1/m^2$? A detailed review of this problem is given in Ref. [28]. The most popular choice is

$$\omega^2 = -(\mathbf{p}_1^2 - \mathbf{p}_1'^2)(\mathbf{p}_2^2 - \mathbf{p}_2'^2)/4m_1m_2. \tag{32}$$

For the scattering in the physical region, the energy conservation implies that $|\mathbf{q}| = |\mathbf{q}'|$, and, taking into account that $\mathbf{P} = \mathbf{P}'$, we have from (32) that

$$\omega^2 = (\mathbf{P}\cdot\mathbf{Q})^2/M_0^2. \tag{32'}$$

Although both expressions (32) and (32′) lead to the same scattering amplitude in the physical region, the potentials obtained with the use of (32) and (32′) differ each other.

It is well known that in QED and QCD the problem of choosing the explicit expression for ω can be evaded by taking the expression for the photon or gluon propagator in the Coulomb gauge, and the result is equivalent to the choice (32). Therefore, comparing (29) and (31) for such a choice we find that $\mathbf{B}(\mathbf{P},\mathbf{q},\mathbf{q}')$ depends actually only on $\mathbf{Q}$ and

$$\mathbf{B}(\mathbf{Q}) = \kappa\frac{m_1 - m_2}{M_0^2}\frac{\partial v^{\rm nr}(Q)}{\partial\mathbf{Q}}, \tag{33}$$

where $\kappa = 1/2$. However, in the general case the universal prescription for choosing the expression for ω does not exist, and, as we shall see below, in the case of scalar interaction the choice (32) leads to (33) where again $\kappa = 1/2$, while the choice (32′) leads to the same expression with $\kappa = 1$ [in addition, the results for $v(\mathbf{q},\mathbf{q}')$ also differ each other].

We see that the interaction corresponding to the one-gluon exchange in the $1/m^2$ approximation indeed can be described in the instant form, and, in the general case, the interaction must be present not only in the mass operator but also in the operator $\hat{\mathcal{U}}$. The fact that $\hat{\mathcal{U}}$ is not equal to $\mathcal{U}$ does not play a role for mesons since $\mathbf{B}$ does not contribute to $\hat{\mathcal{U}}$ when $\mathbf{P} = 0$ and, as noted above, $A_0 = 1$. But in the case of baryons the operators $\hat{\mathcal{U}}$ for each pair of quarks enter into the three-quark mass operator (see Sec. V) and thus in the general case we cannot neglect the dependence of $\hat{\mathcal{U}}$ on the quark-quark interaction.

Let $\mathbf{r}_{12} = \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ be the relative distance between particles 1 and 2. The operator $\mathbf{r}$ is canonically conjugated with $\mathbf{q}$, i.e., in the $\mathbf{r}$ representation $\mathbf{q} = -i\partial/\partial\mathbf{r}$. The $\mathbf{q}$ and $\mathbf{r}$ representations are connected by the usual Fourier transformation. We note that since we assume that the volume element in the $\mathbf{r}$ space is chosen as $d^3\mathbf{r}$ then the volume element in the $\mathbf{q}$ space must be chosen as $d^3\mathbf{q}/(2\pi)^3$. Taking into account this remark, it is easy to show that the operator with the kernel (29) has the following form in the $\mathbf{r}$ representation:

$$V = v - \frac{\mathbf{P}^2}{2M_0^2}v^{\rm nr}(r) + \frac{(\mathbf{P}\cdot\mathbf{r})^2}{2M_0^2 r}\frac{dv^{\rm nr}}{dr} - \frac{i}{2M_0}\left(\frac{1}{m_1} - \frac{1}{m_2}\right)\left\{\frac{(\mathbf{P}\cdot\mathbf{r})\mathbf{r}}{r}\frac{dv^{\rm nr}}{dr}, \frac{\partial}{\partial\mathbf{r}}\right\}$$
$$+\frac{\mathbf{P}\times\mathbf{r}}{2M_0 r}\left(\frac{\mathbf{s}_1}{m_1} - \frac{\mathbf{s}_2}{m_2}\right)\frac{dv^{\rm nr}}{dr} + \frac{M_0}{2m_1m_2}\left[\frac{\partial^2}{\partial\mathbf{r}^2}, \mathbf{P}\cdot\mathbf{B}\right] + \mathbf{P}\cdot\mathbf{B}_1 \tag{34}$$

where we use curly brackets to denote the anticommutator, square brackets to denote the commutator, and the operator $v^{\rm nr}$ in the $\mathbf{r}$ representation is the operator of multiplication by the function

$$v^{\rm nr}(r) = \int v^{\rm nr}(Q)e^{i\mathbf{Q}\cdot\mathbf{r}}\frac{d^3\mathbf{Q}}{(2\pi)^3}. \tag{35}$$

It is also easy to see that if the kernel of the operator $\mathbf{B}$ depends only on $\mathbf{Q}$, then, in the $\mathbf{r}$ representation,

$$\mathbf{B} = \mathbf{B}(\mathbf{r}) = \int \mathbf{B}(\mathbf{Q})e^{i\mathbf{Q}\cdot\mathbf{r}}\frac{d^3\mathbf{Q}}{(2\pi)^3} = \frac{-i\mathbf{r}(m_1 - m_2)}{M_0^2}u^{\rm nr}(r), \tag{36}$$

where $u^{\rm nr}(r)$ is some function which, as follows from the above discussion, can be written, for example, as

$$u^{\rm nr}(r) = \kappa_V v_V^{\rm nr}(r) + \kappa_S v_S^{\rm nr}(r) \tag{37}$$

and the subscript V or S shows that the corresponding quantity is related to the vector or scalar coupling (the pseudoscalar, axial vector, and antisymmetric tensor couplings are of no interest for us since they do not contain terms with $\mathbf{P}$ [29,28]).

The problem of determining the order-$(1/m^2)$ corrections to the nonrelativistic potential is intensively discussed in the literature and the most often used reduction to the potential description is such that the system of two particles is described in the instant form. Therefore, we can compare our results with those existing in the literature. Reviewing several results on the vector and scalar coupling Gromes considers Eqs. (2.48) and (2.49) in his review [28] as the ultimately correct ones. For the case of the vector coupling Eq. (2.48) in Ref. [28] can be written as

$$V = (\cdots) + \frac{\mathbf{P}\times\mathbf{r}}{2M_0 r}\cdot\left(\frac{\mathbf{s}_1}{m_1} - \frac{\mathbf{s}_2}{m_2}\right)\frac{dv^{\rm nr}}{dr}$$
$$-\frac{1}{2m_1m_2}\left\{\left\{\mathbf{p}_1 v^{\rm nr}(r)\mathbf{p}_2 - \mathbf{p}_1\frac{\mathbf{r}}{r}\frac{dv^{\rm nr}}{dr}\mathbf{r}\mathbf{p}_2\right\}\right\}, \tag{38}$$

where we use large parentheses to denote some contribution not depending on $\mathbf{P}$, and the double-curly

brackets notation introduced in Ref. [29] implies that $\{\{p_iFp_j\}\} = (p_iFp_j + p_jFp_i + p_ip_jF + Fp_jp_i)/4$ where F is some function of $\mathbf{r}$. Expressing $\mathbf{p}_1$ and $\mathbf{p}_2$ in (38) in terms of $\mathbf{P}$ and $\mathbf{q}$ it is easy to show that (38) can be written as

$$V = v - \frac{\mathbf{P}^2}{2M_0^2}v^{\rm nr}(r) + \frac{(\mathbf{P}\cdot\mathbf{r})^2}{2M_0^2 r}\frac{dv^{\rm nr}}{dr} + \frac{i}{4M_0} \times\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\left(\left\{v^{\rm nr}(r),\mathbf{P}\frac{\partial}{\partial\mathbf{r}}\right\} -\left\{\frac{(\mathbf{P}\cdot\mathbf{r})\mathbf{r}}{r}\frac{dv^{\rm nr}}{dr},\frac{\partial}{\partial\mathbf{r}}\right\}\right). \quad (39)$$

Comparing Eqs. (34) and (39), it is easy to see that they are compatible with each other if $\mathbf{B}$ is given by (36) with [see (37)] $\kappa_V = 1/2$ what agrees with the result for the one-gluon-exchange potential. This was expected since in fact different calculations (see, for example Refs. [30–35]) were based either on the assumption that it is possible to eliminate the ω dependence from the temporal component of the propagator by appropriate gauge transformation or on the prescription (32). Meanwhile in the general case of vector coupling this is not always clear (for example, if the particle transferring the interaction has the nonzero mass).

Let us now consider the scalar coupling. Equation (2.49) in Ref. [28] has the form

$$V = v^{\rm nr}(r) - \frac{1}{2r}\left(\frac{(\mathbf{r}\times\mathbf{p}_1)\cdot\mathbf{s}_1}{m_1^2} - \frac{(\mathbf{r}\times\mathbf{p}_2)\cdot\mathbf{s}_2}{m_2^2}\right)\frac{dv^{\rm nr}}{dr} -\frac{1}{2m_1^2}\{\{\mathbf{p}_1 v^{\rm nr}(r)\cdot\mathbf{p}_1\}\} -\frac{1}{2m_2^2}\{\{\mathbf{p}_2 v^{\rm nr}(r)\cdot\mathbf{p}_2\}\}. \quad (40)$$

It is obvious that the term quadratic in $\mathbf{P}$ in (40) has the form $-v^{\rm nr}(r)\mathbf{P}^2/M_0^2$, and, therefore the kernel of V in momentum representation contains only the following term quadratic in $\mathbf{P}$: $-v^{\rm nr}(Q)\mathbf{P}^2/M_0^2$. Comparing this term with the term quadratic in $\mathbf{P}$ in (29) we see that the compatibility of Eqs. (29) and (40) cannot be ensured by any choice of $\mathbf{B}(\mathbf{P},\mathbf{q},\mathbf{q}')$.

Equation (40) was obtained in Refs. [30–32] and other works but not always in the most general form. In some partial cases this expression can be correct. For example, in the c.m. frame (when $\mathbf{P}=0$) Eq. (40) corresponds to the prescription (32′) [see Eq. (42)]. However, the general form (40) is incompatible with relativistic invariance. This result was also criticized by Miller and Olsson [33]. They obtained the expression for the scalar coupling,

$$V = v^{\rm nr}(r) - \frac{1}{2r}\left(\frac{(\mathbf{r}\times\mathbf{p}_1)\cdot\mathbf{s}_1}{m_1^2} - \frac{(\mathbf{r}\times\mathbf{p}_2)\cdot\mathbf{s}_2}{m_2^2}\right)\frac{dv^{\rm nr}}{dr} -\frac{1}{2m_1^2}\{\{\mathbf{p}_1 v^{\rm nr}(r)\mathbf{p}_1\}\} -\frac{1}{2m_2^2}\{\{\mathbf{p}_2 v^{\rm nr}(r)\mathbf{p}_2\}\} +\frac{1}{2m_1m_2}\left\{\left\{\mathbf{p}_1\left(v^{\rm nr}(r)+\mathbf{r}\frac{1}{r}\frac{dv^{\rm nr}(r)}{dr}\mathbf{r}\right)\mathbf{p}_2\right\}\right\}, \quad (41)$$

and this result was then obtained from another considerations in Refs. [34,35]. It is easy to show that Eq. (41) can be written in the form (39) and, therefore the corresponding result for $\mathbf{B}$ is also given by (36) and (37) with $\kappa_S = 1/2$.

We see that Eq. (41) is more reasonable than Eq. (40) since Eq. (41) satisfies relativistic invariance at least at a particular choice of $\mathbf{B}$. The question is whether Eq. (41) is most general. This expression was in turn criticized by Barnes and Ghandour [30] because it did not give the correct limit for $v^{\rm nr} = {\rm const} \ll m_1, m_2$. We shall not discuss this question but note that it is not obvious since in the given case the kernel of $v^{\rm nr}$ in momentum representation depends not only on Q. On the other hand we can correct Eq. (40) as

$$V = v^{\rm nr}(r) - \frac{1}{2r}\left(\frac{(\mathbf{r}\times\mathbf{p}_1)\cdot\mathbf{s}_1}{m_1^2} - \frac{(\mathbf{r}\times\mathbf{p}_2)\cdot\mathbf{s}_2}{m_2^2}\right)\frac{dv^{\rm nr}}{dr} -\frac{1}{2m_1^2}\{\{\mathbf{p}_1 v^{\rm nr}(r)\mathbf{p}_1\}\} -\frac{1}{2m_2^2}\{\{\mathbf{p}_2 v^{\rm nr}(r)\mathbf{p}_2\}\} +\frac{(\mathbf{P}\cdot\mathbf{r})^2}{2M_0^2 r}\frac{dv^{\rm nr}}{dr} + \frac{\mathbf{P}^2}{2M_0^2}v^{\rm nr}(r). \quad (42)$$

This expression can be obtained using the prescription (32′), and, comparing (34) and (42) we see that Eq. (42) implies that $\kappa_S = 1$. Therefore, this expression already satisfies relativistic invariance. At the same time, neither Eq. (41) nor Eq. (42) describe the very simple case when v =const not only in the nonrelativistic limit but exactly, since in this case $v = v^{\rm nr}$ =const. Therefore, neither (32) nor (32′) are the most general prescriptions.

We see that even the problem of correct expression of v in terms of $v^{\rm nr}$ is open. In addition, if we take into account that V is determined not only by v but also by $\mathbf{B}$ then the question arises whether a universal (and correct) expression of V in terms of $v^{\rm nr}$ in first order in $1/m^2$ exists at all if we require that V is obtained from some Lagrangian. In other words, is this the case that in first order in $1/m^2$ $v = f_1(v^{\rm nr})$, $\mathbf{B} = f_2(v^{\rm nr})$ where the functions f_1 and f_2 are the same for all possible Lagrangians? In view of the above discussion such a possibility seems very unlikely. For this reason in the case of the one-gluon-exchange potential we can use the well known expression of v in terms of $v^{\rm nr}$ and put $\kappa_V = 1/2$, but in the scalar case it is reasonable to find the explicit expression of v in terms of $v^{\rm nr}$ and the value of κ_S from the point of view of the best description of experimental data.

Summarizing our discussion, we see that the expression for V (in the $1/m^2$ approximation) describing the contribution of vector and scalar couplings can be written as [see (34) and (36)]

$$V = v - \frac{\mathbf{P}^2}{2M_0^2} v^{\rm nr}(r) + \frac{(\mathbf{P}\cdot\mathbf{r})^2}{2M_0^2 r}\frac{dv^{\rm nr}}{dr} + \frac{\mathbf{P}\times\mathbf{r}}{2M_0 r}\cdot\left(\frac{\mathbf{s}_1}{m_1} - \frac{\mathbf{s}_2}{m_2}\right)\frac{dv^{\rm nr}}{dr}$$
$$+ \frac{i(m_2 - m_1)}{2M_0 m_1 m_2}\left\{\frac{\partial}{\partial \mathbf{r}}, \mathbf{P}u^{\rm nr}(r) + \frac{(\mathbf{P}\cdot\mathbf{r})\mathbf{r}}{r}\frac{d[u^{\rm nr}(r) - v^{\rm nr}(r)]}{dr}\right\}, \tag{43}$$

where $v^{\rm nr}(r) = v_V^{\rm nr}(r) + v_S^{\rm nr}(r)$.

It is important to note that though the condition of relativistic invariance imposes direct restrictions only on the terms with $\mathbf{P}$ in V, this condition can also be useful in investigating the dynamics of two particles in the c.m. frame. For the illustration of this point let us consider the following example. In Ref. [36] Eichten and Feinberg obtained the following expression for the $q\bar{q}$ confining potential in the $1/m^2$ approximation:

$$V = \varepsilon(r) + \frac{1}{2r}\left(\frac{(\mathbf{r}\times\mathbf{p}_1)\cdot\mathbf{s}_1}{m_1^2} - \frac{(\mathbf{r}\times\mathbf{p}_2)\cdot\mathbf{s}_2}{m_2^2}\right)[\varepsilon'(r) + 2V_1'(r)]$$
$$+ \frac{1}{r}\left(\frac{(\mathbf{r}\times\mathbf{p}_1)\cdot\mathbf{s}_2}{m_1 m_2} - \frac{(\mathbf{r}\times\mathbf{p}_2)\cdot\mathbf{s}_1}{m_1 m_2}\right)V_2'(r) + \frac{1}{m_1 m_2}\left(\frac{(\mathbf{r}\cdot\mathbf{s}_1)(\mathbf{r}\cdot\mathbf{s}_2)}{r^2} - \frac{\mathbf{s}_1\cdot\mathbf{s}_2}{3}\right)$$
$$+ \frac{\mathbf{s}_1\cdot\mathbf{s}_2}{3m_1 m_2}V_4(r) + \text{ spin-independent corrections}\,. \tag{EF1}$$

Expressing this potential as vacuum expectation value of insertions into a Wilson loop, belonging to the quark at $\mathbf{r}_1$ and antiquark at $\mathbf{r}_2$ and a time interval between $-T/2$ and $T/2$ Gromes [37] has proved that there exists the relation between $\varepsilon(r)$, $V_1(r)$, and $V_2(r)$:

$$\varepsilon'(r) + V_1'(r) - V_2'(r) = 0. \tag{G}$$

This result follows from (43) in a much more simple way. Indeed, let us consider in (EF1) only the terms with $\mathbf{P}$. They obviously have the form

$$V = \frac{\mathbf{P}\times\mathbf{r}}{M_0 r}\cdot\left(\frac{\mathbf{s}_1}{m_1} - \frac{\mathbf{s}_2}{m_2}\right)\left(\frac{\varepsilon'(r)}{2} + V_1'(r) - V_2'(r)\right). \tag{EF2}$$

It is also obvious that $\varepsilon(r)$ in (EF1) is just the nonrelativistic potential $v^{\rm nr}(r)$. Therefore, comparing the terms depending on $\mathbf{P}$ and spins in (EF2) and (43) we can see that Eq. (G) is indeed satisfied.

V. THE SYSTEM OF THREE INTERACTING PARTICLES

The system of three particles of spin 1/2 and masses m_1, m_2, and m_3 can be described by the wave function $\varphi(\mathbf{p}_1,\sigma_1,\mathbf{p}_2,\sigma_2,\mathbf{p}_3,\sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\varphi(\mathbf{p}_1,\sigma_1,\mathbf{p}_2,\sigma_2,\mathbf{p}_3,\sigma_3)|^2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_3 < \infty. \tag{44}$$

Now we use $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$ to denote the total momentum of the three-particle system and $\mathbf{k}_j$ $(j = 1,2,3)$ to denote the particle momenta in the c.m. frame, i.e., in the reference frame where $\mathbf{P} = 0$. By analogy with Eq. (6) we set

$$\mathbf{k}_j = \mathbf{p}_j - \frac{\mathbf{P}\omega_j(\mathbf{p}_j)}{M} + \frac{(\mathbf{P}\cdot\mathbf{p}_j)\mathbf{P}}{M(M+E)}, \tag{45}$$

where now M stands for the free mass operator of the three-particle system and E is the corresponding energy operator: $E = \omega_1(\mathbf{p}_1) + \omega_2(\mathbf{p}_2) + \omega_3(\mathbf{p}_3)$, $M = (E^2 - \mathbf{P}^2)^{1/2} = \omega_1(\mathbf{k}_1) + \omega_2(\mathbf{k}_2) + \omega_3(\mathbf{k}_3)$. From (45) it follows that $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ as expected. In the variables $\mathbf{P}, \mathbf{k}_j$ the three-particle Hilbert space $H^{(3)}$ of functions satisfying (44) is realized as the space of functions $\varphi(\mathbf{P},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\varphi(\mathbf{P},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\sigma_1,\sigma_2,\sigma_3)|^2\,\frac{M\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)\omega_3(\mathbf{p}_3)}{E\omega_1(\mathbf{k}_1)\omega_2(\mathbf{k}_2)\omega_3(\mathbf{k}_3)}\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3 d^3\mathbf{P} < \infty, \tag{46}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, are expressed in terms of $\mathbf{P}$, $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. The role of the "internal" three-particle space $H_{\rm int}^{(3)}$ is now played by the space of functions $\chi(\mathbf{k}_1,\sigma_1,\mathbf{k}_2,\sigma_2,\mathbf{k}_3,\sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\chi(\mathbf{k}_1,\sigma_1,\mathbf{k}_2,\sigma_2,\mathbf{k}_3,\sigma_3)|^2$$
$$\times\,\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3 < \infty. \tag{47}$$

Let us consider the pair of particles i and j in the c.m. frame of three particles. By analogy with Eq. (6) we introduce the quantities

$$\mathbf{K}_{ij} = \mathbf{k}_i + \mathbf{k}_j, \quad \mathbf{k}_{ij} = \mathbf{k}_i - \frac{\mathbf{K}_{ij}\omega_i(\mathbf{k}_i)}{M_{ij}} + \frac{(\mathbf{K}_{ij}\cdot\mathbf{k}_i)\mathbf{K}_{ij}}{M_{ij}(M_{ij}+E_{ij})}, \tag{48}$$

where $E_{ij} = \omega_i(\mathbf{k}_i) + \omega_j(\mathbf{k}_j)$, $M_{ij} = (E_{ij}^2 - \mathbf{K}_{ij}^2)^{1/2}$. We can choose these quantities as two independent variables in $H_{\rm int}^{(3)}$ and realize $H_{\rm int}^{(3)}$ as the space of functions $\chi(\mathbf{K}_{ij}, \mathbf{k}_{ij}, \sigma_1, \sigma_2, \sigma_3)$ such that

$$\sum_{\sigma_1\sigma_2\sigma_3}\int |\chi(\mathbf{K}_{ij}, \mathbf{k}_{ij}, \sigma_1, \sigma_2, \sigma_3)|^2 \times \frac{M_{ij}\omega_i(\mathbf{k}_i)\omega_j(\mathbf{k}_j)}{E_{ij}\omega_i(\mathbf{k}_{ij})\omega_j(\mathbf{k}_{ij})} d^3\mathbf{k}_{ij} d^3\mathbf{K}_{ij} < \infty, \tag{49}$$

where [compare with (7)]

$$M_{ij} = M_{ij}(\mathbf{k}_{ij}) = \omega_i(\mathbf{k}_{ij}) + \omega_j(\mathbf{k}_{ij}),\ E_{ij} = E_{ij}(\mathbf{K}_{ij}, \mathbf{k}_{ij}) = [M_{ij}(\mathbf{k}_{ij})^2 + \mathbf{K}_{ij}^2]^{1/2},$$
$$\mathbf{k}_i = \mathbf{k}_{ij} + \frac{\mathbf{K}_{ij}\omega_i(\mathbf{k}_{ij})}{M_{ij}} + \frac{(\mathbf{K}_{ij}\cdot\mathbf{k}_{ij})\mathbf{K}_{ij}}{M_{ij}(M_{ij}+E_{ij})},$$
$$\mathbf{k}_j = -\mathbf{k}_{ij} + \frac{\mathbf{K}_{ij}\omega_j(\mathbf{k}_{ij})}{M_{ij}} - \frac{(\mathbf{K}_{ij}\cdot\mathbf{k}_{ij})\mathbf{K}_{ij}}{M_{ij}(M_{ij}+E_{ij})}. \tag{50}$$

Now we need two formal but simple notions, the sense of which will be clear soon. First we introduce the operator of reduction from $H^{(3)}$ onto $H_{\rm int}^{(3)}$. This operator I is defined as follows. If $\varphi(\mathbf{p}_1, \sigma_1, \mathbf{p}_2, \sigma_2, \mathbf{p}_3, \sigma_3) \in H^{(3)}$, then

$$I\varphi(\mathbf{p}_1, \sigma_1, \mathbf{p}_2, \sigma_2, \mathbf{p}_3, \sigma_3) = \varphi(\mathbf{k}_1, \sigma_1, \mathbf{k}_2, \sigma_2, \mathbf{k}_3, \sigma_3) \in H_{\rm int}^{(3)}.$$

Suppose also that $\mathcal{O}$ is some operator in $H^{(3)}$ commuting with $\mathbf{P}$. Then we can define the reduction of $\mathcal{O}$ onto $H_{\rm int}^{(3)}$ which we denote as $\tilde{\mathcal{O}}$: if $\varphi \in H^{(3)}, \chi \in H_{\rm int}^{(3)}, I\varphi = \chi$ then the action of $\tilde{\mathcal{O}}$ is defined as $\tilde{\mathcal{O}}\chi = I\mathcal{O}\varphi$. As follows from the von Neumann theorem (see, for example, Ref. [38]), if $\mathcal{O}$ is self-adjoint in $H^{(3)}$ then $\tilde{\mathcal{O}}$ is self-adjoint in $H_{\rm int}^{(3)}$.

The sense of reducing some operator $\mathcal{O}$ from $H^{(3)}$ onto $H_{\rm int}^{(3)}$ is that we consider the action of $\mathcal{O}$ only in the subspace of functions with $\mathbf{P} = 0$. Let V_{ij} be the interaction-energy operator for the pair ij [see Eq. (27)]. The action of V_{ij} in $H^{(3)}$ is defined as in Secs. III and IV assuming that V_{ij} does not act through the variables $\mathbf{p}_k, \sigma_k$ $(k \neq i \neq j \neq k)$. Then, as is easily seen, the action of $\tilde{V}_{ij}$ in the variables $\mathbf{k}_i, \sigma_i, \mathbf{k}_j, \sigma_j$ is defined by the same expressions, as the action of V_{ij} in the variables $\mathbf{p}_i, \sigma_i, \mathbf{p}_j, \sigma_j$.

Let A_{ij} be the operator A for pair ij. We can analogously consider the action of A_{ij} in $H^{(3)}$ and obtain $\tilde{A}_{ij}$ by reducing A_{ij} onto $H_{\rm int}^{(3)}$. Namely, if the action of A_{ij} is defined by Eqs. (21) and (22) then

$$\tilde{A}_{ij} = \exp(\mathbf{K}_{ij}\cdot\tilde{\mathbf{B}}_{ij}),\ \tilde{\mathbf{B}}_{ij}\chi(\mathbf{K}_{ij}, \mathbf{k}_{ij}) = \int \mathbf{B}_{ij}(\mathbf{K}_{ij}; \mathbf{k}_{ij}, \mathbf{k}'_{ij})\chi(\mathbf{K}_{ij}, \mathbf{k}'_{ij}) d^3\mathbf{k}'_{ij}. \tag{51}$$

Then, as follows from (19) and (27)

$$\tilde{V}_{ij} = \tilde{A}_{ij}[(M_{ij} + \tilde{W}_{ij})^2 + \tilde{K}_{ij}^2]^{1/2}\tilde{A}_{ij}^{-1} - (M_{ij}^2 + \tilde{K}_{ij}^2)^{1/2}, \tag{52}$$

where, as follows from (20), the operator $\tilde{W}_{ij}$ acts in $H_{\rm int}^{(3)}$ as

$$\tilde{W}_{ij}\chi(\mathbf{K}_{ij}, \mathbf{k}_{ij}) = \int \left[\frac{E_{ij}M'_{ij}\omega_i(\mathbf{k}_{ij})\omega_j(\mathbf{k}_{ij})\omega_i(\mathbf{k}'_i)\omega_j(\mathbf{k}'_j)}{M_{ij}E'_{ij}\omega_i(\mathbf{k}_i)\omega_j(\mathbf{k}_j)\omega_i(\mathbf{k}'_{ij})\omega_j(\mathbf{k}'_{ij})}\right]^{1/2} U_{ij}(\mathbf{K}_{ij}, \mathbf{k}_{ij})v_{ij}(\mathbf{k}_{ij}, \mathbf{k}'_{ij})U_{ij}(\mathbf{K}_{ij}, \mathbf{k}'_{ij})^{-1}\chi(\mathbf{K}_{ij}, \mathbf{k}'_{ij})d^3\mathbf{k}'_{ij}, \tag{53}$$

where [see (50)]

$$M_{ij} = M_{ij}(\mathbf{k}_{ij}),\ E_{ij} = E_{ij}(\mathbf{K}_{ij}, \mathbf{k}_{ij}),$$
$$M'_{ij} = M_{ij}(\mathbf{k}'_{ij}),\ E'_{ij} = E_{ij}(\mathbf{K}_{ij}, \mathbf{k}'_{ij}),$$

and [see (9) and (10)]

$$U_{ij}(\mathbf{K}_{ij}, \mathbf{k}_{ij}) = \gamma(\mathbf{K}_{ij}/M_{ij}, \boldsymbol{\sigma}_i, \mathbf{k}_{ij}/m_i) \times\gamma(\mathbf{K}_{ij}/M_{ij}, \boldsymbol{\sigma}_j, -\mathbf{k}_{ij}/m_j).$$

The problem we are faced in the case of three interacting particles is the following. We must construct the Poincaré group representation generators satisfying (1) and the cluster separability. The solution of this problem using the method of Sokolov packing operators [5,12] was found by different authors [5–7,17–20]. If the operators v_{ij} and the three-particle interaction operator v_{123} are given then the solution is not still unique since the three-body mass operator can be constructed in different ways and the choice of the so-called "symmetrized

product of packing operators" (see Refs. [5–7] for details) also has a considerable arbitrariness. However, if we restrict ourselves to the consideration of the baryon spectrum and the internal baryon wave functions, then we can consider our three-body system only in its c.m. frame. In this case the choice of the "symmetrized product" does not play a role (see the detailed discussion in Ref. [15]) and only the uncertainty in constructing the mass operator remains. Following Coester [16] we can choose the relativistic three-body mass operator in the form

$$\hat{M} = M + \sum_{i<j} \tilde{V}_{ij} + v_{123}, \tag{54}$$

where v_{123} depends only on the internal three-particle variables, commutes with the system spin operator $\mathbf{S} = \boldsymbol{l}(\mathbf{K}_{ij}) + \boldsymbol{l}(\mathbf{k}_{ij}) + \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3$, and vanishes if all interactions involving any particle are turned off. As shown in Ref. [15], though the solutions for the mass operator in Refs. [5–7,17–20] were found from different considerations and in different forms of dynamics, all these solutions are unitarily equivalent to (54) and therefore lead to the same physical consequences. In addition, as shown in Refs. [13,15], the solution (54) agrees with quantum field theory in the sense that this solution corresponds to the linear composition of interaction Lagrangians in the full system Lagrangian. For these reason we assume that (54) is the reasonable choice of the relativistic three-body mass operator and we propose to use just this operator in relativistic three-quark calculations.

At first sight the impression is that the difference between (54) and (2) is only in the kinetic energy (since M can formally be written as $m_1 + m_2 + m_3 + T$ where T is the relativistic kinetic energy of three particles in their c.m. frame), and thus we have come to the "semirelativistic" prescription. The role of the operators v_{123} in both expressions is indeed similar, but the operators $\tilde{V}_{ij}$ in (54) considerably differ from the operators $\tilde{v}_{ij}$ in (2). The main difference is that the operators v_{ij} describing the interaction of particles i and j in their c.m. frame enter into $\tilde{V}_{ij}$ under the square root [see (52) and (53)]. This fact was first pointed out by Coester [16] and can be explained as follows. Since the cluster separability requires the additivity of the energy operators for noninteracting systems and not the mass operators, we should construct the three-particle energy operator from the energy operators of different pairs. The three-particle mass operator is the reduction of the three-particle energy operator onto $H^{(3)}_{\rm int}$ and, therefore, the three-particle mass operator is determined by reducing the operators $\hat{E}_{ij}$ onto $H^{(3)}_{\rm int}$. Since $\hat{E}_{ij} = (\hat{M}^2_{ij} + \mathbf{P}^2_{ij})^{1/2}$, then v_{ij} must enter into $\hat{M}$ under the square root.

The difference between $\tilde{V}_{ij}$ and $\tilde{v}_{ij}$ is also manifested as follows. The operator $\tilde{V}_{ij}$ depends not only on the internal variables corresponding to pair ij but also on the total momentum of this pair in the c.m. frame of the three particles. Secondly, the two-body internal interaction operator v_{ij} enters into W_{ij} [see (52) and (53)] being unitarily transformed by the Wigner rotations $U_{ij}(\mathbf{K}_{ij}, \mathbf{k}_{ij})$. Finally, to fully determine the operators $\tilde{V}_{ij}$ we must know not only the interaction of the corresponding two particles in their c.m. frame but some extra operators $\tilde{A}_{ij}$ which cannot be determined if we are only in the c.m. frame of these particles.

VI. ALGORITHM OF CONSTRUCTING THE RELATIVISTIC THREE-BODY MASS OPERATOR

Let us summarize the results concerning the problem of constructing the relativistic three-body mass operator. Suppose that for each pair ij the action of the two-body interaction operator v_{ij} according to (18) and the action of the two-body unitary operators A_{ij} according to (21) and (22) are known. Then the action of the three-body mass operator $\hat{M}$ must be defined in $H^{(3)}_{\rm int}$, in the space of functions satisfying Eq. (47). Using Eq. (48), we also can realize $H^{(3)}_{\rm int}$ as the space of functions satisfying Eq. (49) for each ij. We introduce the operator $\tilde{W}_{ij}$, the action of which is defined by Eq. (53). We also introduce the operator $\tilde{A}_{ij}$ the action of which is defined by Eq. (51). Then we define the operators $\tilde{V}_{ij}$ for each pair ij according to Eq. (52). Finally, if M is the free three-body mass operator and v_{123} is the three-body interaction operator which is assumed to be known from some considerations, then the operator $\hat{M}$ is defined by Eq. (54).

VII. DISCUSSION: THE SYSTEM OF THREE CONSTITUENT QUARKS IN QCD-INSPIRED MODELS

A. On the fully relativistic three-quark problem

In CQM's the same set of phenomenological parameters is used for parametrization of the operators v_{ij} for both quark-antiquark and quark-quark interactions. In the former case the operators v_{ij} determine the meson spectroscopy. They are defined in the internal two-particle space of functions $\chi(\mathbf{r}_{ij}, \sigma_i, \sigma_j)$ where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is the operator of relative distance between particles i and j. This operator is canonically conjugated with the quantity $\mathbf{q}_{ij} = (m_j\mathbf{p}_i - m_i\mathbf{p}_j)/(m_i + m_j)$. Therefore, knowing the action of the v_{ij} on the functions $\chi(\mathbf{r}_{ij}, \sigma_i, \sigma_j)$ we can determine the kernels $v_{ij}(\mathbf{q}_{ij}, \mathbf{q}'_{ij})$ which define the action of the v_{ij} in momentum representation. Now we take into account that, as follows from Eq. (6), in the c.m. frame of two particles the quantities $\mathbf{q}_{ij}$ and $\mathbf{k}_{ij}$ are equal to each other. Therefore the kernels $v_{ij}(\mathbf{k}_{ij}, \mathbf{k}'_{ij})$ entering into the three-quark problem should be parametrized in accordance with the parametrization of the kernels $v_{ij}(\mathbf{q}_{ij}, \mathbf{q}'_{ij})$ entering into the quark-antiquark problem (of course, we must also take into account the dependence of the v_{ij} on the factor β—see Sec. IV).

In QCD-inspired models v_{ij} must contain a part cor-

responding to the one-gluon-exchange (and, possibly, the second-order corrections to α_s [39]) and a part corresponding to the confinement. In Refs. [10,2] it is described in detail the process of smearing and relativization of v_{ij}. In the absence of a more elaborated dynamical scheme such a procedure is admissible. We note only that different modifications of v_{ij} are possible only within the space $H_{\rm int}$ for the corresponding pair of particles. Then in our approach the dependence of $\tilde{W}_{ij}$ on $\mathbf{K}_{ij}$ is fully defined by (53). In particular, the dependence of the Thomas precession on $\mathbf{K}_{ij}$ originates from the Wigner rotations U_{ij} in (53).

Let us now discuss the operator $\tilde{A}_{ij}$ in (51). The investigation carried out in Sec. IV shows that we cannot set $A_{ij}=1$ since even in first order in $1/m^2$ $\mathbf{B}_{ij}$ is not equal to zero and the reasonable choice of $\mathbf{B}_{ij}$ in this order is [see (36)]

$$\mathbf{B}_{ij}=\frac{-i(m_i-m_j)\mathbf{r}_{ij}}{(m_i+m_j)^2}u^{\rm nr}(r_{ij}) \tag{55}$$

(we do not write indices i,j in $u^{\rm nr}$ since in QCD the gluon field does not act on flavor variables). The appearance of the factor (m_i-m_j) in (55) can be explained as follows. If $\mathbf{B}_{ij}$ does not act on the variables $\mathbf{K}_{ij},\sigma_1,\sigma_2$ then it is the vector operator depending only on $\mathbf{r}_{ij}$. But $\mathbf{r}_{ij}$ changes sign under the interchange of particles i and j while $\mathbf{B}_{ij}$ must be symmetric under this interchange. Therefore, $\mathbf{B}_{ij}$ must contain the factor (m_i-m_j). However, in principle we cannot exclude that in higher orders in $1/m$ $\mathbf{B}_{ij}$ may depend on spins and, therefore, we cannot prove that $\mathbf{B}_{ij}=0$ in all orders in $1/m$ if $m_i=m_j$. Nevertheless, this is the reasonable assumption, for example, in the few nucleon problem (see Ref. [15] for details). At the same time we by no means can neglect the contribution of $\mathbf{B}_{ij}$ in relativistic CQM's when quarks have different masses. In this case we should choose for $\mathbf{B}_{ij}$ (and therefore for A_{ij}) a model satisfying the condition that in first order in $1/m^2$ $\mathbf{B}_{ij}$ is given by (55), but in higher orders the $\mathbf{B}_{ij}$ is arbitrary since we cannot make contact with QCD in these orders.

Now we shall briefly discuss the operator v_{123} in (54). As noted in Sec. V, this operator obeys the same restrictions as in the nonrelativistic theory. Therefore, as in Ref. [2], it is not necessary to subject v_{123} to smearing and relativization though there may exist relativistic dynamical corrections to v_{123} [40]. In Ref. [2] the operator v_{123} was formally obtained by subtracting a part related to the two-quark interactions from the string interaction between three quarks. However, v_{123} obtained in such a way has not the physical sense of the pure three-quark interaction. This is a problem (see the discussion in Ref. [2]), but, as pointed out in Ref. [2], the contribution of v_{123} was always small. Another version of v_{123} has been proposed in Ref. [41] for the nonrelativistic CQM, and this has turned out to be important in considering the "Roper puzzle."

We have described all the operators entering into (51)–(54) and, therefore, the relativistic three-quark problem is fully defined by the algorithm given in Sec. VI. However there are considerable technical complications in comparison with the nonrelativistic problem. For example, in the nonrelativistic theory the sets

$$(\mathbf{K}_{12},\mathbf{q}_{12}),(\mathbf{K}_{13},\mathbf{q}_{13}),(\mathbf{K}_{23},\mathbf{q}_{23})$$

are related to each other by the simple linear expressions. In the relativistic case we must relate the sets

$$(\mathbf{K}_{12},\mathbf{k}_{12}),(\mathbf{K}_{13},\mathbf{k}_{13}),(\mathbf{K}_{23},\mathbf{k}_{23}).$$

This can be done using (48) and the condition $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$, but the resulting expressions are more complicated than in the nonrelativistic case.

One more serious complication in numerical calculations is that we must explicitly determine the square root in (52). For this purpose we can first consider the operator

$$R_{ij}=\tilde{A}_{ij}(M_{ij}+\tilde{W}_{ij})^2\tilde{A}_{ij}^{-1}-M_{ij}^2 \tag{56}$$

and, since $\tilde{A}_{ij}$ commutes with $\mathbf{K}_{ij}$, we can write (52) as

$$\tilde{V}_{ij}=(M_{ij}^2+\mathbf{K}_{ij}^2+R_{ij})^{1/2}-(M_{ij}^2+\mathbf{K}_{ij}^2)^{1/2}. \tag{57}$$

Using the standard technique we can calculate the operator

$$\begin{aligned}T_{ij}(s)&=R_{ij}-R_{ij}G_{ij}(s)R_{ij}\\&=\left(1+R_{ij}\frac{1}{M_{ij}^2+\mathbf{K}_{ij}^2+s}\right)^{-1}R_{ij},\end{aligned} \tag{58}$$

where $G_{ij}(s)=(M_{ij}^2+\mathbf{K}_{ij}^2+R_{ij}+s)^{-1}$ is the resolvent of the operator $(M_{ij}^2+\mathbf{K}_{ij}^2+R_{ij})$. Let us use the familiar expression for the square root of a positive operator,

$$(M_{ij}^2+\mathbf{K}_{ij}^2+R_{ij})^{1/2}=\frac{1}{\pi}\int_0^\infty\frac{ds}{s^{1/2}}[1-sG_{ij}(s)] \tag{59}$$

(the integral is understood as the strong limit of the Riemann integral sums), and the analogous expression for the square root of the operator $M_{ij}^2+\mathbf{K}_{ij}^2$, the resolvent of which is denoted as $G_{ij}^0(s)=(M_{ij}^2+\mathbf{K}_{ij}^2+s)^{-1}$. Using also the familiar expression

$$G_{ij}(s)-G_{ij}^0(s)=-G_{ij}^0(s)T_{ij}(s)G_{ij}^0(s) \tag{60}$$

we find from (57)–(60) that

$$\tilde{V}_{ij}=\frac{1}{\pi}\int_0^\infty s^{1/2}(M_{ij}^2+\mathbf{K}_{ij}^2+s)^{-1}T_{ij}(s)(M_{ij}^2+\mathbf{K}_{ij}^2+s)^{-1}ds. \tag{61}$$

It is easy to see that the kernel of the operator $T_{ij}(s)$ has the form

$$\begin{aligned}&T_{ij}(\mathbf{K}_{ij},\mathbf{k}_{ij};\mathbf{K}'_{ij},\mathbf{k}'_{ij};s)\\&\qquad=\delta^{(3)}(\mathbf{K}_{ij}-\mathbf{K}'_{ij})T_{ij}(\mathbf{K}_{ij};\mathbf{k}_{ij},\mathbf{k}'_{ij};s)\end{aligned}$$

and, therefore, as follows from (61), the kernel of $\tilde{V}_{ij}$ is expressed in terms of the kernel of T_{ij} as

$$\tilde{V}_{ij}(\mathbf{K}_{ij},\mathbf{k}_{ij};\mathbf{K}'_{ij},\mathbf{k}'_{ij}) = \delta^{(3)}(\mathbf{K}_{ij}-\mathbf{K}'_{ij})\frac{1}{\pi}\int_0^\infty \frac{T_{ij}(\mathbf{K}_{ij};\mathbf{k}_{ij},\mathbf{k}'_{ij};s)s^{1/2}ds}{[M_{ij}(\mathbf{k}_{ij})^2+\mathbf{K}_{ij}^2+s][M_{ij}(\mathbf{k}'_{ij})^2+\mathbf{K}_{ij}^2+s]}. \tag{62}$$

In Ref. [42] we proposed the similar way of calculating the square root from the corresponding operator entering into the relativistic few-nucleon problem.

Having calculated the operators $\tilde{V}_{ij}$ we can numerically solve the relativistic three-body problem by the well-elaborated methods used in solving the nonrelativistic three-body problem in momentum representation. In particular, instead of the eigenvalue problem for the operator $\hat{M}$ we can solve the system of Faddeev equations. In this case we have to know the formulas connecting the angular momentum states in bases $(ij)k$ (when particle k is the spectator) and $(ik)j$ (when particle j is the spectator). In the relativistic case these formulas are considerably more complicated than in the nonrelativistic case since we must take into account the Wigner rotations and more complex kinematics. For the case of particles with equal masses, all needed transition coefficients were explicitly expressed in terms of the Clebsch-Gordan ones in Ref. [42].

We can conclude that there are no principal obstacles for performing systematic calculations in the relativistic three-quark problem. However, in view of the above discussion, we can expect that the full relativistic calculation will require a much larger amount of machine time than the calculations made so far.

B. On the three-quark problem in first order in $1/m^2$

As noted above, the relativistic three-quark calculations in first order in $1/m^2$ are not expected to be reliable for systems of light quarks. Nevertheless the formal consideration of this case is useful since it sheds light on some nontrivial aspects of the fully relativistic problem. Using (54), (55), and (43) we obtain the final result for $\hat{M}$ in the $1/m^2$ approximation:

$$\begin{aligned}\hat{M} = &\sum_i m_i + T^{\rm nr} + \sum_{i<j}\tilde{v}_{ij} + T' \\ &+\sum_{i<j}\left(-\frac{\mathbf{K}_{ij}^2}{2(m_i+m_j)^2}v^{\rm nr}(r_{ij}) + \frac{(\mathbf{K}_{ij}\cdot\mathbf{r}_{ij})^2}{2(m_i+m_j)^2 r_{ij}}\frac{dv^{\rm nr}(r_{ij})}{dr_{ij}} + \frac{(\mathbf{K}_{ij}\cdot\mathbf{r}_{ij})}{2(m_i+m_j)r_{ij}}\left(\frac{\mathbf{s}_i}{m_i}-\frac{\mathbf{s}_j}{m_j}\right)\frac{dv^{\rm nr}(r_{ij})}{dr_{ij}}\right. \\ &\left.-\frac{i(m_i-m_j)}{2(m_i+m_j)m_im_j}\left\{\frac{\partial}{\partial\mathbf{r}_{ij}},\mathbf{K}_{ij}u^{\rm nr}(r_{ij}) + \frac{(\mathbf{K}_{ij}\cdot\mathbf{r}_{ij})\mathbf{r}_{ij}}{r_{ij}}\frac{d[u^{\rm nr}(r_{ij}]-v^{\rm nr}(r_{ij}))}{dr_{ij}}\right\}\right) + v_{123},\end{aligned} \tag{63}$$

where

$$\begin{aligned}T^{\rm nr} &= \frac{(m_i+m_j)\mathbf{q}_{ij}^2}{2m_im_j} + \frac{\mathbf{K}_{ij}^2(m_i+m_j+m_k)}{2(m_i+m_j)m_k}, \\ T' &= \frac{\mathbf{q}_{ij}^2(\mathbf{K}_{ij}\cdot\mathbf{q}_{ij})(m_i-m_j)}{2m_i^2m_j^2} - \frac{(\mathbf{K}_{ij}\cdot\mathbf{q}_{ij})^2}{2(m_i+m_j)m_im_j} - \frac{\mathbf{q}_{ij}^4}{8}\left(\frac{1}{m_i^3}+\frac{1}{m_j^3}\right) \\ &\quad -\frac{\mathbf{K}_{ij}^2\mathbf{q}_{ij}^2}{4(m_i+m_j)m_im_j} - \frac{\mathbf{K}_{ij}^4}{8}\left(\frac{1}{(m_i+m_j)^3}+\frac{1}{m_k^3}\right) \quad (k\neq i\neq j\neq k).\end{aligned} \tag{64}$$

Note that $\mathbf{K}_{ij}$ is canonically conjugated with

$$\mathbf{R}_{ij} = \frac{m_i\mathbf{r}_i+m_j\mathbf{r}_j}{m_i+m_j} - \mathbf{r}_k \quad (k\neq i\neq j\neq k).$$

The operator T' is the order-$(1/m^2)$ relativistic correction to $T^{\rm nr}$. We recall that $\tilde{v}_{ij}$ contains not only the nonrelativistic term $v^{\rm nr}(r_{ij})$ but also the order-$(1/m^2)$ relativistic corrections which depend on the internal variables $\mathbf{r}_{ij},\sigma_i,\sigma_j$.

Comparing Eqs. (2) and (63) we can conclude that the correct treatment of the order-$(1/m^2)$ relativistic corrections leads to the terms in (63) depending on $\mathbf{K}_{ij}$ not only in the kinetic energy, but also in the interaction operator. For example, it is easy to see that the terms with $\mathbf{K}_{ij}^2 v^{\rm nr}$ in the last sum in (63) emerge from relativistic kinematics, the terms with $\mathbf{K}_{ij}\times\mathbf{r}_{ij}$ describe the Thomas precession, the terms with $(\mathbf{K}_{ij}\cdot\mathbf{r}_{ij})^2\,dv^{\rm nr}/dr^{ij}$ emerge from the difference between the vectors $\mathbf{k}_{ij}$ and $\mathbf{q}_{ij}$ and the other terms in this sum have partially the same and partially the dynamical origin.

In Refs. [43,2] the interaction-dependent terms with $\mathbf{K}_{ij}$ were taken into account only in the Thomas precession, while a detailed study of such terms in CQM's in first order in $1/m^2$ was carried out in Ref. [35]. As noted in Sec. IV, the results of Ref. [35] agree with those of Refs. [33,34] and are based on the prescription (32) for

both vector and scalar couplings. However, as argued in Sec. IV, this prescription is not most general. It is important to note that different prescriptions for choosing ω will play a role not only when quarks have different masses [as one might think looking at Eq. (63)] but also for baryons consisting of lightest quarks, since such a prescription affects also the operators v_{ij}.

The interaction-dependent terms with $\mathbf{K}_{ij}$ in first order in $1/m^2$ were also taken into account in Ref. [44], but the spin-independent terms of such a type were taken into account only for the scalar potential according to Eq. (40). However, as noted in Sec. IV, Eq. (40) cannot be correct.

As shown in Refs. [10,2], the unification of mesons and baryons can be achieved only in fully relativistic models (note, however, that some authors advocate the opposite point of view [45]). Therefore, we cannot expect that the expansion in powers of $(1/m^2)$ in the model of Ref. [2] will be correct. Nevertheless, let us estimate the correction to the nucleon mass emerging from the interaction-dependent terms with $\mathbf{K}_{ij}$ in (63). According to Ref. [2], for a rough estimation we can choose the simple harmonic-oscillator nucleon wave function $\psi \sim \exp[-\alpha^2(\rho^2+\lambda^2)/2]$ (see the notations in Ref. [2]) and take the nonrelativistic quark-quark potential in the form $v^{\rm nr} = v_{\rm Coul} + v_0 + v_1$ where

$$v_{\rm Coul}(r) = -\frac{2}{3}\frac{\bar\alpha_s}{r},\ v_0 = -\frac{1}{2}C_{q\bar q},\ v_1 = fbr \tag{65}$$

(the parameters are given in Ref. [2]). For simplicity we work not with the running coupling constant, but with some its average value $\bar\alpha_s$. Let us note also that we interpret the constant C_{qqq} in Ref. [2] as $C_{qqq} = 3C_{q\bar q}/2 + C_3$ where $3C_{q\bar q}/2$ corresponds to the two-body interactions and C_3—to the three-body ones. Calculating the correction to the nucleon mass ΔM as the average value of the interaction-dependent terms with $\mathbf{K}_{ij}$ in (63) over the nucleon wave function we find

$$\Delta M = \frac{27\alpha^2}{32m^2}\left\langle -\frac{5}{3}v_{\rm Coul} - v_0 - \frac{1}{3}v_1 \right\rangle$$
$$= \frac{27\alpha^2}{32m^2}\left[\frac{20}{9}\alpha\bar\alpha_s(2/\pi)^{1/2} + \frac{1}{2}C_{q\bar q} - \frac{1}{3\alpha}fb(2/\pi)^{1/2}\right], \tag{66}$$

where m is the mass of the u and d quarks and we use $\langle A\rangle$ to denote the average value of the operator A over the nucleon wave function. Equation (66) shows that in comparison with the nonrelativistic problem the order-$(1/m^2)$ correction to the nucleon mass is determined in more extent by the Coulomb-like contribution [the factor 5/3 in (66)] and in less extent by the terms linear in r_{ij} [the factor 1/3 in (66)]. If we assume further that α is determined from the nonrelativistic expression for the proton mean square charge radius and $\bar\alpha_s = 0.4$ [the average value between $\alpha_s(0) = 0.6$ and $\alpha_s(2\ {\rm GeV}) = 0.2$] then with the parameters of Ref. [2] we obtain from (66) that $\Delta M \approx 60$ MeV. The fact that $\Delta M > 0$ means that the correction under consideration diminishes the nucleon mass.

Let us now consider the another example—the Isgur-Karl model [46] which is mainly nonrelativistic. Here $v^{\rm nr}(r) = Kr^2/2$, and we can find the contribution of the interaction terms with $\mathbf{K}_{ij}$ in Eq. (63). However, in the given case we must also take into account the contribution caused by T' [see Eqs. (63) and (64)]. Using the same nucleon wave function we find

$$\Delta M = -\frac{7071\alpha^4}{1024m^3} + \frac{9\omega^2}{256m}, \tag{67}$$

where $\omega = (3K/m)^{1/2}$. The second term on the right-hand side of Eq. (67) is rather small: for example, if we take the values $\omega = 250$ MeV and $m = 350$ MeV used in Ref. (46), then this term is of about 6 MeV. At the same time the first term (caused by the contribution of T') is of about -90 MeV if α corresponds to the proton mean square charge radius. Note however that the results given by Eqs. (66) and (67) crucially depend on the choice of α and become much greater for the values of α which were actually used in Refs. [46,2].

Of course, these results can be regarded only as a rough estimation but one can expect that taking into account the interaction-dependent terms with $\mathbf{K}_{ij}$ in realistic calculations can considerably alter some theoretical predictions in baryon spectroscopy.

VIII. CONCLUSIONS

Let us briefly discuss the results of the present work. In Sec. III we have shown that the relativistically invariant introduction of the interaction into the system of two particles is determined in the general case not only by replacing $M \to M + v$ in Eq. (12) but also by some unitary operator $A = \exp(\mathbf{P}\cdot\mathbf{B})$. For the purpose of understanding the structure of the operators $\mathbf{B}_{ij}$ for different pairs ij in the three-particle system we investigate in Sec. IV the two-body energy operator in first order in $1/m^2$. It is shown that the condition of relativistic invariance imposes considerable restrictions on the dependence of the two-particle energy operator on the total momentum, and we explicitly write down this dependence in different cases. We also argue that there is no universal expression for V in terms of $v^{\rm nr}$ in this order. Owing to the gauge invariance, it is possible to obtain the unique Breit-Fermi Hamiltonian in QED, QCD, and general relativity but there are no reasons to believe that this is also the case in the scalar theories. As for the operators $\mathbf{B}_{ij}$, we show that in the general case they are not equal to zero. This fact was taken into account only by some authors in first order in $1/m^2$ (see the discussion in the preceding section) while in fully relativistic calculations these operators were not taken into account.

In Secs. V and VI we have explicitly defined all quantities entering into the relativistic three-body mass operator. From the discussion in Sec. VII it is clear that the relativistic three-quark problem is much more complicated than the nonrelativistic one, but in principle, full relativistic calculations can be carried out, and their results will undoubtedly be very important for under-

standing the quark structure of hadrons.

Relativistic models for the three-quark system are now used not only in spectroscopic calculations but also in investigating some electromagnetic processes (see, for example, Refs. [14,21,35,44,47]). For this reason it is interesting to investigate the role of the operators $\mathbf{B}_{ij}$ not only in the c.m. frame of three quarks but also in cases when the three-quark system has a nonzero total momentum. We suppose to consider this problem in future publications.

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