# Flavor unification and discrete non-Abelian symmetries

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Grand unified theories with fermions transforming as irreducible representations of a discrete non-Abelian flavor symmetry can lead to realistic fermion masses, without requiring small fundamental parameters. We construct a specific example of a supersymmetric GUT based on the flavor symmetry $\Delta(75)$, a subgroup of SU(3), which can explain the observed quark and lepton masses and mixing angles. The model predicts $\tan\beta\simeq 1$ and gives a $\tau$ neutrino mass $m_\nu \simeq M_p/G_F M_{\rm GUT}^2 = 10$ eV, with other neutrino masses much lighter. Combined constraints of light quark masses and perturbative unification place flavor symmetry-breaking near the GUT scale; it may be possible to probe these extremely high energies by continuing the search for flavor-changing neutral currents.

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## I. INTRODUCTION

Particle physics seems to be at a stage similar to chemistry before Mendeleev or spectroscopy before Balmer—we are confronted with apparent patterns in quark and lepton masses and mixing angles, yet have no compelling explanation for them. It is likely that the difficulty is due to several simultaneous effects contributing to the observed mass relations. These effects could include radiative corrections in scaling from short distances, Clebsch factors from gauge groups, mass matrix "textures," and Clebsch factors from flavor symmetry groups, flavor symmetry-breaking vacuum alignment, and higher dimension operators induced by quantum gravity. Aside from the observed masses, the only experimental evidence we have to guide us is the absence of flavor-changing neutral currents (FCNC's). In order to make headway in the face of such ignorance, it is necessary to have aesthetic prejudices for guidance; in this paper we adopt several. The first prejudice is that the fundamental theory not contain parameters less than $\sim(10^{-1})$. The second is the principle of "flavor democracy" [2], namely, that all fermions with identical gauge charges have the same or similar short distance interactions, with the observed diversity in masses arising from dynamics. Third, we only consider theories where the gauge interactions are unifiable, in order to adopt the successes in explaining the equality of the proton and positron charges, as well as predicting $\sin^2\theta_W$ and the relations between quark and lepton masses [3–5].

As we will show, these three prejudices naturally lead us to consider theories with non-Abelian discrete flavor symmetries. Such symmetries allow us to understand many features of the quark and lepton masses, such as why the down-type quarks are lighter than up quarks in all but the first generation, and why the Cabibbo angle is much larger than the other Kobayashi-Maskawa (KM) angles. The type of theories we consider typically require flavor symmetry breaking to be near the grand unified theory (GUT) scale and offer the tantalizing prospect of probing GUT-scale physics through searches for flavor-changing neutral currents. They also suggest that the neutrinos are massive, with the $\tau$ neutrino mass naturally in the range favored for dark matter.

The principles we adopt force us to think carefully about flavor symmetries. In order to explain in a natural way a small mass ratio such as $m_e/m_t \sim 3\times 10^{-6}$ in terms of the parameters $\epsilon \sim 10^{-1}$, we must assume that the mass ratios arise as high powers of $\epsilon$. These powers of $\epsilon$ can arise naturally if $\epsilon$ measures mixing between ordinary fermions and massive exotic fermions through soft flavor symmetry breaking [6]. Then $\epsilon \sim g\langle X\rangle/M$, where $g$ is a coupling constant, $\langle X\rangle$ is a soft flavor symmetry-breaking parameter, and $M$ is the heavy fermion mass. The invariant tensors of the broken flavor symmetry group and pattern of symmetry breaking naturally impose a texture on the effective Yukawa couplings of the low energy theory.[1] The goal then is to find models which lead to a phenomenologically acceptable texture. Most previous work in this direction has focused on Abelian flavor symmetries [U(1) or $Z_N$] which allow one to "dial" the fermion mass matrices by judiciously choosing the charges for each fermion; for a recent example consistent with current phenomenology, see Ref. [9]. Pouliot and Seiberg have also constructed a non-Abelian example of such models, based on $O(2)\times U(1)$ [10], with the quarks in reducible representations. Since all of these models have quarks and leptons in reducible flavor representations, the different generations are distinguished by their flavor charges and have different interactions. However, this is not compatible with our goal of flavor democracy, which can only be achieved by putting all particles of like gauge charge in irreducible flavor representations. Furthermore, existing approaches do not lend themselves readily to a unification of gauge forces.

In order to unify the three families into irreducible flavor triplets, we are compelled to search for a non-

[1]There has been much recent interest in investigating acceptable and predictive mass matrix textures; see, for example, [7,8].

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Abelian flavor symmetry $G_f$ with one or more three-dimensional representations. For continuous symmetries, this only allows groups with at least one factor of SO(3), SU(2), or SU(3). A further restriction is found by considering the top quark, whose mass must arise at $O(\epsilon^0)$ if it is to have perturbative interactions. Thus the operator

$$QU^cH_u \tag{1.1}$$

must be a $G_f$ invariant and lead to a rank-1 mass matrix. If $Q$ and $U^c$ are to be triplets of $G_f$ and $H_u$ is some irreducible representation, then we can rule out the possibilities $G_f=\mathrm{SU}(2)$ and $G_f=\mathrm{SO}(3)$—for those groups the operator (1.1) yields a mass matrix that is either the unit matrix or traceless and, hence, at least rank 2. Similar reasoning excludes $G_f=\mathrm{SU}(3)$ unless $Q$ and $U^c$ transform as 3's and $H_u$ is a $\overline{6}$ with $\langle H_u\rangle = v\delta_{33}$. A semisimple group such as $G_f=\mathrm{SU}(3)\times\mathrm{SU}(3)$ with $Q=(3,1)$, $U^c=(1,3)$, and $H_u=(\overline{3},\overline{3})$ is a possibility, as are groups with more factors.

The difficulty with the continuous flavor symmetries described above is that they contain few low dimensional representations, and therefore there are few invariant tensors that are of use in building up the fermion mass matrix in powers of $\epsilon$. In contrast, if one is willing to consider non-Abelian discrete groups for $G_f$ one can find groups with an arbitrarily large number of triplet representations, for example. With such a symmetry, there are many invariant tensors which can arise without resorting to a multitude of exotic particles. In this paper we consider the $\Delta(3n^2)$ dihedral subgroups of SU(3), which contain an arbitrary number of triplet representations. The explicit model we give is based on $\Delta(75)$, a group with eight triplet and three singlet representations.

## II. NON-ABELIAN DISCRETE SYMMETRIES

The representations of discrete groups with $^\circ G$ elements satisfy the relation $\sum_i d_i^2 = {^\circ G}$, where $d_i$ is the dimension of the $i$th representation. Thus finite groups have a finite number of finite dimensional representations. Among the non-Abelian discrete groups most familiar to physicists, namely, the crystallographic symmetries, the ones with more than one triplet representation are the octahedral and icosahedral groups. The octahedral group **O** has 24 elements and representations $\{1,1',2,3,3'\}$. We could consider constructing an SU(5)×O grand unified theory, for example, by having the $Q$, $U$, and $E^c$ fermions transform as a (10,3). However, one finds that

$$3\otimes 3 = 3_a \oplus 3'_s \oplus 2_s \oplus 1_s\ . \tag{2.1}$$

Evidently, the 5 of SO(3) decomposes as a $3'_s\otimes 2_s$ under **O**. This does not help to solve the problem encountered with SO(3) as a flavor group, since each of these couplings leads to a rank-2 mass matrix again: The 3′ and 2 decompositions of $3\otimes 3$ consist of

$$(3\otimes 3)|_{3'} = \begin{bmatrix} 3\lambda_6 3 \\ 3\lambda_4 3 \\ 3\lambda_1 3 \end{bmatrix},\quad (3\otimes 3)|_2 = \begin{bmatrix} 3\lambda_3 3 \\ 3\lambda_8 3 \end{bmatrix}, \tag{2.2}$$

where the $\lambda_a$ are the Gell-Mann SU(3) matrices and $3\lambda_a 3 = 3_i(\lambda_a)_{ij}3_j$. The same conclusion holds for the icosahedral group.

What is needed to explain the top mass operator (1.1) is a group which contains a triplet $3=\{x,y,z\}$ as well as a 3′ representation contained in $3\otimes 3$ with $3\otimes 3|_{3'} = \{x^2,y^2,z^2\}$. Then the top mass arises at the tree level if the Higgs field transforms as $H_u = 3'^*$ with a vacuum expectation value (VEV) only in the third family component. This is only possible if the 3 representation is complex, since otherwise $x^2+y^2+z^2$ is a singlet. It follows that $G_f$ cannot be a subgroup of SO(3), and we turn to discrete subgroups[2] of SU(3).

The discrete subgroups of SU(3) are the irregular groups $\Sigma$ and the dihedral groups $\Delta(3n^2)$ and $\Delta(6n^2)$ for all integers $n$. The $\Delta(3n^2)$ groups are particularly interesting since their representations consist solely of triplets and singlets. These groups are of order $3n^2$ and are generated by the matrices

$$E_{00} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.3}$$

and

$$A_{pq} = \begin{bmatrix} (\eta_n)^p & 0 & 0 \\ 0 & (\eta_n)^q & 0 \\ 0 & 0 & (\eta_n)^{-(p+q)} \end{bmatrix}, \tag{2.4}$$

where $\eta_n$ is the $n$th root of unity,

$$\eta_n = e^{2\pi i/n}\ , \tag{2.5}$$

and $p,q$ are integers.

The irreducible representations of the $\Delta(3n^2)$ groups consist of (i) nine singlets and $(n^2-3)/3$ triplets for $n$ a multiple of three and (ii) three singlets and $(n^2-1)/3$ triplets otherwise. The large number of inequivalent triplet representations in these groups is invaluable for building a model of fermion masses, starting with flavor democracy at short distances. In this paper we will focus on a particular discrete symmetry in order to exhibit some of the general features of model building with non-Abelian discrete symmetries. The symmetry we discuss is $\Delta(75)$ [i.e., $\Delta(3n^2)$ with $n=5$], which is apparently the smallest of the dihedral groups with sufficient structure to be interesting.

### A. Δ(75)

The irreducible representations of $\Delta(75)$ include one real singlet $A_1$, one complex singlet $A_2$, and four complex triplets $T_1,\ldots,T_4$. The character table may be constructed from the generators (2.3),(2.4) with $n=5$ and is given in Table I. (For an explanation of discrete symmetries and character tables, see, for example, Ref. [11].)

[2]All of our discussion of discrete SU(3) subgroups is based on Ref. [1].

TABLE I. Character table for $\Delta(75)$, computed from Ref. [1]. The quantities $\chi$ and $\omega$ are defined as $\chi_{pq}=(\eta_5)^p+(\eta_5)^q+(\eta_5)^{-p-q}$ and $\omega=\eta_3$, where $\eta_n=e^{2\pi i/n}$.

| $\Delta(75)$ | $E$ | $3A_{10}$ | $3A_{20}$ | $3A_{30}$ | $3A_{40}$ | $3A_{11}$ | $3A_{22}$ | $3A_{33}$ | $3A_{44}$ | $25C$ | $25E$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega$ | $\bar{\omega}$ |
| $T_1$ | 3 | $\chi_{10}$ | $\chi_{20}$ | $\chi_{20}$ | $\chi_{10}$ | $\chi_{11}$ | $\chi_{22}$ | $\bar{\chi}_{22}$ | $\bar{\chi}_{11}$ | 0 | 0 |
| $T_2$ | 3 | $\chi_{20}$ | $\chi_{10}$ | $\chi_{10}$ | $\chi_{20}$ | $\chi_{22}$ | $\bar{\chi}_{11}$ | $\chi_{11}$ | $\bar{\chi}_{22}$ | 0 | 0 |
| $T_3$ | 3 | $\chi_{11}$ | $\chi_{22}$ | $\bar{\chi}_{22}$ | $\bar{\chi}_{11}$ | $\chi_{20}$ | $\chi_{10}$ | $\chi_{10}$ | $\chi_{20}$ | 0 | 0 |
| $T_4$ | 3 | $\chi_{22}$ | $\bar{\chi}_{11}$ | $\chi_{11}$ | $\bar{\chi}_{22}$ | $\chi_{10}$ | $\chi_{20}$ | $\chi_{20}$ | $\chi_{10}$ | 0 | 0 |

The defining representation is taken to be $T_1$, and we have labeled the conjugacy classes after generators contained in that class for the $T_1$ representation. For example, the class labeled $3A_{10}$ contains the group elements $A_{10}$, $A_{04}$, and $A_{41}$,

$$\begin{bmatrix}(\eta_5)^1 & 0 & 0\\ 0 & (\eta_5)^0 & 0\\ 0 & 0 & (\eta_5)^4\end{bmatrix},\quad \begin{bmatrix}(\eta_5)^0 & 0 & 0\\ 0 & (\eta_5)^4 & 0\\ 0 & 0 & (\eta_5)^1\end{bmatrix},\quad \begin{bmatrix}(\eta_5)^4 & 0 & 0\\ 0 & (\eta_5)^1 & 0\\ 0 & 0 & (\eta_5)^0\end{bmatrix}, \tag{2.6}$$

in the $T_1$ representation, while the class $25E$ contains the 25 elements

$$E_{pq}=\begin{bmatrix}0 & \eta_5^p & 0\\ 0 & 0 & \eta_5^q\\ \eta_5^{-(p+q)} & 0 & 0\end{bmatrix}. \tag{2.7}$$

The $25C$ class contains the square of the $E_{pq}$ matrices.

From the character table, it is possible to determine the decomposition of the product of any two representations. Evidently, $A_1$ is the trivial representation, while

$$A_2\otimes A_2=\bar{A}_2,\quad A_2\otimes\bar{A}_2=A_1,\quad A_2\otimes T_i=\bar{A}_2\otimes T_i=T_i\ , \tag{2.8}$$

where $i=1,\ldots,4$. Less obvious are the products of two triplet representations, whose decompositions are given in Table II.

Since we wish to construct explicit models with particle couplings obeying $\Delta(75)$ symmetry, we need to choose a basis for all of the representations and construct the invariant tensors. We have chosen a basis defined by

$$T_1\otimes T_1|_{T_2}=\begin{bmatrix}x^2\\ y^2\\ z^2\end{bmatrix},\quad T_1\otimes\bar{T}_1|_{T_3}=\begin{bmatrix}y\bar{z}\\ z\bar{x}\\ x\bar{y}\end{bmatrix},\quad T_2\otimes\bar{T}_2|_{T_4}=\begin{bmatrix}b\bar{c}\\ c\bar{a}\\ a\bar{b}\end{bmatrix}, \tag{2.9}$$

where we have written $T_1=\{x,y,z\}$, $T_2=\{a,b,c\}$. This basis has the virtue that the generator $E_{00}$ is the same matrix (2.3) in all of the triplet representations. Thus, when any two triplets $T_i$ and $T_j$ (or their conjugates) are combined into a third triplet $T_k$, the elements of $T_k$ must cyclically permute when the elements of $T_i$ and $T_j$ are simultaneously cyclically permuted; therefore, all of the components of $T_k$ are specified when the first component is known. The decomposition of all products of triplets in this basis is given in the Appendix.

### B. Symmetry breaking

We now turn to ways to spontaneously break the $\Delta(75)$ symmetry in a supersymmetric theory. One reason we choose to focus on supersymmetry is that the flavor-breaking patterns can be more interesting: In a supersymmetric theory, one can have different symmetry-breaking patterns in different sectors of the theory which communicate only through higher dimension operators and not through radiative corrections. Nongeneric flavor symmetry breaking can lead to interesting structure, as

TABLE II. Decomposition of the product of two triplets. Triplets $T_n$ and $\bar{T}_n$ are represented by $n$ and $\bar{n}$, respectively, while $A\equiv A_1\oplus A_2\oplus\bar{A}_2$. For example, $T_3\otimes\bar{T}_1=\bar{T}_1\oplus T_2\oplus T_4$ and $T_1\otimes\bar{T}_1=A_1\oplus A_2\oplus\bar{A}_2\oplus T_3\oplus\bar{T}_3$.

| $\Delta(75)$ | 1 | $\bar{1}$ | 2 | $\bar{2}$ | 3 | $\bar{3}$ | 4 | $\bar{4}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $\bar{1}\bar{1}2$ | $A3\bar{3}$ | $\bar{2}3\bar{3}$ | $\bar{1}4\bar{4}$ | $1\bar{2}4$ | $1\bar{2}\bar{4}$ | $234$ | $2\bar{3}\bar{4}$ |
| 2 | $\bar{2}3\bar{3}$ | $14\bar{4}$ | $\overline{122}$ | $A4\bar{4}$ | $\bar{1}3\bar{4}$ | $\bar{1}\bar{3}4$ | $12\bar{3}$ | $123$ |
| 3 | $1\bar{2}4$ | $\bar{1}24$ | $\bar{1}3\bar{4}$ | $13\bar{4}$ | $\bar{3}\bar{3}4$ | $A2\bar{2}$ | $2\bar{2}\bar{4}$ | $1\bar{1}\bar{3}$ |
| 4 | $234$ | $\bar{2}34$ | $12\bar{3}$ | $\overline{123}$ | $2\bar{2}\bar{4}$ | $1\bar{1}3$ | $\bar{3}\bar{4}4$ | $A1\bar{1}$ |

we will show. Here we give a couple of toy models showing different symmetry-breaking patterns.

The first toy model we consider has $\Delta(75)$ breaking down to $Z_3$ generated by $E_{00}$ alone [Eq. (2.3)]. We include the singlet fields $S$, $\phi$, and $\bar{\phi}$ transforming as the $A_1$, $A_2$, and $\bar{A}_2$ representations, respectively, as well as $Z$ and $\bar{Z}$ triplets transforming as $T_1$ and $\bar{T}_1$. The (nonrenormalizable) superpotential is taken to be

$$W=\alpha S(-3\mu^2+\bar{Z}Z)+\beta\phi\bar{Z}Z+\gamma\bar{\phi}\bar{Z}Z+\frac{g}{3}\bar{Z}^3+\frac{Z^5}{5M^2}\ . \tag{2.10}$$

Written in terms of components, the above interactions read (see the Appendix)

$$\begin{aligned} W=&\alpha S(-\mu^2+\bar{Z}_1Z_1+\bar{Z}_2Z_2+\bar{Z}_3Z_3)\\ &+\beta\phi(\bar{Z}_1Z_1+\omega\bar{Z}_2Z_2+\omega^2\bar{Z}_3Z_3)\\ &+\gamma\bar{\phi}(\bar{Z}_1Z_1+\omega^2\bar{Z}_2Z_2+\omega\bar{Z}_3Z_3)+g(\bar{Z}_1\bar{Z}_2\bar{Z}_3)\\ &+(Z_1^5+Z_2^5+Z_3^5)/5M^2 \end{aligned} \tag{2.11}$$

(where $\omega\equiv e^{2i\pi/3}$), with several isolated supersymmetric minima; all have $\phi=\bar{\phi}=0$. One of the vacua takes the values

$$Z=\mu\delta\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \bar{Z}=\frac{\mu}{3\delta}\begin{bmatrix}1\\1\\1\end{bmatrix},\quad S=-\frac{g\mu}{9\alpha\delta^3}\ , \tag{2.12}$$

with

$$\delta=\left[\frac{gM^2}{27\mu^2}\right]^{1/8}\ .$$

Our second example has $\Delta(75)\times U(1)$ broken to $Z_5\times Z_5$ by giving a triplet a VEV in a single component. The toy model includes the following superfields that transform as irreducible representations under $\Delta(75)\times U(1)$, where the U(1) is gauged:

$$\begin{aligned} &S=(A_1)_0,\quad Z=(T_1)_1,\quad \bar{Z}=(\bar{T}_1)_{-1}\ ,\\ &R=(T_1)_{-2},\quad \bar{R}=(\bar{T}_1)_2\ . \end{aligned} \tag{2.13}$$

From these fields we construct the renormalizable superpotential

$$W=\alpha S(-\mu^2+\bar{Z}Z)-MR\bar{R}+\beta RZZ+\gamma\bar{R}\bar{Z}\bar{Z}\ . \tag{2.14}$$

In terms of component fields,

$$\begin{aligned} W=&\alpha S(-\mu^2+\bar{Z}_1Z_1+\text{c.p.})-M(\bar{R}_1R_1+\text{c.p.})\\ &+\beta(R_1Z_2Z_3+\text{c.p.})+\gamma(\bar{R}_1\bar{Z}_2\bar{Z}_3+\text{c.p.})\ , \end{aligned} \tag{2.15}$$

where c.p. stands for cyclic permutation of each triplet's indices (see the Appendix). Minimizing the scalar potential [including the $D$ term from the gauged U(1)] yields three families of supersymmetric vacua, including the isolated solution

$$S=R=\bar{R}=0,\quad Z=\bar{Z}=\begin{bmatrix}0\\0\\\mu\end{bmatrix}\ . \tag{2.16}$$

### C. Fermion mass texture

Flavor symmetry breaking can be communicated to the Yukawa couplings of the light fermions in two ways: either through the mixing of light and heavy fermions or through the Higgs potential. We have seen that in flavor unification the large top quark mass requires that the Higgs fields $H_u$ transform under flavor at short distances and have direct (unsuppressed) flavor symmetry-breaking VEV's. Keeping in mind that the successful GUT prediction for $\sin^2\theta_W$ assumes that there are only two Higgs doublets below the GUT scale, it is natural to suppose that flavor symmetry breaking occurs at the GUT scale or above and that all but these two Higgs doublets acquire large masses.

For example, suppose $H_u$ and $H_d$ are Higgs doublets that are both flavor triplets in the $\bar{T}_2$ and $\bar{T}_1$ representations of $\Delta(75)$, respectively, and that they couple to the left-chiral superfield triplets $Z=T_3$ and $\bar{Z}=\bar{T}_3$, which are gauge singlets. There are two couplings:

$$\begin{aligned} W&=\lambda ZH_uH_d+\lambda'\bar{Z}H_uH_d\\ &=\lambda(Z_1H_{u2}H_{d1}+\text{c.p.})+\lambda'(\bar{Z}_1H_{u3}H_{d1}+\text{c.p.})\ . \end{aligned} \tag{2.17}$$

If $Z$ and $\bar{Z}$ get the VEV's $\{\mu,0,0\}$ and $\{0,\mu,0\}$, respectively, where $\mu$ is some very heavy scale, then only the Higgs doublets $H_{u3}$ and $H_{d3}$ remain light and are able to eventually develop SU(2)×U(1) breaking VEV's. What has happened is that $\Delta(75)\times U(1)_{PQ}$ has been broken down to a diagonal $Z_5$, where $U(1)_{PQ}$ is the Peccei-Quinn symmetry in the interactions (2.17). The three components of both of the Higgs doublets carry $Z_5$ charges that allow two of the Higgs flavors to pair up and become heavy, while protecting the third.

We now incorporate these ideas into a toy model based on $\Delta(75)\times U(1)$ that leads to an interesting fermion mass hierarchy, ignoring gauge interactions for the moment. The "matter" fields are

$$F=(T_1)_1,\quad \psi=(\bar{T}_4)_1,\quad \bar{\psi}=(T_4)_1\ ,$$

where $F$ will play the role of three families of quarks and leptons, while $\psi$ and $\bar{\psi}$ are three vectorlike exotic families that will become heavy when the U(1) is broken. This occurs at a scale $xM$ when the singlet field $S$ develops a VEV:

$$S=(A_1)_{-2}=M\ .$$

At a somewhat lower scale, $\Delta(75)$ is broken, and we assume that this is due to the fields

$$X=(\bar{T}_3)_{-2}=xM\begin{bmatrix}1\\1\\1\end{bmatrix},\quad Y=(\bar{T}_1)_{-2}=yM\begin{bmatrix}1\\1\\1\end{bmatrix}\ ,$$

where $x$ and $y$ are small numbers. The fermions $F$ only get a mass when the "Higgs" field $H$ gets a VEV, and we assume that

$$H=(\bar{T}_2)_{-2}=\begin{bmatrix}0\\0\\v\end{bmatrix}\ ,$$

where $v\ll M$ is the "weak scale," envisaging a mecha-


FIG. 1. Leading supergraph contributions to the effective Yukawa coupling of the $F$ superfield in Eq. (2.19). The internal dotted lines indicate $\psi$ and $\bar{\psi}$ superfields with mass $M$. The unlabeled external lines are the light fermions $\mathcal{F}$.

nism such as described above that renders all but the third family component of $H$ heavy at the scale $xM$.

The most general renormalizable superpotential $W_m$ describing the interactions of the matter fields with $S$, $X$, $Y$, and $H$ is given by

$$\begin{aligned} W_m &= S\bar{\psi}\psi + X\bar{\psi}F + Y\bar{\psi}\psi + H(FF+F\psi) \\ &= S(\bar{\psi}_1\psi_1) + X_1\bar{\psi}_3F_3 + Y_1\bar{\psi}_3\psi_2 \\ &\quad + H_3(F_3F_3 + F_2\psi_3) + \text{c.p.} \end{aligned} \tag{2.18}$$

(For simplicity, we have omitted coupling constants, assumed to all be of order 1.) At the scale $M$, the $\psi$ field gets a mass and is integrated out of the theory, giving rise to the effective theory

$$W_{\text{eff}} = Y_{ij}H_3F_iF_j \ . \tag{2.19}$$

The Yukawa coupling $Y_{ij}$ can be computed by summing the diagrams in Fig. 1, making use of the invariant tensors discussed in the Appendix. The result is

$$Y_{ij} \sim \begin{bmatrix} 0 & xy^2 & 0 \\ xy^2 & xy & x \\ 0 & x & 1 \end{bmatrix} . \tag{2.20}$$

In addition, there are wave function renormalization graphs which give effective $D$ terms which eliminate the zeros in the above matrix, but they are negligible: The $\{13\}$ and $\{31\}$ entries in $Y_{ij}$ receive $O(|x|^2y^*)$ contributions, while the $\{11\}$ entry is $O(|x|^4y^{*2})$. $Y_{ij}$ exhibits an obvious hierarchical structure, and with $x \sim y \sim \frac{1}{20}$, it could provide a reasonable description of the Yukawa coupling matrix of the up-type quarks at the GUT scale [4]. In the next section, we incorporate this toy model into SO(10) and SU(5) grand unified theories.

## III. SUPERSYMMETRIC SO(10)$\times\Delta(75)$ GUT

In this section we show how to use non-Abelian discrete flavor symmetries to construct a GUT in which the gauge and flavor symmetries are separately unified. In particular, we show how to incorporate the toy model (2.18) into an SO(10) grand unified theory. To get realistic quark masses, it is necessary that the $Y_D$ Yukawa coupling of the down quark matrix look quite different from $Y_U$; we achieve this by having the Higgs fields $H_u$ and $H_d$ transform as different flavor representations. The representations are chosen so that (i) down-type quarks get masses at higher order in symmetry breaking, explaining the small $b/t$ mass ratio without requiring unnaturally large $\tan\beta$, and (ii) $\{22\}$ and $\{12\}$ entries of the down mass matrix are susceptible to large corrections from higher dimension operators which arise from Planck scale physics, accounting for $m_s/m_b \gg m_c/m_t$ and the large Cabibbo angle.

### A. Fields and interactions

The model we offer as an example is an SO(10)$\times\Delta(75)$ supersymmetric GUT, where $\Delta(75)$ is the flavor group.[3] This example is an extension of the toy model (2.18), containing both "matter superfields" which do not get VEV's and "Higgs superfields" which do. The matter fields consist of three ordinary chiral families,

$$F = (16, T_1) \ , \tag{3.1}$$

as well as exotic fields,

$$\begin{aligned} &\psi = (16, \bar{T}_4), \quad \bar{\psi} = (\overline{16}, T_4) \ , \\ &\chi = (10, \bar{T}_2), \quad \bar{\chi} = (10, T_2) \ . \end{aligned} \tag{3.2}$$

There are several fields associated with symmetry breaking. To break SO(10) down to SU(3)$\times$SU(2)$\times$U(1) at $M_{\text{GUT}} \simeq 10^{16}$ GeV in the most economical fashion requires both a **45** and a **16** of Higgs fields, and we include a conjugate partner for the latter. These fields are assumed to come in $\Delta(75)$ triplets:

$$\Sigma = (45, \bar{T}_4), \quad \Omega = (16, T_2), \quad \bar{\Omega} = (\overline{16}, \bar{T}_2) \ . \tag{3.3}$$

There are also gauge singlets which get VEV's at a similar scale: namely,

$$X = (1, \bar{T}_3), \quad Y = (1, \bar{T}_1), \quad Z = (1, T_2) \ . \tag{3.4}$$

Finally, there are singlet fields $S$ and $S'$ which are invariant under both SO(10) and $\Delta(75)$; their VEV's are responsible for the masses of the vectorlike fermion families $\psi$ and $\chi$ and occur over an order of magnitude above $M_{\text{GUT}}$.

To break the weak interactions, we require a **10** of Higgs fields; we will take three families of these Higgs fields as well. In order to construct a model without the fine-tuning problems associated with large $\tan\beta = \langle H_u/H_d \rangle$ [14], we have the up and down Higgs doublets reside in different **10**'s:

$$H_u = (10, \bar{T}_2), \quad H_d = (10, \bar{T}_1) \ . \tag{3.5}$$

As we will show below, the flavor quantum numbers of $H_d$ are chosen so that the down-type quarks have naturally suppressed Yukawa couplings.

SO(10)$\times\Delta(75)$ symmetry allows us to write down the renormalizable superpotential

$$\begin{aligned} W_m &= S\bar{\psi}\psi + S'\bar{\chi}\chi + X\bar{\psi}F + Y\bar{\psi}\psi + \chi[FF+F\psi] \\ &\quad + H_u[FF+F\psi] + H_d\bar{\chi}Y \ . \end{aligned} \tag{3.6}$$

For notational simplicity we have not indicated coupling constants for these operators, which are all assumed to be of order 1. Note that we have omitted an $S\bar{\chi}H_u$ opera-

[3] SO(10) GUT's have been discussed extensively in the literature. See [12] and, for recent references, [13].

tor, which can be done by choosing suitable definitions of the $\chi$ and $H_u$ fields, which have the same quantum numbers. Other operators allowed by SO(10)$\times\Delta(75)$ but absent from (3.6), such as $M_P\bar{\psi}\psi$, operators involving $Z$, $\Sigma$, and $\Omega$, etc., may be naturally excluded by imposing an additional U(1) or $Z_N$ symmetry to the theory which commutes with flavor and has no SO(10) anomalies. The choices of charges under this symmetry are not unique, and in fact the symmetry can be either an $R$ symmetry or ordinary. It is the spontaneous violation of this Abelian symmetry by $\langle S\rangle$ and $\langle S'\rangle$ that determines the masses of the heavy fermions $\psi$ and $\chi$.

Although the fields $\Sigma$, $Z$, and $\Omega$ do not have renormalizable couplings to the matter fields $F$, $\psi$, and $\chi$, they will interact through operators of dimension 5 and higher suppressed by powers of $M_P$. By means of the same Abelian symmetry controlling operators in the renormalizable sector of the theory, the allowed dimension-5 operators can be restricted to

$$W_{\rm grav}=\frac{1}{M_P}[H_dFFZ+\Sigma H_dFF+FF\bar{\Omega}\bar{\Omega}]\ . \tag{3.7}$$

As we will show below, the first two operators give important contributions to the down quark mass matrix, while the third operator is responsible for giving an interesting pattern of neutrino masses. Furthermore, in an SU(5) version of this model, the second operator can explain the ratio of down quark masses to charged lepton masses in the manner of Georgi and Jarlskog [15].

In order to generate realistic masses for the quarks and leptons, it is necessary to make certain assumptions about the symmetry-breaking pattern of the fields that get VEV's. We make the following assumptions, along the lines of our discussion of symmetry breaking in the previous section.

(1) The $S$ and $S'$ fields get VEV's at a scale which is about $(20-50)M_{\rm GUT}$, giving large masses to the $\psi$ and $\chi$ fields.

(2) The $X$, $Y$, and $Z$ fields get VEV's on the order of $M_{\rm GUT}$ in each component, inducing mass mixing between the heavy fermions $\psi,\chi$ and the light fermions $F$.

(3) SO(10) is broken to SU(3)$\times$SU(2)$\times$U(1) at the GUT scale by VEV's of the $\Omega$, $\bar{\Omega}$, and $\Sigma$ fields. We assume that each flavor component of the $\bar{\Omega}$ and at least the second flavor component of $\Sigma$ develop VEV's.

(4) Of the $H_u$ and $H_d$ triplets, only the $Y=-\frac{1}{2}$ weak doublet from $(H_u)_3$ and the $Y=+\frac{1}{2}$ weak doublet from $(H_d)_3$ remain lighter than $M_{\rm GUT}$ and develop SU(2)$\times$U(1) breaking VEV's.

The reason we take the flavor symmetry-breaking scale to be so high is dictated by the desire to keep interactions perturbative up to scales near the Planck mass. This is a generic feature of models of flavor unification where masses arise through mixing with heavy fermions: Such theories will have at least an extra set of fermion families as well as their mirrors, which, with the Higgs fields, render the gauge theory asymptotically unfree above the flavor unification scale. Thus the scale of flavor physics is forced to lie within a few decades of the Planck scale. Furthermore, it is interesting to note that gauge interactions are often strong very near the scale where quantum gravity is expected to be relevant.

### B. Quark masses

The effective quark Yukawa couplings are generated in this model when the $\psi$ and $\chi$ fields are integrated out of the theory at the scales $\langle S\rangle$ and $\langle S'\rangle$, taken to lie above $M_{\rm GUT}$, and the symmetry-breaking fields $X$, $Y$, $Z$, and $\Sigma$ acquire their VEV's. The diagrams arising from the renormalizable interactions (3.6) that contribute to an effective superpotential are shown in Fig. 2. Denoting

$$\langle X/S\rangle\equiv x,\quad \langle Y/S\rangle\equiv y,\quad \langle Y/S'\rangle\equiv y'\ ,$$

and ignoring both the order 1 coefficients in (3.6), the effective Yukawa couplings generated from these diagrams are

$$Y_u\sim\begin{bmatrix}0 & xy^2 & 0\\ xy^2 & xy & x\\ 0 & x & 1\end{bmatrix},\quad Y_d\sim y'\begin{bmatrix}0 & xy^2 & 0\\ xy^2 & xy & x\\ 0 & x & 1\end{bmatrix}, \tag{3.8}$$

where $Y_u$ and $Y_d$ are the coefficients of the effective operators $H_uFF$ and $H_dFF$, respectively. One sees that there is a natural hierarchical structure to the masses and that down-type quarks are automatically a factor of $y'$ more weakly coupled to the Higgs doublet than are up-type quarks. The two matrices are not simply proportional to each other [due to the omitted order 1 coefficients of (3.6)], so that there are nonzero mixing angles, although there may be partial cancellations leading to a small $V_{cb}$.

Additional important contributions to $Y_u$ and $Y_d$ come from the dimension-5 operators (3.7), which enter the effective Yukawa couplings through the diagrams pictured in Fig. 3. The first two graphs in Fig. 3 contribute to the $d$ and $s$ quark masses, as well as the Cabibbo angle. Denoting

$$\delta_z\equiv\langle Z/\Lambda\rangle,\quad \delta_\Sigma\equiv\langle\Sigma/M_p\rangle\ ,$$

Eq. (3.8) is modified to read

$$Y_u\sim\begin{bmatrix}0 & xy^2 & 0\\ xy^2 & xy & x\\ 0 & x & 1\end{bmatrix},\quad Y_d\sim\begin{bmatrix}0 & \delta_z & 0\\ \delta_z & \delta_\Sigma & xy'\\ 0 & xy' & y'\end{bmatrix} \tag{3.9}$$

for the Yukawa couplings at the GUT scale. We have only given the leading contributions to each entry and ig-

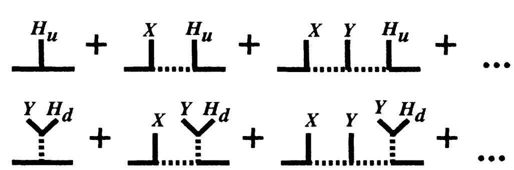

FIG. 2. Leading supergraph contributions to quark and lepton Yukawa couplings. The internal lines indicate $\psi$, $\bar{\psi}$, $\chi$, and $\bar{\chi}$ superfields. The unlabeled external lines are the light fermions $F$. The top row of diagrams contributes to $Y_u$, while the bottom row contributes to $Y_d$.

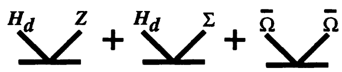


FIG. 3. Supergraphs involving the dimension-5 operators [Eq. (3.7)] contributing to quark and lepton Yukawa couplings.

nore the negligible contributions from wave function renormalization to the $\{13\}$, $\{31\}$, and $\{11\}$ entries. Taking scaling effects into account, these matrices can lead to realistic quark masses for the values

$$x \sim y \sim \tfrac{1}{20}, \quad y' \sim \tfrac{1}{50}$$

and imply

$$\tan\beta \simeq 1$$

for a top quark mass $m_t \simeq 160$ GeV. This fit assumes that the couplings in $W_m$ [Eq. (3.6)] are all of order 1 and work best if the couplings in $W_{\rm grav}$ [Eq. (3.7)] are actually $\simeq 0.5$ (i.e., so that the characteristic scale of nonrenormalizable gravitational interactions is $2M_P$).

### C. Lepton masses

The third diagram in Fig. 3 gives the right-handed neutrino a Majorana mass

$$M_\nu \sim \frac{\langle \bar\Omega \rangle^2}{M_P} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} , \tag{3.10}$$

where the entries denoted as "1" are to be understood as of order 1. By identifying the $B-L$ breaking scale with the GUT scale, the fact that $F$ couples to $\bar\Omega$ only through a dimension-5 operator naturally predicts a Majorana mass of $M_{\rm GUT}^2/M_P$. The seesaw mechanism [16] then leads to a $\tau$ neutrino mass of roughly $M_P/(G_F M_{\rm GUT}^2)$, where $G_F$ is the Fermi constant, which gives rise to a mass hierarchy for neutrinos that is of interest for both dark matter and neutrino oscillations.

The charged lepton masses do not work in the SO(10) model described above, but do in a similar SU(5) version, where $F \to \bar{5} + 10 + 1$, $H_d \to \bar{5}$, $H_u \to 5$, $\Sigma \to 24$, and so forth. In this model the $\{22\}$ entry in $Y_d$ in Eq. (3.9) involves SU(5) breaking through the coupling to the $\bar{5} \otimes \overline{45}$ in $H_d \Sigma = \bar{5} \times 24$. If the coupling is primarily in the $\overline{45}$ channel, then the mass matrices are similar to the Georgi-Jarlskog form and yield the successful GUT-scale mass relations [15]

$$\frac{m_b}{m_\tau} \simeq 1, \quad \frac{m_s}{m_\mu} \simeq \frac{1}{3}, \quad \frac{m_d}{m_e} \simeq 3 . \tag{3.11}$$

We do not bother writing down the SU(5) model, since it is in almost every respect identical to the SO(10) version described above. The reason why the Georgi-Jarlskog mechanism does not work in the SO(10) version of the model is that $H_d \Sigma = \mathbf{10} \times \mathbf{45}$ can only couple to $FF$ as a $\mathbf{10}$, which does not split the down quark from lepton masses.

## IV. FLAVOR-CHANGING NEUTRAL CURRENTS

In the standard model, FCNC's must proceed through dimension-6 operators, and so experiments are insensitive to physics above $\sim 1000$ TeV. In contrast, FCNC's enter supersymmetry through dimension-2 squark mass matrices and are sensitive to physics at very short distances [17]. Limits on FCNC's from the neutral $K$ and $B$ mesons require that the squarks must be mass eigenstates in very nearly the same flavor basis as are the quarks [18,19]. To discuss these constraints, we use the notation and analysis from [19].

The $6 \times 6$ squark mass-squared matrix may be written as

$$\tilde{M}^{q2} = \begin{bmatrix} \tilde{M}^{q2}_{LL} & \tilde{M}^{q2}_{LR} \\ \tilde{M}^{q2\dagger}_{LR} & \tilde{M}^{q2}_{RR} \end{bmatrix} , \tag{4.1}$$

where $L$ and $R$ refer to the chirality of the associated quarks. Assuming that the $SU(2)\times U(1)$ violating $LR$ components of $\tilde{M}^{q2}$ are smaller than the diagonal components, then FCNC experiments limit the quantities

$$\delta^q_{AB} = \frac{V^q_A \tilde{M}^{q2}_{AB} V^{q\dagger}_B}{\tilde{m}^2} , \tag{4.2}$$

where $V^u_{L,R}$ and $V^d_{L,R}$ are the unitary matrices which diagonalize the $u$ and $d$ quark mass matrices. The $[\delta^d_{AB}]_{12}$'s are constrained to be less than $few \times 10^{-3}$, while the $[\delta^d_{AB}]_{13}$'s and $[\delta^u_{AB}]_{12}$'s are constrained to be smaller than $few \times 10^{-2}$. Various explanations of how these small numbers arise naturally have been proposed, such as squark universality and horizontal flavor symmetries. Universality, as invoked in minimal supergravity [20], is quite unnatural, since there is no reason why the physics that gives diverse Yukawa couplings to the different families would not also give diverse squark masses, but models have been proposed where squark universality is a natural consequence of their identical gauge interactions [21]. Explanations for small FCNC's based on horizontal symmetries [19,22] simply ensure that the inevitable breaking of flavor symmetry in the squark sector is small enough for symmetry reasons not to have been observed. The model we are describing here falls into this second category.

Our $\Delta(75)$ model has small FCNC effects due to the non-Abelian flavor symmetry, as long as the order parameter for supersymmetry (SUSY) breaking is flavor neutral. First, consider the $LR$ sector of the squark mass matrix. One contribution is proportional to the Yukawa coupling and is diagonal in the quark mass eigenstate basis. The other contribution arises through the soft SUSY-violating trilinear couplings of the squarks to the Higgs doublets. These couplings are assumed to arise from a dimension-5 superpotential $W' \sim W\phi/M_P$, where $\phi$ is a chiral superfield whose $F$ component breaks supersymmetry at an intermediate scale and the tilde means that there is a one-to-one correspondence between operators, although the order 1 coupling constants are not assumed to be the same. This implies that at low energy the effective trilinear couplings are

$$\tilde{m}[\tilde{Y}_u \tilde{Q} H_u \tilde{u}^c + \tilde{Y}_d \tilde{Q} H_d \tilde{d}^c] , \tag{4.3}$$

where the $\tilde{Y}$ matrices have the same texture as the Yukawa coupling matrices. Thus, in the flavor basis where the quark masses are diagonal, the $\{ij\}$ component of $\tilde{M}^{q2}_{LR}$ is at most of order $\tilde{m}\sqrt{m_i m_j}$, where $m_i$ are the corresponding quark masses, and so their contributions to the constrained parameters $\delta^q_{LR}$ are very small.

The $LL$ and $RR$ parts of the squark mass matrix also get two contributions. The first is proportional to $Y^\dagger Y$ and is diagonal in the quark mass eigenstate basis. The second arises from the dimension-6 $D$ terms:

$$\frac{\phi^*\phi}{M_P^2}[c_1 F^* F + c_2 \psi^* \psi + c_3 \bar{\psi}^* \bar{\psi} + \cdots ]_D , \tag{4.4}$$

where the $\Delta(75)$ symmetry dictates that there is universality in the coupling of the three families. These terms alone give contributions to the $LL$ and $RR$ components of $\tilde{M}^{q2}$ which are proportional to the unit matrix and hence diagonal in any basis. FCNC effects can exist in dimension-8 operators arising directly from the Planck scale

$$\left.\frac{\phi^*\phi S^*(F^* X F)}{M_P^4}\right|_D , \tag{4.5}$$

inducing off-diagonal contributions to $\delta^q_{LL,RR}$ of order $\langle S \rangle M_{\rm GUT}/M_P^2 \simeq 2\times 10^{-5}$. Larger contributions arise from dimension-8 operators generated by integrating out the heavy $\psi$ field as in Fig. 4, leading to the operator

$$\left.\frac{\phi^*\phi(F^* X X^* F)}{M_P^2 \langle S \rangle^2}\right|_D . \tag{4.6}$$

Since $\langle X/S \rangle \equiv x \simeq \frac{1}{20}$, this operator would appear to contribute to FCNC's at the $3\times 10^{-3}$ level. However, $\langle X^* X \rangle$ in the above operator is flavor diagonal in the $\Delta(75)$ basis we have been using and therefore gives rise to off-diagonal contributions in $\delta^q_{LL,RR}$ of order $x^2\theta$, where $\theta$ is the relevant mixing angle. In the kaon system, for example, this gives $\delta^d \simeq \theta_C/400 = 5\times 10^{-4}$. Thus FCNC's in a model such as this one are below current limits, but only by about an order of magnitude, even though flavor physics occurs up at the GUT scale.

It is interesting to note that FCNC effects increase in supersymmetric models as the flavor symmetry-breaking scale gets closer to the Planck scale. Thus it is conceivable that improved searches for FCNC's could in fact probe physics in the region between the GUT and Planck scales. This is peculiar to models such as supersymmetry in which Glashow-Iliopoulos-Maiani (GIM) violation can proceed through soft operators.

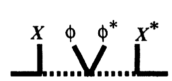


FIG. 4. Supergraph contributing to flavor-changing squark masses. The dotted line is the $\psi/\bar{\psi}$ field, and $\phi$ is the field giving rise to supersymmetry breaking.

## V. CONCLUSIONS

In this paper we are advocating using non-Abelian discrete flavor symmetries for unifying flavor at short distances. The example we have given, a supersymmetric GUT with a $\Delta(75)$ flavor symmetry, can account for the diversity of quark and lepton masses and mixings without small fundamental parameters, other than the hierarchy of the mass scales $M_p, M_{\rm GUT}$ and an intermediate scale associated with the masses of vectorlike families. This particular model predicts mixing angles to be approximately equal to their observed values, as well as $\tan\beta \simeq 3$. The model also predicts a seesaw mechanism for neutrino masses, with the $\tau$ neutrino mass given approximately by $M_p/G_F M^2_{\rm GUT} \simeq 10$ eV. The two lighter neutrino masses scale like the up-type quark masses squared (at the GUT scale) and are much lighter.

We believe that our $\Delta(75)$ model exhibits a number of features that will be generic in flavor unification models that do away with an explicit fermion mass hierarchy put in by hand. These include the following.

(i) Because of the extra families added in such schemes, the gauge group $\beta$ function changes sign at short distances. This requires that flavor symmetry breaking occur near the GUT scale or higher or that there are larger gauge groups at low energies than usually envisioned. Typically, gauge interactions are strong near $M_P$ in these models. It is intriguing that a model of flavor physics favors strongly interacting physics at the Planck scale.

(ii) With flavor symmetry breaking occurring at a high scale, the light quark masses and mixings are sensitive to operators suppressed by powers of $M_P$. In the model described here, the relatively large Cabibbo angle is due to a dimension-5 operator.

(iii) Flavor-changing neutral currents are typically suppressed enough to be acceptable in such models, as a result of the non-Abelian flavor symmetry. However, the proximity of the flavor symmetry-breaking scale to $M_P$ means that FCNC effects from these ultrashort distance scales could be detectable.

(iv) Because of supersymmetry, the most generic operators consistent with flavor symmetry are *not* generated when heavy particles are integrated out of the theory. This suggests that an effective Lagrangian approach is no substitute for a model of short distance flavor physics.

In models with the short distance flavor democracy we are advocating, Higgs fields typically carry family quantum numbers, and understanding symmetry breaking becomes a more pressing issue. An important problem sidestepped in this paper has been the doublet-triplet splitting of the Higgs doublet, which now becomes entangled with the problem of flavor. Other issues that remain to be addressed in detail are neutrino masses and $CP$ violation.

## ACKNOWLEDGMENTS

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### APPENDIX: TRIPLET DECOMPOSITION IN Δ(75)

Here we give the decomposition of the products of triplet representations shown in Table II, consistent with the basis defined in Eq. (2.9). As discussed in Sec. II, the generator $\hat{E}_{00}$ has the same representation matrix $D_R(E_{00})$ for all of the triplet representations $R$:

$$D_R(E_{00})=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\quad R=\{T_1,\ldots,\bar{T}_4\}\ . \tag{A1}$$

The representation matrices corresponding to the generator $\hat{A}_{10}$ are given by $D_1(\hat{A}_{10})=A_{10}$ and

$$D_2(\hat{A}_{10})=A_{20},\quad D_3(\hat{A}_{10})=A_{13},\quad D_4(\hat{A}_{10})=A_{21}\ , \tag{A2}$$

where $D_n$ is the representation matrix for the triplet $T_n$ and the $A_{pq}$ matrices are defined in Eq. (2.4). The above representations follow from the conventions (2.9). This is enough information to determine all of the invariant tensors of the group.

From Table II one sees that $T_n\otimes\bar{T}_n$ always contains all three singlet representations, for $n=1,\ldots,4$. Writing $T_n$ as $\{x,y,z\}$, one finds these singlets to be

$$\begin{aligned}T_n\otimes\bar{T}_n|_{A_1}&=x\bar{x}+y\bar{y}+z\bar{z}\ ,\\T_n\otimes\bar{T}_n|_{A_2}&=x\bar{x}+\omega y\bar{y}+\omega^2 z\bar{z}\ ,\\T_n\otimes\bar{T}_n|_{\bar{A}_2}&=x\bar{x}+\omega^2 y\bar{y}+\omega z\bar{z}\ ,\end{aligned} \tag{A3}$$

where $\omega\equiv e^{2i\pi/3}$.

For the decomposition of a product of two triplets into a third triplet, it suffices to give the structure of all of the three-triplet invariants. Because of Eq. (A1), all invariants of three triplets $(ABC)$ can be specified by three numbers $\{ijk\}$ signifying that $(ABC)=A_iB_jC_k+$c.p., where c.p. stands for cyclic permutation of each representation's index. For example, $(ABC)=\{112\}$ denotes that $(A_1B_1C_2+A_2B_2C_3+A_3B_3C_1)$ is a Δ(75) singlet. Table II reveals that the product of three triplets of a given representation always contains two invariants. These are given by

$$(T_nT_nT_n)=\{123\}+\{213\}\ . \tag{A4}$$

Thus, for example, if one wants to find the $\bar{T}_1$'s contained in $T_1\otimes T_1$, one finds them to be

$$T_1\otimes T_1'|_{\bar{T}_1}=\begin{bmatrix}yz'\\zx'\\xy'\end{bmatrix},\ \begin{bmatrix}zy'\\xz'\\yx'\end{bmatrix} \tag{A5}$$

or any linear combination of the two. There remain 16 independent invariants with 3 triplets, and their structure is found to be

$$\begin{aligned}&\{111\}:(11\bar{2})(122),(334),(344)\ ,\\&\{112\}:(1\bar{3}2),(14\bar{3}),(\bar{2}34),(\bar{2}41)\ ,\\&\{113\}:(132),(1\bar{4}3),(234),(24\bar{1})\ ,\\&\{123\}:(3\bar{1}1),(\bar{4}14),(4\bar{2}2),(2\bar{3}3)\ .\end{aligned} \tag{A6}$$

Thus, for example, if one wants to find the invariant formed from $\bar{T}_2\otimes T_4\otimes T_1$, one notes that $(\bar{T}_2T_4T_1)$ is an invariant of the $\{112\}$ type, so that

$$\bar{T}_2\otimes T_4\otimes T_1|_{A_1}=\bar{a}\alpha y+\bar{b}\beta z+\bar{c}\gamma x\ , \tag{A7}$$

where we have taken $T_1=\{x,y,z\}$, $\bar{T}_2=\{\bar{a},\bar{b},\bar{c}\}$, and $T_4=\{\alpha,\beta,\gamma\}$. Similarly, if one wants to find the $T_4$ contained in $\bar{T}_1\otimes T_2$, the same $\{112\}$ invariant yields

$$\bar{T}_1\otimes T_2|_{T_4}=\begin{bmatrix}\bar{y}a\\\bar{z}b\\\bar{x}c\end{bmatrix}\ . \tag{A8}$$

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