Multipole amplitudes in the one-photon radiative transitions of the singlet D state of charmonium

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We calculate in potential models the angular momentum helicity amplitudes and from them the radiation multipole amplitudes of the one-photon radiative transitions of the charmonium singlet D state, namely, $1\,{}^1D_2 \to 1\,{}^1P_1+\gamma$ and $1\,{}^1D_2 \to 1\,{}^3S_1+\gamma$. The $M2$ and the $E3$ multipole amplitudes in the parity-changing transition ${}^1D_2 \to {}^1P_1$ are of order v^2/c^2 compared to the $E1$ amplitude and are independent of the specific potential we use. They only depend on k/m where k is the photon energy and m is the mass of the quark. On the other hand the $M1$, $E2$, and $M3$ multipole amplitudes in the parity-conserving transition ${}^1D_2 \to {}^3S_1$ are all of order v^2/c^2 and depend on radial integrals which could depend on the potentials. We numerically evaluate these multipole amplitudes in the potential model of Gupta, Radford, and Repko and find that the $E2$ amplitude is the most dominant one. We also calculate the decay rate for the transition $1\,{}^1D_2 \to 1\,{}^3S_1+\gamma$ and find it to be about 63 keV. So if the $1\,{}^1D_2$ state of charmonium is narrow, as expected in potential models, this transition will have a significant branching ratio, although the most dominant transition is probably $1\,{}^1D_2 \to 1\,{}^1P_1+\gamma$.

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I. INTRODUCTION

The singlet D state of charmonium, namely, the ${}^1D_2$, is interesting in several respects. First, even though its mass is supposed to be above the charm threshold, it is expected to have a narrow width since the strong transition ${}^1D_2 \to D+\bar{D}$ is forbidden by conservation of parity and the predicted mass [1] of ${}^1D_2$ is such that the strong transition ${}^1D_2 \to D+\bar{D}^*$ or $D^*+\bar{D}$ is energetically forbidden. The ${}^1D_2$ states can be directly formed in $\bar{p}p$ collisions now being studied at Fermilab by the E760 group [2]. The predominant radiative decay modes of the state are ${}^1D_2 \to {}^1P_1+\gamma$, which has a width of about 600–700 keV [1], and ${}^1D_2 \to 1\,{}^3S_1+\gamma$, which as we shall see has a width of about 60 keV. From angular momentum and parity conservation, the radiation multipole amplitudes involved in ${}^1D_2 \to {}^1P_1+\gamma$ are $E1$, $M2$, and $E3$ and those of ${}^1D_2 \to 1\,{}^3S_1+\gamma$ are $M1$, $E2$, and $M3$. In a recent paper [3], we have shown that all the multipole amplitudes in the transition ${}^1D_2 \to {}^1P_1+\gamma$ can be extracted from the angular distribution of the two photons in the cascade process

$$\bar{p}p \to {}^1D_2 \to {}^1P_1+\gamma_1 \to {}^1S_0+\gamma_1+\gamma_2 .$$

We have also shown [3] that the multipole amplitudes of the transition ${}^1D_2 \to 1\,{}^3S_1+\gamma$ can be extracted from the angular distribution of the photon and electron in the cascade process

$$\bar{p}p \to {}^1D_2 \to 1\,{}^3S_1+\gamma \to e^+ + e^- + \gamma .$$

In this paper we calculate these multipole amplitudes in potential models. In potential models the $M2$ and $E3$ amplitudes in the ${}^1D_2 \to {}^1P_1$ transition are of order v^2/c^2 or k/m compared to the $E1$ amplitude. In fact, we will find that the $M2$ and $E3$ amplitudes depend only on k/m, where k is the energy of the photon and m is the mass of the quark, and not on the wave functions or the specific nature of the potential. On the other hand, all the multipole amplitudes $M1$, $E2$, and $M3$ are of order v^2/c^2. In other words, in the nonrelativistic limit this transition will not occur. Also, these multipole amplitudes depend on certain integrals involving radial wave functions. Hence they could depend on the specific form of the potential. We numerically evaluate these multipole amplitudes in the potential model of Gupta, Radford, and Repko [4] (GRR). We find that the $E2$ amplitude is about 12 times as large as the $M1$ amplitude and about 4 times the size of the $M3$ amplitude. We also derive an expression for the total decay rate of ${}^1D_2 \to 1\,{}^3S_1+\gamma$ including the contribution from all the multipole amplitudes. This should be contrasted with the expression we derived in Ref. [1] where we only took into account the $M1$ contribution to the decay rate. It turns out that the $E2$ and $M3$ contributions to the decay rate are quite significant, and these contributions change the previously [1] calculated decay rate of 2.1 keV into 63 keV.

The format of the rest of the paper is as follows. In Sec. II we present our expressions for the parity-changing and -conserving one-photon transition amplitudes of quarkonium to relative order $1/c^2$ in an arbitrary potential model. Using the results of Sec. II, in Sec. III we derive expressions for the angular momentum helicity amplitudes and from there the expressions for the $E1$, $M2$, and $E3$ multipole amplitudes for the transition $1\,{}^1D_2 \to 1\,{}^1P_1+\gamma$. In Sec. IV we derive the angular momentum helicity amplitudes and the $M1$, $E2$, and $M3$ multipole amplitudes for the parity-conserving one-photon transition $1{}^1D_2 \to 1\,{}^3S_1+\gamma$. We also give the expression for the total decay rate of this transition, including contributions from all the multipole amplitudes. In Sec. V we make numerical estimates of the multipole amplitudes and of the total decay rate of $1\,{}^1D_2 \to 1\,{}^3S_1+\gamma$ using the potential model of Gupta, Radford, and Repko

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[4]. Finally, the Sec. VI we make some concluding remarks.

II. PARITY-VIOLATING AND -CONSERVING ONE-PHOTON TRANSITION AMPLITUDES OF QUARKONIUM IN AN ARBITRARY POTENTIAL MODEL

Let H_0 be the Hamiltonian of the isolated quarkonium state and H_I the interaction Hamiltonian which represents the interaction of the quarkonium with the quantized radiation field. To relative order $1/c^2$, the interaction Hamiltonian H_I can be written as

$$H_I=i\sum_{j=1}^{2}\frac{e_j}{2c}(\mathbf{A}_j\cdot[\mathbf{r}_j,H_0]+[\mathbf{r}_j,H_0]\cdot\mathbf{A}_j)-\sum_{j=1}^{2}\frac{e_j}{m_jc}\mathbf{s}_j\cdot\mathbf{B}_j-\sum_{j=1}^{2}\frac{e_j}{2m_j^2c^2}\mathbf{s}_j\cdot(\mathbf{E}_j\times\mathbf{p}_j)-i\sum_{j=1}^{2}\frac{e_j}{4m_j^2c^2}\mathbf{s}_j\cdot(\nabla_j\times\mathbf{E}_j)+\sum_{j=1}^{2}\frac{e_j}{4m_j^3c^3}[p_j^2,\mathbf{s}_j\cdot\mathbf{B}_j]_+ \ . \tag{1a}$$

In writing Eq. (1a), we assumed that the quarks have no anomalous magnetic moments. It is also understood that, in Eq. (1a),

$$A_j=A(\mathbf{r}_j,t) \tag{1b}$$

and so on.

In Eq. (1a), as well as throughout this paper, we put $\hbar=c=1$. For quarkonium, $e_1=-e_2=+e_q$ and $m_1=m_2=m$. The commutator term between $\mathbf{r}_j$ and H_0 includes all the relativistic correction terms in H_I except for some spin-dependent terms. The contributing spin-dependent terms are given by the terms after the first term in Eq. (1a). Next we express the interaction Hamiltonian in terms of the relativistic internal and center-of-mass variables of Krajcik and Foldy [5]. In terms of these variables, the Hamiltonian of the isolated quarkonium takes the form $H_0=\sqrt{h^2+P^2}$ expanded to relative order v^2/c^2, where h is the internal Hamiltonian and $\mathbf{P}$ is the total momentum of the quarkonium. Since h and $\mathbf{P}$ commute, the eigenstates of H_0 can now be written as

$$H_0|A\rangle=E_A|A\rangle \ , \tag{2a}$$

where

$$|A\rangle=|A\rangle_1\otimes|\mathbf{P}\rangle_{\text{c.m.}} \tag{2b}$$

and

$$E_A=\sqrt{(E_A^I)^2+P^2}\simeq E_A^I+\frac{P^2}{2E_A^I}-\frac{P^4}{8(E_A^I)^3} \ . \tag{3}$$

Let us assume that the initial state is an eigenstate of h with eigenvalue E_A^I and with total momentum equal to zero, whereas the final state after the photon emission is an eigenstate of h with eigenstate E_B^I and momentum equal to $-\mathbf{k}$, where $\mathbf{k}$ is the momentum of the emitted photon:

$$|i\rangle=|A\rangle_I\otimes|0\rangle_{\text{c.m.}}\otimes|0\rangle_{\text{ph}} \ , \tag{4}$$

$$|f\rangle=|B\rangle\otimes|\mathbf{k},\hat{\boldsymbol{\varepsilon}}_\alpha\rangle_{\text{ph}}=|B\rangle_I\otimes|-\mathbf{k}\rangle_{\text{c.m.}}\otimes|\mathbf{k},\hat{\boldsymbol{\varepsilon}}_\alpha\rangle_{\text{ph}} \ , \tag{5}$$

where $|\mathbf{k},\hat{\boldsymbol{\varepsilon}}_\alpha\rangle_{\text{ph}}$ is a single-photon state with momentum $\mathbf{k}$ and polarization vector $\hat{\boldsymbol{\varepsilon}}_\alpha$. If the system consisting of the quarkonium and radiation field is in state $|i\rangle$ at time $t=0$, the probability amplitude of finding it in state $|f\rangle$ at time t, in first order time-independent perturbation theory, can be written as

$$T(t)=-i\int_0^t\langle f|H_I(t')|i\rangle dt' \tag{6}$$

in the interaction picture. Next we write this transition amplitude in terms of the relativistic internal variables and separate the parity-odd and -even amplitudes. If T_o and T_e are the parity-odd and -even amplitudes, respectively, they are given by the expressions [6]

$$T_o(t)=\frac{1}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}\omega_{BA}^I\ {}_I\!\left\langle A\left|\hat{\boldsymbol{\varepsilon}}_\alpha\cdot\left\{e_q\mathbf{r}-\frac{ie_q}{4mk}[(\mathbf{k}\cdot\mathbf{r})^2\mathbf{p}-i(\mathbf{k}\cdot\mathbf{r})\mathbf{k}+2(\mathbf{k}\cdot\mathbf{r})(\mathbf{S}\times\mathbf{k})-k^2(\mathbf{S}\times\mathbf{r})]\right\}\right|B\right\rangle_I\times\int_0^t e^{i(\omega-\omega_{BA})t'}dt' \tag{7}$$

and

$$T_e(t)=\frac{1}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}{}_I\!\left\langle A\left|\frac{e_q}{m}(\mathbf{k}\times\hat{\boldsymbol{\varepsilon}}_\alpha)\cdot\mathbf{s}\left[1+\frac{k}{2m}-\frac{p^2}{2m^2}-\frac{1}{8}(\mathbf{k}\cdot\mathbf{r})^2\right]+ik\frac{e_q}{4m^2}(\mathbf{k}\cdot\mathbf{r})\hat{\boldsymbol{\varepsilon}}_\alpha\cdot(\mathbf{s}\times\mathbf{p})-\frac{e_q}{4m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}(\mathbf{k}\cdot\mathbf{r})\hat{\boldsymbol{\varepsilon}}_\alpha\cdot(\mathbf{s}\times\mathbf{r})+\frac{e_q}{2m^3}(\hat{\boldsymbol{\varepsilon}}_\alpha\cdot\mathbf{p})\mathbf{k}\cdot(\mathbf{s}\times\mathbf{p})\right|B\right\rangle_I\int_0^t e^{i(\omega-\omega_{BA})t'}dt' \ . \tag{8}$$

We assume box normalization in a box of volume V with periodic boundary conditions. In Eqs. (7) and (8), $\mathbf{r}$ and $\mathbf{p}$ are internal variables [5,6] defined as

$$\mathbf{r}=\lim_{\mathbf{P}\to 0}(\mathbf{r}_1-\mathbf{r}_2) \ , \tag{9}$$

$$\mathbf{p}=\lim_{\mathbf{P}\to 0}\mathbf{p}_1=-\lim_{\mathbf{P}\to 0}\mathbf{p}_2 \ , \tag{10}$$

where $\mathbf{P}$ is the total momentum. Also, in Eqs. (7) and (8), the variables ω or k represent the energy of the actual emitted photon and

$$\omega^I_{BA}=E^I_B-E^I_A \ , \tag{11}$$

whereas

$$\omega=\omega_{BA}=E_B-E_A \ . \tag{12}$$

The relation between E_A and E^I_A or E_B and E^I_B is given by Eq. (3). Furthermore,

$$\mathbf{S}=\mathbf{s}_1+\mathbf{s}_2 \tag{13}$$

and

$$\mathbf{s}=\mathbf{s}_1-\mathbf{s}_2 \ , \tag{14}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are the spin operators of the quark and antiquark in the c.m. frame where $\mathbf{P}=0$.

In Eq. (7), the first term $e_q\mathbf{r}$ is the dominant term which comes from the commutator term in Eq. (1a) where in the expansion of the vector potential $\mathbf{A}_j$ we replace $e^{i\mathbf{k}\cdot\mathbf{r}_j}$ by 1. The second and third terms also come from the commutator term, where the exponential term $e^{i\mathbf{k}\cdot\mathbf{r}_j}$ is expanded to second order in $(\mathbf{k}\cdot\mathbf{r}_j)$. These are the so-called finite size corrections. They are of relative order v^2/c^2 compared to the dominant term $e_q\mathbf{r}$ in the transition operator. The fourth and fifth terms in Eq. (7) come from the second and third terms in Eq. (1a).

In Eq. (8) we have only included terms proportional to $\mathbf{s}=\mathbf{s}_1-\mathbf{s}_2$ as they alone can connect the spin-singlet $^1D_2$ state and spin-triplet $^3S_1$ state. We also assumed that the interaction-dependent part of the Lorentz boost operator [5–7] $\mathbf{W}^{(1)}$ and the terms of relative order $1/c^2$ (called $h^{(1)}$) in the internal Hamiltonian are independent of $\mathbf{s}=\mathbf{s}_1-\mathbf{s}_2$. These assumptions are certainly valid in any potential model of quarkonium so far used. The only nonrelativistic term in the transition operator of Eq. (8) is the term involving 1 in the first term. Obviously, this term does not contribute between $1\,^1D_2$ and $1\,^3S_1$ states since the spatial wave functions are orthogonal in this case. All the other terms in the transition operator are of relative order v^2/c^2. So all the nonvanishing multipole amplitudes $M1$, $E2$, and $M3$ in the transition $1\,^1D_2\to1\,^3S_1+\gamma$ are of relative order v^2/c^2. This situation should be contrasted with that of $1\,^1D_2\to1\,^1P_1+\gamma$ where the dominant multipole amplitude is $E1$ since it survives even in the nonrelativistic limit, whereas the higher multipoles $M2$ and $E3$ are of relative order v^2/c^2. It is interesting to note that the last term on the right-hand side of Eq. (8) comes from the relativistic terms of order $1/c^2$ in the relativistic relations between the constituent and center-of-mass variables of Krajcik and Foldy [5].

In Secs. III and IV, we apply Eqs. (7) and (8) to calculate the helicity and multipole amplitudes in the transitions $1\,^1D_2\to1\,^1P_1+\gamma$ and $1\,^1D_2\to1\,^3S_1+\gamma$ of charmonium.

III. HELICITY AND MULTIPOLE AMPLITUDES IN THE TRANSITION $1\,^1D_2\to1\,^1P_1+\gamma$

Let us assume that the $^1D_2$ is formed at rest in the $\bar{p}p$ collisions. Even if it is not, we can always calculate everything in the $^1D_2$ rest frame and compare the results with the experiment through a Lorentz transformation. Let $^1P_1$ and the photon γ be emitted in the $+Z$ and $-Z$ directions, respectively. The component of the angular momentum of $^1D_2$ in the $+Z$ direction is called ν. The helicities of $^1P_1$ and γ are called σ and μ, respectively. By angular momentum conservation

$$\nu=\sigma-\mu \ . \tag{15}$$

The transition amplitude of $^1D_{2\nu}\to{}^1P_{1\sigma}+\gamma_\mu$ in this kinematical configuration is called the angular momentum helicity amplitude A^2_ν or simply A_ν of our previous paper [3]. In Ref. [3] we had shown, using parity invariance,

$$A_\nu=A_{-\nu}\quad(\nu=0,1,2) \ . \tag{16}$$

So there are three independent helicity amplitudes in this decay. We normalize them so that

$$\sum_{\nu=0}^{2}|A_\nu|^2=1 \ . \tag{17}$$

The relation between the helicity and multipole amplitudes a_k is given by [8,9]

$$A_\nu=\sum_{k=1}^{3}a_k\left[\frac{2k+1}{5}\right]^{1/2}\langle k1;1\nu-1|2\nu\rangle \ . \tag{18}$$

The coefficients of this transformation form a real orthogonal matrix so that

$$\sum_{\nu=0}^{2}|A_\nu|^2=\sum_{k=1}^{3}|a_k|^2=1 \ . \tag{19}$$

a_1 is the $E1$ amplitude, a_2 the $M2$ amplitude, and a_3 the $E3$ amplitude.

In order to calculate the angular momentum helicity amplitude A_2, we will consider the process

$$^1D_{22}\to{}^1P_{11}+\gamma_{-1} \ ,$$

where $^1P_1$ and γ have the helicities $+1$ and -1, respectively. Since the photon is moving in the $-Z$ direction with negative helicity,

$$\mathbf{k}=-k\hat{\mathbf{z}} \ ,\qquad \hat{\boldsymbol{\varepsilon}}_\alpha=-\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})^*=-\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-i\hat{\mathbf{y}}) \ . \tag{20}$$

We will represent the state of $^1D_2$ and $^1P_1$ by their angular momentum quantum numbers jm so that

$$|^1D_{22}\rangle_I=|22\rangle \ ,\qquad |^1P_{11}\rangle=|11\rangle \ . \tag{21}$$

The transition probability amplitude of Eq. (7) now becomes

$$T(t)=\frac{e_q}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}\omega^I_{BA}\left\langle 11\left|-\frac{1}{\sqrt{2}}(x-iy)+\frac{i}{4\sqrt{2}m}kz^2(p_x-ip_y)\right|22\right\rangle\int_0^t e^{i(\omega-\omega_{BA})t'}dt' . \quad (22)$$

In deriving Eq. (22) from Eq. (7), we have made use of the fact that the $^1D_2$ and $^1P_1$ states are spin-singlet states and therefore the terms involving the total spin **S** do not contribute to the matrix element. Next we will express the transition operator in terms of the irreducible tensor operator components. For this purpose we define the following highest weight tensor components:

$$T_{33}=x_+^2p_+,\quad T_{22}=x_+(\mathbf{r}\times\mathbf{p})_+,\quad T_{11}=(\tfrac{2}{5})^{1/2}r^2p_+,\quad T'_{11}=(\tfrac{2}{5})^{1/2}x_+(\mathbf{r}\cdot\mathbf{p}),\quad x_{11}=x_+ . \quad (23)$$

For any vector operator **A**, we define the components

$$A_+=-\frac{1}{\sqrt{2}}(A_x+iA_y),\quad A_-=\frac{1}{\sqrt{2}}(A_x-iA_y),\quad A_3=A_z . \quad (24)$$

Once we have a tensor operator T_{kq} of rank k and component q, we can find the other components from the highest weight component by the formula

$$[J_-,T_{kq}]=\sqrt{k(k+1)+q(1-q)}\,T_{k,q-1} . \quad (25)$$

After some lengthy algebra, we are able to express the transition helicity amplitude as

$$T(t)=\frac{e_q}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}\omega^I_{BA}\left\langle 11\left|\left\{-x_{1-1}+\frac{ik}{2m}\left[\frac{1}{\sqrt{10}}T_{1-1}-\frac{1}{2\sqrt{10}}T'_{1-1}\right]\right\}+\frac{ik}{6m}T_{2-1}+\frac{ik}{2m}\frac{1}{\sqrt{15}}T_{3-1}\right|22\right\rangle\int_0^t e^{i(\omega-\omega_{BA})t'}dt' . \quad (26)$$

Next we use the Wigner-Eckart theorem

$$\langle jm|T_{kq}|j'm'\rangle=\langle jm;kq|j'm'\rangle\langle j||T_k||j'\rangle . \quad (27)$$

Now the matrix element in Eq. (26) can be written as

$$A'_2=\langle 11|1-1;22\rangle\left[-\langle 1||x_1||2\rangle+\frac{ik}{2m}\frac{1}{\sqrt{10}}(\langle 1||T_1||2\rangle-\tfrac{1}{2}\langle 1||T'_1||2\rangle)\right]+\langle 11|2-1;22\rangle\left[\frac{ik}{6m}\langle 1||T_2||2\rangle\right]+\langle 11|3-1;22\rangle\left[\frac{ik}{2m}\frac{1}{\sqrt{15}}\langle 1||T_3||2\rangle\right] . \quad (28)$$

The angular momentum helicity amplitude given by Eq. (28) is called A'_2 since it is not normalized as in Eq. (19).

The arbitrarily normalized helicity amplitude A'_1 can be calculated from the transition amplitude of the process $^1D_{21}\rightarrow{}^1P_{10}+\gamma_{-1}$. Since the photon is moving in the $-Z$ direction with negative helicity, the transition operator will be the same and only the initial and final states will change. The helicity amplitude A'_1 can now be written as

$$A'_1=\langle 10|1-1;21\rangle C_1+\langle 10|2-1;21\rangle C_2+\langle 10|3-1;21\rangle C_3 , \quad (29)$$

where C_1, C_2, and C_3 are the quantities in the square brackets in Eq. (28). Similarly,

$$A'_0=\sum_{k=1}^{3}\langle 1-1|k,-1;20\rangle C_k . \quad (30)$$

The unnormalized multipole amplitudes a'_k are related to the helicity amplitudes A'_ν by the equation [8,9]

$$A'_\nu=\sum_{k=1}^{3}a'_k\left[\frac{2k+1}{5}\right]^{1/2}\langle k1;1,\nu-1|2\nu\rangle . \quad (31)$$

Using the symmetry properties of the Clebsch-Gordan coefficients,

$$\langle j_1m_1;j_2m_2|j_3m_3\rangle=(-1)^{j_1-m_1}\left[\frac{2j_3+1}{2j_2+1}\right]^{1/2}\langle j_3m_3;j_1-m_1|j_2m_2\rangle , \quad (32)$$

we can also write Eq. (31) for $j'=2$ as

$$A'_\nu=\sum_{k=1}^{3}a'_k\left[\frac{2k+1}{3}\right]^{1/2}\langle 1,\nu-1|k,-1;2\nu\rangle . \quad (33)$$

Comparing Eq. (33) for $\nu=2$ with Eq. (28), we obtain

$$a'_1=-\langle 1||x_1||2\rangle+\frac{ik}{2m}\left[\frac{1}{\sqrt{10}}(\langle 1||T_1||2\rangle-\tfrac{1}{2}\langle 1||T'_1||2\rangle)\right] , \quad (34)$$

$$a_2'=\left[\tfrac{3}{5}\right]^{1/2}\frac{ik}{6m}\langle 1\|T_2\|2\rangle , \quad (35)$$

$$a_3'=\frac{ik}{2m}\frac{1}{\sqrt{35}}\langle 1\|T_3\|2\rangle . \quad (36)$$

The multipole amplitudes a_k normalized as in Eq. (19) are related to a_k' as

$$a_k=\frac{a_k'}{\pm[\sum_j |a_j'|^2]^{1/2}} . \quad (37)$$

Since a_2' and a_3' are already of order k/m, we can neglect $(a_2')^2$ and $(a_3')^2$ to order k/m in Eq. (37). Also, since a_1 by definition is chosen to be positive, we have to choose the positive square root in Eq. (37):

$$a_k \simeq \frac{a_k'}{a_1'} . \quad (38)$$

Moreover, to first order in k/m, we can neglect the terms proportional to k/m in Eq. (34) in evaluating a_2 and a_3 to first order in k/m. Then,

$$a_1 \simeq 1 , \quad (39)$$

$$a_2 \simeq -\frac{ik}{6m}\left[\frac{3}{5}\right]^{1/3}\frac{\langle 1\|T_2\|2\rangle}{\langle 1\|x_1\|2\rangle} , \quad (40)$$

$$a_3 \simeq -\frac{ik}{2m}\frac{1}{\sqrt{35}}\frac{\langle 1\|T_3\|2\rangle}{\langle 1\|x_1\|2\rangle} . \quad (41)$$

The reduced matrix elements in Eqs. (40) and (41) were evaluated before [10], and they can be expressed in terms of the two radial integrals I_2 and I_4 defined before [10]:

$$I_2=\int_0^\infty r^3 dr R_{1D}(r) R_{1P}(r) ,$$
$$I_4=\int_0^\infty r^4 dr \frac{dR_{1D}}{dr} R_{1P}(r) . \quad (42)$$

In terms of these integrals,

$$\langle 1\|x_1\|2\rangle = +\frac{1}{3\sqrt{10}} I_2 ,$$
$$\langle 1\|T_2\|2\rangle = -\tfrac{1}{2}(\tfrac{3}{5})^{1/2} I_2 , \quad (43)$$
$$\langle 1\|T_3\|2\rangle = -\frac{i\sqrt{2}}{5\sqrt{7}}(3I_4+8I_2) .$$

Substituting Eqs. (43) in Eqs. (40) and (41), we find

$$a_2 \simeq i\frac{k}{m}\frac{9}{10\sqrt{6}} , \quad (44)$$

$$a_3 \simeq -\frac{k}{m}\left[\frac{24}{35}\right]\left[1+\frac{3}{8}\frac{I_4}{I_2}\right] . \quad (45)$$

The $M2$ amplitude a_2 is purely imaginary and depends only on the ratio of the photon energy to the quark mass, namely k/m, and is independent of the specific potential used, except through its effect on the quark mass used. The $E3$ amplitude a_3 may depend on the specific potential since the ratio of the radial integrals I_2 and I_4 may depend on the potential used. As we will see in Sec. V for the GRR [4] and Buchmuller-Tye [11] models, these radial integrals have practically the same numerical values. We also find that the relative strengths of both the $M2$ and $E3$ amplitudes are rather small, of the order of about 8% of the $E1$ amplitude.

IV. HELICITY AND MULTIPOLE AMPLITUDES IN THE TRANSITION $1\,{}^1D_2 \to 1\,{}^3S_1+\gamma$

In this transition, to conserve parity, the photon γ has even parity since the ${}^1D_2$ and ${}^3S_1$ states both have parity $P=(-1)^{L+1}=-1$. This should be contrasted with the case we discussed in Sec. III, namely, $1\,{}^1D_2 \to 1\,{}^1P_1+\gamma$, where the photon had the odd parity to conserve parity in the transition.

As before, we will assume that the ${}^1D_2$ state is formed at rest in $\bar{p}p$ collisions. Let the $1\,{}^3S_1$ state (otherwise known as ψ) and the photon γ be emitted in the $+Z$ and $-Z$ directions, respectively. The component of the angular momentum of ${}^1D_2$ in the direction of Z or ψ momentum is called ν. The helicities of ψ and γ are called σ and μ, respectively. The transition amplitude of the process ${}^1D_{2\nu} \to \psi_\sigma + \gamma_\mu$ in this kinematical configuration is called the helicity amplitude A_ν. Even though ν can take five integer values from -2 to $+2$, only three A_ν's are independent since A_ν is equal to $A_{-\nu}$ because of parity invariance [12]. We will take the three independent angular momentum helicity amplitudes to be A_0, A_1, and A_2. Even though we are representing the helicity amplitudes in the two decays $1\,{}^1D_2 \to 1\,{}^1P_1+\gamma$ and $1\,{}^1D_2 \to 1\,{}^3S_1+\gamma$ by the same symbols, they are, of course, quite different. In the former case they are parity odd and in the latter they are parity even amplitudes. These angular momentum helicity amplitudes A_ν are related to the $M1$, $E2$, and $M3$ multipole amplitudes a_1, a_2, and a_3 through Eq. (18), and they satisfy the normalization conditions of Eqs. (17) and (19).

In order to calculate the helicity amplitude A_2, we will consider the process

$${}^1D_{22} \to \psi_1 + \gamma_{-1} ,$$

where ψ and γ have the helicities $+1$ and -1, respectively. So just as in Sec. III the photon's momentum vector $\mathbf{k}$ and the polarization vector $\hat{\boldsymbol{\varepsilon}}_\alpha$ satisfy Eqs. (20) and Eq. (8) now reduces to

$$T_e(t)=\frac{1}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}{}_I\left\langle A\left|\frac{e_q k}{\sqrt{2}m}i(s_x-is_y)\left[1+\frac{k}{2m}-\frac{p^2}{2m^2}-\frac{1}{8}k^2z^2\right]+\frac{ik^2}{\sqrt{2}}\frac{e_q}{4m^2}z\left\{(\mathbf{s}\times\mathbf{p})_x-i(\mathbf{s}\times\mathbf{p})_y\right\}\right.\right.$$
$$\left.\left.-\frac{e_q k}{4\sqrt{2}m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}z\left\{(\mathbf{s}\times\mathbf{r})_x-i(\mathbf{s}\times\mathbf{r})_y\right\}+\frac{e_q k}{2\sqrt{2}m^3}(p_x-ip_y)(\mathbf{s}\times\mathbf{p})_y\right|B\right\rangle_I$$
$$\times\int_0^t e^{i(\omega-\omega_{BA})t'}dt' . \quad (46)$$

Next we will express the transition operator in Eq. (46) in terms of the irreducible tensor operator components. For this purpose we define the following highest weight tensor components:

$$p_{22}=p_+(\mathbf{s}\times\mathbf{p})_+,\quad Q_{22}=x_+(\mathbf{s}\times\mathbf{r})_+,\quad M_{11}=\left[1+\frac{k}{2m}-\frac{p^2}{2m^2}\right]s_+,\quad Q_{33}=x_+x_+s_+\ ,$$
$$L_{22}=x_+(\mathbf{s}\times\mathbf{p})_+,\quad p_{11}=[\mathbf{p}\times(\mathbf{s}\times\mathbf{p})]_+,\quad Q_{11}=(\mathbf{r}\cdot\mathbf{s})x_+,\quad L_{11}=[(\mathbf{s}\times\mathbf{p})\times\mathbf{r}]_+,\quad Q'_{11}=r^2s_+\ . \tag{47}$$

Now the transition operator can be expressed entirely in terms of the different components of the irreducible tensor operators defined in Eq. (47). Using the Wigner-Eckart theorem of Eq. (27), we can express the matrix element of Eq. (46) entirely in terms of the reduced matrix elements of the tensor operators and the Clebsch-Gordan coefficients. After collecting together tensors of the same rank, we finally obtain for the matrix element of Eq. (46) the expression (which we will call A'_2, since it is not normalized)

$$\begin{aligned}A'_2=\langle 11|1-1;22\rangle&\left[-\frac{ie_qk}{4\sqrt{2}m^3}\langle 1||p_1||2\rangle+\frac{ie_qk}{8\sqrt{2}m^2}\left\langle 1\left|\left|\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_1\right|\right|2\right\rangle+\frac{ie_qk}{m}\langle 1||M_1||2\rangle\right.\\&\left.+\frac{ie_qk^3}{40m}\langle 1||Q_1||2\rangle-\frac{ie_qk^3}{20m}\langle 1||Q'_1||2\rangle+\frac{e_qk^2}{8\sqrt{2}m^2}\langle 1||L_1||2\rangle\right]\\&+\langle 11|2-1;22\rangle\left[\frac{e_qk}{2\sqrt{2}m^3}\langle 1||p_2||2\rangle-\frac{e_qk}{4\sqrt{2}m^2}\left\langle 1\left|\left|\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_2\right|\right|2\right\rangle-\frac{e_qk^3\sqrt{2}}{24m}\langle 1||Q_2||2\rangle\right.\\&\left.+\frac{ie_qk^2}{4\sqrt{2}m^2}\langle 1||L_2||2\rangle\right]+\langle 11|3-1;22\rangle\left[\frac{(-i)e_qk^3}{4\sqrt{15}m}\langle 1||Q_3||2\rangle\right]\ .\end{aligned} \tag{48}$$

In deriving Eq. (48) from the matrix element of Eq. (46), we represented the state vectors of $1\,^1D_{22}$, $1\,^3S_{11}$, and γ_{-1} by the symbols $|22\rangle$, $|11\rangle$, and $|1-1\rangle$, respectively.

The arbitrarily normalized angular momentum helicity amplitude A'_1 can be calculated from the transition amplitude of the process $1\,^1D_{21}\rightarrow 1\,^3S_{10}+\gamma_{-1}$. Since the photon is moving in the $-Z$ direction with negative helicity, the transition operator will be the same as in Eq. (46) and only the initial and final states will change. The helicity amplitude A'_1 can now be written as

$$A'_1=\langle 10|1-1;21\rangle C_1+\langle 10|2-1;21\rangle C_2+\langle 10|3-1;21\rangle C_3\ , \tag{49}$$

where C_1, C_2, and C_3 are the quantities in the square brackets in Eq. (48).

Similarly, the arbitrarily normalized helicity amplitude A'_0 can be calculated from the decay $1\,^1D_{20}\rightarrow 1\,^3S_{1-1}+\gamma_{-1}$ and it can be written as

$$A'_0=\sum_{k=1}^{3}\langle 1-1|k,-1;20\rangle C_k\ . \tag{50}$$

The unnormalized multipole amplitudes a'_1, a'_2, and a'_3, the $M1$, $E2$, and $M3$, respectively, are related to A'_ν $(\nu=0,1,2)$ by the relation of Eq. (33). So we get

$$\begin{aligned}a'_1=&-\frac{ie_qk}{4\sqrt{2}m^3}\langle 1||p_1||2\rangle+\frac{ie_qk}{8\sqrt{2}m^2}\left\langle 1\left|\left|\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_1\right|\right|2\right\rangle+\frac{ie_qk}{m}\langle 1||M_1||2\rangle\\&+\frac{ie_qk^3}{40m}\langle 1||Q_1||2\rangle-\frac{ie_qk^3}{20m}\langle 1||Q'_1||2\rangle+\frac{e_qk^2}{8\sqrt{2}m^2}\langle 1||L_1||2\rangle\ ,\end{aligned} \tag{51}$$

$$\begin{aligned}a'_2=\left[\frac{3}{5}\right]^{1/2}&\left[\frac{e_qk}{2\sqrt{2}m^3}\langle 1||p_2||2\rangle-\frac{e_qk}{4\sqrt{2}m^2}\left\langle 1\left|\left|\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_2\right|\right|2\right\rangle\right.\\&\left.-\frac{e_qk^3\sqrt{2}}{24m}\langle 1||Q_2||2\rangle+\frac{ie_qk^2}{4\sqrt{2}m^2}\langle 1||L_2||2\rangle\right]\ ,\end{aligned} \tag{52}$$

$$a'_3=\left[\frac{3}{7}\right]^{1/2}(-i)\frac{e_qk^3}{4\sqrt{15}m}\langle 1||Q_3||2\rangle\ . \tag{53}$$

Using the definition of the highest weight components of the spherical tensors and calculating their matrix elements using wave functions from any potential model, we can express all the reduced matrix elements in Eqs. (51)–(53) in terms of four radial integrals given in terms of the nonrelativistic radial wave functions of the $1S$ and $1D$ states of charmonium. The four radial integrals are

$$\begin{aligned} J_1 &= \frac{\sqrt{2}k^2}{10}\int_0^\infty R_{1S}R_{1D}r^4 dr\ , \\ J_2 &= \frac{k}{2mc}\int_0^\infty \frac{dR_{1S}}{dr}R_{1D}r^3 dr\ , \\ J_3 &= -\frac{1}{m^2c^2}\int_0^\infty \left[\frac{d^2R_{1S}}{dr^2}-\frac{1}{r}\frac{dR_{1S}}{dr}\right]R_{1D}r^2 dr\ , \\ J_4 &= \frac{1}{2mc^2}\int_0^\infty R_{1S}\frac{\partial U^{(0)}}{\partial r}R_{1D}r^3 dr\ . \end{aligned} \tag{54}$$

In terms of these radial integrals, the arbitrarily normalized $M1$, $E2$, and $M3$ amplitudes are

$$a_1' = -\frac{ie_q k}{12mc}(J_1+J_2+J_3+J_4)\ , \tag{55}$$

$$a_2' = -\frac{ie_q k}{12mc}\left[\sqrt{5}J_1+\frac{6}{\sqrt{10}}J_2 -6\left[\frac{2\pi}{5}\right]^{1/2}J_3+\frac{6}{\sqrt{10}}J_4\right]\ , \tag{56}$$

$$a_3' = -\frac{ie_q k}{12mc}(2J_1)\ . \tag{57}$$

The multipole amplitudes normalized according to Eq. (19) are given by

$$a_k = \frac{a_k'}{\sqrt{|a_1'|^2+|a_2'|^2+|a_3'|^2}}\quad (k=1,2,3)\ . \tag{58}$$

If we define

$$\eta = \left[|J_1+J_2+J_3+J_4|^2 + \left|\sqrt{5}J_1+\frac{6}{\sqrt{10}}J_2-6\left[\frac{2\pi}{5}\right]^{1/2}J_3+\frac{6}{\sqrt{10}}J_4\right|^2 +4|J_1|^2\right]^{1/2}\ , \tag{59}$$

then Eq. (58) leads to the following expressions for a_1, a_2, and a_3:

$$a_1 = -i\eta(J_1+J_2+J_3+J_4)\ , \tag{60}$$

$$a_2 = -i\eta\left[\sqrt{5}J_1+\frac{6}{\sqrt{10}}J_2-6\left[\frac{2\pi}{5}\right]^{1/2}J_3+\frac{6}{\sqrt{10}}J_4\right]\ , \tag{61}$$

$$a_3 = -i\eta(2J_1)\ . \tag{62}$$

Equations (60)–(62) give the normalized $M1$, $E2$, and $M3$ amplitudes entirely in terms of the radial integrals defined by Eqs. (54). Unlike the $E1$, $M2$, and $E3$ amplitudes in $1\,^1D_2 \to 1\,^1P_1+\gamma$ where the $M2$ and $E3$ amplitudes are of order k/m compared to the $E1$ amplitude, the $M1$, $E2$, and $M3$ amplitudes in the relativistic $M1$ transition $1\,^1D_2 \to 1\,^3S_1+\gamma$ are all of the same order. In fact, in the GRR potential model [4] the $E2$ amplitude turns out to be numerically the largest as we will see in the next section.

The total decay rate for the transition $1\,^1D_2 \to 1\,^3S_1+\gamma$ can be written in terms of the arbitrarily normalized multipole amplitudes a_1', a_2', and a_3' as

$$\mathcal{W} = \frac{8}{5}\frac{\omega_{BA}}{c}[|a_1'|^2+|a_2'|^2+|a_3'|^2]\ . \tag{63}$$

When we substitute the expressions (56)–(58) for a_1', a_2', and a_3' in Eq. (63), we obtain

$$\mathcal{W}_{^1D_2\to {}^3S_1+\gamma} = \frac{1}{90}\alpha\left[\frac{e_q}{e}\right]^2\left[\frac{\omega}{mc^2}\right]^2\omega\left[|J_1+J_2+J_3+J_4|^2+\left|\sqrt{5}J_1+\frac{6}{\sqrt{10}}J_2-6\left[\frac{2\pi}{5}\right]^{1/2}J_3+\frac{6}{\sqrt{10}}J_4\right|^2+|2J_1|^2\right]\ . \tag{64}$$

When we compare Eq. (64) with Eq. (18) of Ref. [1], we find that in Ref. [1] the contribution of the $M2$ and $E3$ amplitudes to the decay rate was neglected. The contribution of the $M2$ and $E3$ amplitudes to the total one-photon decay rate of the $^1D_2$ state is quite significant. In fact, as we will see in the next section, including the contributions of the $M2$ and $E3$ amplitudes, the total decay rate for $1\,^1D_2 \to 1\,^3S_1+\gamma$ of charmonium turns out to be about 63 keV. On the other hand, neglecting the $M2$ and $E3$ contributions, the decay rate predicted in Ref. [1] was only about 2 keV. So including the $M2$ and $E3$ contributions is crucial in getting a reliable estimate of the $1\,^1D_2 \to 1\,^3S_1+\gamma$ transition rate.

V. NUMERICAL ESTIMATE OF THE MULTIPOLE AMPLITUDES AND THE DECAY RATES OF THE $1\,^1D_2 \to 1\,^1P_1+\gamma$ AND $1\,^1D_2 \to 1\,^3S_1+\gamma$ TRANSITIONS OF CHARMONIUM

We use the potential model of Gupta, Radford, and Repko [4] to calculate the multipole amplitudes and decay rates of the transitions $1\,^1D_2 \to 1\,^1P_1+\gamma$ and $1\,^1D_2 \to 1\,^3S_1+\gamma$ of charmonium. We used the values $m_c=1.32$ GeV, $\mu=1.94$ GeV, $\alpha_s=0.36$, $k=0.15$ GeV^2, and $c=0.392$ GeV for the parameters in their model since they give an extremely good fit of the energy spectrum of charmonium. The predicted mass of the $1\,^1D_2$

state in this model is 3822 MeV [1]. The wave functions were obtained by the variational calculation using a trial wave function whose radial wave function is a polynomial of degree 10 times an exponential function. The variational parameters are the coefficients in the polynomial function. The constant in the exponent is determined by satisfying the virial theorem [4]. We obtain the following numerical results for the two radiative transitions.

A. $1\,^1D_2 \to 1\,^1P_1 + \gamma$

Using the wave functions of the GRR model [4], we obtain the following numerical values for the radial integrals I_2 and I_4 defined by Eqs. (42):

$$I_2 = 0.664 \text{ fm} , \quad I_4 = -0.890 \text{ fm} . \tag{65}$$

If we take the mass of the $1\,^1P_1$ state to be 3526 MeV [2,13,14], the energy of the emitted photon will be approximately

$$k \simeq M_{1_{D_2}} - M_{1_{P_1}} = 3822 \text{ MeV} - 3526 \text{ MeV} = 296 \text{ MeV} . \tag{66}$$

If the mass of the c quark is 1.32 GeV,

$$\frac{k}{m} \simeq 0.22 . \tag{67}$$

So the $M2$ and $E3$ amplitudes of Eqs. (44) and (45) become

$$a_2 \simeq i \frac{k}{m} \frac{9}{10\sqrt{6}} \simeq i(0.081) , \tag{68}$$

$$a_3 \simeq -\frac{k}{m} \left[\frac{24}{35} \right] \left[1 + \frac{3}{8} \frac{I_4}{I_2} \right] \simeq -0.075 . \tag{69}$$

So the $M2$ amplitude is purely imaginary and the $E3$ amplitude is real and negative. They are both quite small in magnitude compared to the $E1$ amplitude.

The decay rate of the transition $1\,^1D_2 \to 1\,^1P_1 + \gamma$ can be calculated using Eq. (74) of Ref. [15]. If we calculate the reduced matrix elements using the GRR model [4], we get a transition rate of about 650 keV. So this is probably the most dominant transition of the $^1D_2$ state.

We should also mention that the numerical values [10] of the radial integrals I_2 and I_4 are practically the same in the Buchmuller-Tye potential model [11].

B. $1\,^1D_2 \to 1\,^3S_1 + \gamma$

In the GRR model [4] described above, the radial integrals J_1, J_2, J_3, and J_4 take the numerical values

$$J_1 = 0.3872, \quad J_2 = -0.4056 , \quad J_3 = -0.4266, \quad J_4 = 0.1646 . \tag{70}$$

It should be noted that the radial integral J_1 we defined in Eqs. (54) is $\sqrt{2}$ times the radial integral J_1 of Ref. [1]. Using Eqs. (59)–(62), the numerical values of the $M1$, $E2$, and $M3$ multipole amplitudes are

$$a_1 = + \frac{i(0.2804)}{\sqrt{|0.2804|^2 + |3.2778|^2 + |0.7744|^2}} = i(0.083) , \tag{71}$$

$$a_2 = -i \frac{3.2778}{3.380} = -i(0.970) , \tag{72}$$

$$a_3 = -i \frac{0.7744}{3.380} = -i(0.229) . \tag{73}$$

If we take the mass of the $1\,^1D_2$ state to be 3822 MeV [1], the energy of the emitted photon in the $1\,^1D_2 \to 1\,^3S_1$ transition will be 643 MeV and the predicted rate for the transition $1\,^1D_2 \to 1\,^3S_1 + \gamma$, using Eq. (64), will come out to be

$$\Gamma(1\,^1D_2 \to 1\,^3S_1 + \gamma) \simeq 62.6 \text{ keV} . \tag{74}$$

This is an order of magnitude larger than the value predicted in Ref. [1] where the $E2$ and $M3$ contributions to the transition rate were completely neglected, which was a serious error. Since this rate is so large, is should have a significant branching ratio.

VI. CONCLUDING REMARKS

We have calculated the multipole amplitudes in the one-photon radiative transitions of the $1\,^1D_2$ state of charmonium, namely, $1\,^1D_2 \to 1\,^1P_1 + \gamma$ and $1\,^1D_2 \to 1\,^3S_1 + \gamma$, in an arbitrary potential model. The relative strength of the $M2$ multipole amplitude compared to $E1$ in the $1\,^1D_2 \to 1\,^1P_1$ transition is independent of the specific potential and depends only on the ratio of the photon energy to the quark mass. Although the relative $E3$ amplitude [Eq. (45)] depends on the ratio of two radial integrals I_4 and I_2, it is probably very insensitive to the specific potential used. In the $1\,^1D_2 \to 1\,^1P_1$ transition, the $E2$ and $M3$ amplitudes are of relative order v^2/c^2 compared to the $E1$ amplitude.

In the $1\,^1D_2 \to 1\,^3S_1$ transition, the amplitude is parity even. Since this transition is forbidden in the nonrelativistic limit, all the multipole amplitudes $M1$, $E2$, and $M3$ are of relative order v^2/c^2. All the three amplitudes can be expressed in terms of four radial integrals which could depend on the specific potential used. Numerical calculations in the GRR potential model [4] show that the $E2$ amplitude is the most dominant one, about 12 times as large as the $M1$ amplitude and about 4 times the size of the $M3$ amplitude. Most of the contribution to the $1\,^1D_2 \to 1\,^3S_1$ one-photon transition rate comes from the $E2$ and $M3$ amplitudes.

Another interesting thing to point out about our results is that in general the multipole amplitudes are complex. In fact, the $M2$, $M1$, $E2$, and $M3$ amplitudes we calculated are purely imaginary, while the $E3$ amplitude is real. In Ref. [3] we have shown that by studying the angular distribution of the decay products of the $1\,^1D_2$ state formed in the unpolarized $\bar{p}p$ collisions we can only obtain the magnitudes of all the helicity amplitudes A_0, A_1, and A_2 and the cosine of their relative phases or $\text{Re}(A_i A_j^*)$. In order to obtain the real and imaginary parts of the helicity amplitudes A_i $(i=0,1,2)$ or, equivalently, the multipole amplitudes a_k $(k=1,2,3)$, we should also determine $\mathrm{Im}(A_i A_j^*)$. For this we should measure the angular distributions of the decay products of the $1\,^1D_2$ state formed in polarized $\bar{p}p$ collisions [16]. Our results show the importance of doing experiments with polarized proton and antiprotons beams.

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