QCD-based interpretation of the lepton spectrum in inclusive $\bar{B} \to X_u \ell \bar{\nu}$ decays

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We present a QCD-based approach to the end-point region of the lepton spectrum in $\bar{B} \to X_u \ell \bar{\nu}$ decays. A genuinely nonperturbative form factor, the shape function, describes the falloff of the spectrum close to the end point. The moments of this function are related to forward scattering matrix elements of local, higher-dimension operators. We find that nonperturbative effects are dominant over a finite region in the lepton energy spectrum, the width of which is related to the kinetic energy of the b quark inside the B meson. In this region, a resummation of the most singular terms in the operator product expansion is performed. Applications of our method to the extraction of fundamental standard model parameters, among them V_{ub}, are discussed.

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I. INTRODUCTION

Recently, much progress has been achieved in the understanding of inclusive weak decays of hadrons containing a heavy quark Q. Using the theoretical tools of the operator product expansion and the heavy quark effective theory (HQET) [1–5], one can construct a systematic expansion of the (differential) decay distributions in powers of Λ/m_Q, where Λ is a characteristic low-energy scale of the strong interactions [6–10]. Quite remarkably, the parton model emerges as the leading term in this QCD-based expansion, and the nonperturbative corrections to it are suppressed by a factor Λ^2/m_Q^2. The fact that there are no first-order power corrections relies on a particular definition of m_Q, which is provided in a natural way by requiring that there be no residual mass term for the heavy quark in HQET [11, 12]. This definition is unique and can be regarded as a nonperturbative generalization of the concept of a pole mass.

The availability of a systematic, QCD-based expansion of inclusive decay rates raises the hope for a better understanding of these processes in general, and in particular for a more reliable extraction of the standard model parameters m_b, m_c, V_{cb}, and V_{ub}, which was so far hindered by strong model dependence. For a determination of V_{ub}, however, it is essential to understand the end-point region of the lepton spectrum, which is of genuinely nonperturbative nature. Although the new methods developed in Refs. [7–10] provide an important step towards this goal, they are not directly applicable to the end-point region. The difficulties arise from the fact that close to the end point the expansion parameter is no longer Λ/m_b, but $\Lambda/(m_b - 2E_\ell)$, and thus the theoretical prediction becomes singular when the lepton energy approaches the parton model end point $E_{\ell,\max} = m_b/2$. It is then not obvious how to interpret the theoretical results.

To see what the problem is, consider the theoretical prediction for the lepton spectrum in $\bar{B} \to X_u \ell \bar{\nu}$ decays. Including the leading nonperturbative corrections, one obtains [7–9]

$$\frac{1}{2\Gamma_b}\,\frac{d\Gamma}{dy} = F(y)\,\Theta(1-y) - \frac{\lambda_1 + 33\lambda_2}{6m_b^2}\,\delta(1-y) - \frac{\lambda_1}{6m_b^2}\,\delta'(1-y)\,, \tag{1}$$

where

$$y = \frac{2E_\ell}{m_b}\,,\quad \Gamma_b = \frac{G_F^2\,|V_{ub}|^2}{192\pi^3}\,m_b^5\,, \tag{2}$$

and

$$F(y) = (3-2y)\,y^2 + \frac{5y^3}{3}\,\frac{\lambda_1}{m_b^2} + (6+5y)\,y^2\,\frac{\lambda_2}{m_b^2}\,. \tag{3}$$

For simplicity, we do not include perturbative QCD corrections, which have been calculated in Refs. [13, 14]. The hadronic parameters λ_1 and λ_2 are related to the kinetic energy K_b of the heavy quark inside the B meson, and to the mass splitting between B and B^* mesons [15]:

$$K_b = -\frac{\lambda_1}{2m_b}\,,\quad m_{B^*}^2 - m_B^2 = 4\lambda_2\,. \tag{4}$$

The singular structure of the operator product expansion close to the end point at $y = 1$ manifests itself in the appearance of δ-function (and higher) distributions. Certainly, one cannot trust the shape of the theoretical spectrum close to the end point. Nevertheless, integrating (1) with a smooth weight function, one obtains well-behaved results for quantities such as the total decay rate and the average lepton energy:

$$\begin{aligned} \Gamma &= \Gamma_b\left\{1 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2}\right\}, \\ \langle E_\ell\rangle &= \frac{7m_b}{20}\left\{1 - \frac{7\lambda_1 + 57\lambda_2}{14m_b^2}\right\}. \end{aligned} \tag{5}$$

The coefficients of the singular terms give nonvanishing contributions to these integrated quantities. There

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is thus some relevant physical information contained in these terms.

Bigi *et al.* [7] have advocated to integrate over the singularities before confronting the theoretical prediction for the lepton spectrum with data. They argued that one has to integrate over a finite energy interval of at least several hundred MeV, corresponding to a region of order $1/m_b$ in the variable y. This proposal is based on the idea of quark-hadron duality, which implies that when one sums over a sufficient number of exclusive hadronic modes, the decay probability into hadrons equals the decay probability into free quarks.[1] Note that the δ-function term in (1) contributes to the integrated spectrum, but the term proportional to $\delta'(1-y)$ does not. Similarly, more singular terms, which appear at higher orders in the $1/m_b$ expansion, do not contribute.

A slightly modified procedure was proposed by Manohar and Wise [9]. They chose to smear the spectrum with a Gaussian distribution of width Δy. Empirically, they found that $\Delta y \sim 0.2 - 0.5$ is necessary to obtain from the theoretical prediction a smooth lepton spectrum, which can be compared to data. This procedure has the disadvantage that the results depend on the smearing function, and that the choice of Δy is ad hoc. When the smearing function is chosen to be symmetric, it again follows that the term proportional to $\delta'(1-y)$ does not contribute to the smeared spectrum.

The frustrating conclusion of these analyses is that the new theoretical methods are only of very limited use for a more reliable determination of V_{ub}, since the region of the lepton spectrum which is accessible to a measurement is smaller than the region over which the theoretical spectrum has to be integrated in order to obtain a reasonable result.

As proposed in Refs. [7–9], the theoretical description is to a large extent ignorant of the rich physical information contained in the lepton spectrum close to the end point. In this paper, we shall suggest a different approach. It is motivated by a very simple observation: In the parton model, the end-point region of the lepton spectrum is described by a step function, the location of which is determined by the kinematics of a free quark decay. The true physical end point, however, is determined by the decay kinematics of hadrons. Hence, when QCD is trying to tell us something about the redistribution of the end-point region due to nonperturbative effects, it can only do this by the occurrence of singular functions. Our approach will allow us to extract the physical information contained in the singular terms in the QCD-predicted lepton spectrum in a systematic way, and to all orders in the $1/m_b$ expansion. To this end, we shall introduce the concept of a *shape function*, which is a genuinely nonperturbative form factor that describes the falloff of the spectrum in the end-point region. We find that the characteristic width of this region is given by $\sigma_y = (-\lambda_1/3m_b^2)^{1/2}$, corresponding to a *finite* region in the lepton energy. Although there do not appear first-order power corrections in (1), it is important to realize that there exists a small region where the true spectrum is very different from the theoretical prediction. This difference is described by the shape function. We will show that the *moments* of this function can be addressed in QCD, and can be related to hadronic parameters (such as λ_1 and λ_2) that are defined in terms of forward scattering matrix elements of local, higher-dimension operators. To all orders in $1/m_b$, the leading contributions to the moments can be given in closed form.

We believe that our approach will eventually lead to a better understanding of the nonperturbative aspects of the lepton spectrum close to the end point. It establishes the connection between the experimentally observed lepton spectrum and the underlying theory of QCD. This connection works in both directions: Theoretical ideas about the shape function can help to analyze the lepton spectrum and to determine the values of the quark masses and mixing angles. On the other hand, from a precise measurement of the spectrum in the end-point region, one can extract the shape function and with it some fundamental matrix elements of higher-dimension operators in QCD.

Starting from a resummation of the theoretical lepton spectrum, we present in Sec. II a heuristic argument that leads to the notion of a function $S(y)$, which describes the falloff of the spectrum in the end-point region. In Sec. III, we introduce this shape function, discuss its properties, and derive relations for the first two moments of $S(y)$. Section IV is devoted to a formal definition of the shape function to all orders in $1/m_b$. The leading contributions to the moments are related to forward scattering matrix elements of local, higher-dimension operators in HQET. For the purpose of illustration, a simple model calculation of the shape function is presented in Sec. V. In Sec. VI, we summarize our results, indicate possible further applications and improvements of the method, and give some conclusions.

II. RESUMMATION OF THE SINGULAR TERMS

To motivate the concept of a shape function, let us try to resum the singular contributions in (1) into a corrected parton model decay distribution. Obviously, the term proportional to $\delta(1-y)$ can be absorbed by a shift of the argument of the step function in the leading-order term. More interesting is the contribution proportional to $\delta'(1-y)$. It arises at second order in the expansion of the step function. However, since there is no δ-function contribution of first order in $1/m_b$, it follows that one needs more than one step function, resulting in a *dispersion* of the spectrum. In fact, to order $1/m_b^2$, we can rewrite the theoretical spectrum in the following way:

$$\frac{1}{2\Gamma_b}\,\frac{d\Gamma}{dy} = F(y)\,\frac{1}{N}\sum_{i=1}^{N}\Theta(1-y+\varepsilon_i)\,, \qquad (6)$$

where

[1] One expects that duality holds for the lepton spectrum even in the end-point region, which extends over an interval of order $1/m_b$ in y. Only in a tiny region of order $1/m_b^2$ below the physical end point, the spectrum is dominated by a few exclusive modes.

$$\delta y = \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i = -\frac{\lambda_1 + 33\lambda_2}{6m_b^2}\,,$$
$$\sigma_y^2 = \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2 = -\frac{\lambda_1}{3m_b^2}\,. \tag{7}$$

From the second relation it follows that the displacements ε_i are of order $1/m_b$. Thus, the first relation corresponds to a nontrivial cancellation. Note that σ_y can be identified with the characteristic width of the end-point region, i.e., the region over which the spectrum is dominated by nonperturbative effects. This is already a remarkable conclusion: The coefficient of the most singular term in (1), which had no effect in the approaches of Refs. [7, 9], determines the size of the end-point region.

At this point, it is worthwhile to obtain some estimates of the nonperturbative corrections. From the observed value of the mass splitting between B and B^* mesons one obtains $\lambda_2 \simeq 0.12$ GeV^2. The parameter λ_1, on the other hand, is not directly related to an observable. Recently, we have shown that the field-theory analog of the virial theorem relates the kinetic energy of a heavy quark inside a hadron (and thus λ_1) to a matrix element of the gluon field strength tensor [16]. This theorem makes explicit an intrinsic "smallness" of λ_1, which was not taken into account in existing QCD sum rule calculations of this parameter [17–19]. As a consequence, we expect that $(-\lambda_1)$ is considerably smaller than predicted in these analyses. Here we shall use the range $-\lambda_1 = 0.05 - 0.30$ GeV^2. According to its definition, λ_1 is negative, so that the width σ_y in (7) is well defined. Using these numbers, as well as $m_b = 4.8$ GeV, we estimate $\delta y \simeq -0.03$ and $\sigma_y \simeq 0.03 - 0.07$. We can multiply these quantities by $m_b/2$ to obtain the corresponding shift and spread in the lepton energy spectrum. They are $\delta E \simeq -65$ MeV and $\sigma_E = (-\lambda_1/12)^{1/2} \simeq 65 - 160$ MeV. The value of σ_E can be compared to the width of the gap between the parton model end point of the spectrum and the physical end point, which, if we neglect the pion mass, is located at $E_{\ell,\max}^{\text{phys}} = m_B/2$. This width is $\Delta E \simeq (m_B - m_b)/2 \simeq 240$ MeV. These numbers seem quite reasonable. In fact, assuming that the distribution of the displacements ε_i around $y = 1$ is approximately symmetric, we have to require that $\sigma_E < \Delta E$, which is equivalent to $-\lambda_1 < 3(m_B - m_b)^2$. For reasonable values of λ_1, this bound is always satisfied.

In Fig. 1, we show the resummed lepton energy spectrum (6) for $\lambda_1 = -0.2$ GeV^2, $\lambda_2 = 0.12$ GeV^2, $N = 10$, and a particular set of ε_i satisfying the constraints in (7). For this choice of parameters, the dispersion in the spectrum is such that the end point falls close to the physical end point at $y \simeq 1.1$. Our reinterpretation of the QCD-predicted lepton spectrum has led to a reasonable shape which, however, is quite arbitrary. In fact, increasing N, we can generate any decreasing function that satisfies the constraints in (7). In the following section, we will introduce a shape function $S(y)$ instead of the sum over step functions. The constraints will then turn into predictions for the first two moments of this function.

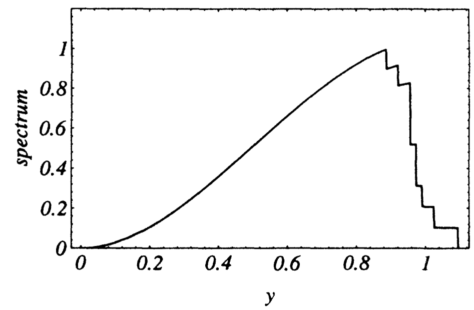

FIG. 1. An example of a resummed lepton energy spectrum according to (6). On the vertical axis, we show $d\Gamma/dy$ in units of $2\Gamma_b$.

III. THE SHAPE FUNCTION

We proceed by replacing the sum over step functions in (6) by a continuous function $\vartheta(y)$, which we furthermore decompose as $\vartheta(y) = \Theta(1-y) + S(y)\,F(1)/F(y)$. The form of the second term is chosen for later convenience. We shall refer to $S(y)$ as the shape function. The support of this function is restricted to a small interval [2] of width 2Δ around $y = 1$, with Δ of order $1/m_b$. Some properties of $S(y)$ can be derived from the physical requirements that the differential decay rate be positive, and that $\vartheta(y)$ be a continuous function. We note that

$$\begin{aligned} S(y) &= 0 \quad \text{if } |y-1| > \Delta\,, \\ S(y) &\geq 0 \quad \text{if } y \geq 1\,, \\ \lim_{\epsilon\to 0} S(1+\epsilon) &- S(1-\epsilon) = 1\,. \end{aligned} \tag{8}$$

Moreover, we expect that $S'(y) \leq 0$ if $y \neq 1$, but we shall not impose this as a condition on $S(y)$.

As emphasized in Refs. [7, 9], because of the singular form of the operator product expansion, one has to integrate the theoretical lepton spectrum with a smooth function before it can be compared to data. On the set of smooth functions (i.e., functions of y which are slowly varying over scales of order $1/m_b$), a rapidly varying function such as $S(y)$, which vanishes outside a small interval around $y = 1$, obeys a singular expansion of the form[3]

$$S(y) = \sum_{n=0}^{\infty} \frac{M_n}{n!}\,\delta^{(n)}(1-y)\,, \tag{9}$$

where the *moments* M_n are defined as

$$M_n = \int_0^\infty dy\,(y-1)^n\,S(y) = \int_{1-\Delta}^{1+\Delta} dy\,(y-1)^n\,S(y)\,. \tag{10}$$

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[2]More precisely, we should require that $S(y)$ be exponentially small outside an interval of width 2Δ.

[3]This procedure is familiar from the multipole expansion of a localized distribution of charges in electrodynamics.

To see that (9) is correct, assume that any reasonable test function $f(y)$ can be Taylor expanded around $y = 1$, with the result that

$$\int_0^\infty dy\, f(y)\, S(y) = \sum_{n=0}^\infty \frac{M_n}{n!}\, f^{(n)}(1)\,. \tag{11}$$

In terms of the shape function, the theoretical lepton spectrum takes the form

$$\frac{1}{2\Gamma_b}\frac{d\Gamma}{dy} = F(y)\,\Theta(1-y) + F(1)\,S(y)\,. \tag{12}$$

From a comparison with (1), we find that the first two moments of $S(y)$ must satisfy

$$M_0 = -\frac{\lambda_1 + 33\lambda_2}{6m_b^2} = \delta y \simeq -2.7\%\,,$$
$$M_1 = -\frac{\lambda_1}{6m_b^2} = \frac{\sigma_y^2}{2} \simeq (0.4 - 2.2)\times 10^{-3}\,. \tag{13}$$

Notice that $M_n = O(1/m_b^{n+1})$ by dimensional analysis. Hence, the QCD prediction that $M_0 = O(1/m_b^2)$ corresponds to a nontrivial cancellation: The area under the shape function (almost) vanishes. The first moment, which is related to the characteristic width of the end-point region, is of the expected order of magnitude.

The concept of a shape function exploits to full extent the physical information contained in the coefficients of the singular terms in the QCD-predicted lepton spectrum. We find that over a region of width $2\Delta \propto 1/m_b$, the spectrum is of genuinely nonperturbative nature and described by a function $S(y)$, the moments of which can be addressed in QCD. When one goes to higher orders in the $1/m_b$ expansion, one can address higher moments. In fact, the moments obey an expansion of the form

$$M_n = \frac{a_n}{m_b^{n+1}} + \frac{b_n}{m_b^{n+2}} + \cdots\,, \tag{14}$$

where so far we know the coefficients $a_0 = 0$, $b_0 = -\lambda_1/6 - 11\lambda_2/2$, and $a_1 = -\lambda_1/6$. With the exception of the moment M_0, where the leading term a_0 vanishes, we may argue that it would be a good approximation to know the leading coefficient a_n for each moment. The corrections involving b_n only change the moments by small amounts. On the other hand, knowledge of a new moment teaches us a new piece of essential information about the shape of the spectrum in the end-point region. The higher moments give a small contribution to integrated quantities such as the total decay rate, simply because in (10) one integrates over a small region. Nevertheless, they can affect the shape of the end-point region in a substantial way. What is relevant to the shape are the rescaled moments

$$\mathcal{M}_n = 2\int_0^\infty dE_\ell\,(2E_\ell - m_b)^n\, S(E_\ell)$$
$$= m_b^{n+1}\, M_n = a_n + \frac{b_n}{m_b} + \cdots\,, \tag{15}$$

which remain nonzero in the limit $m_b \to \infty$. As an illustration of the importance of higher moments, we show in Fig. 2 two shape functions which have the same first two moments M_0 and M_1, but different third (and higher) moments. The total decay rate and the average lepton energy are the same in both cases (up to terms of order $1/m_b^3$), but obviously the behavior close to the end point is quite different.

IV. FORMAL DEFINITION OF THE SHAPE FUNCTION

The above discussion shows that for an understanding of the lepton spectrum in the end-point region, it is insufficient to truncate the theoretical calculation at order $1/m_b^2$. Instead, what one needs to investigate to all orders are the coefficients a_n in (14) and (15). They arise from the most singular terms in the shape function. In this section, we give a formal definition of these terms to all orders in $1/m_b$. This will provide us with a relation between the coefficients a_n and forward scattering matrix elements of local, higher-dimension operators in HQET.

As mentioned in the Introduction, the derivation of the lepton spectrum is based on the operator product expansion in connection with an expansion of hadronic matrix element in powers of $1/m_b$, as provided by HQET. This is explained in detail in Refs. [7–10]. Using the same technology, we can derive a closed expression for the most singular terms of the shape function, where $S(y)$ is

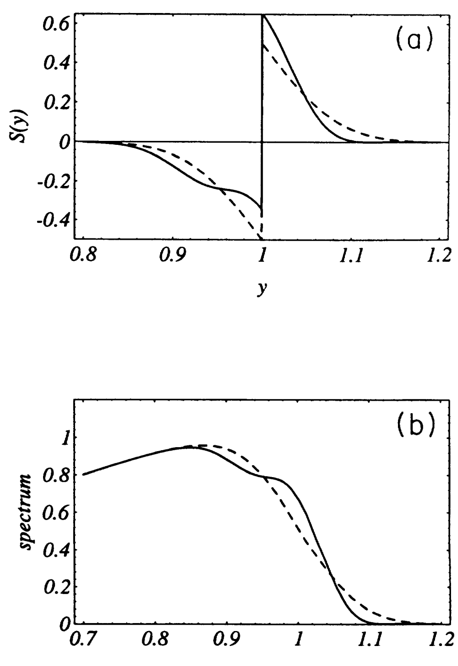

FIG. 2. Two shape functions with identical moments M_0 and M_1 (a), and the corresponding lepton spectra (b).

defined as in (12) as the sum of all terms in the theoretical spectrum that become singular in the limit $y \to 1$. We obtain the formal result

$$S(y) = \left\langle \Theta\left[1 - y + \frac{2}{m_b}\,(v - \hat{p}) \cdot iD\right] - \Theta(1 - y) \right\rangle + \text{less singular terms}, \tag{16}$$

which is valid to all orders in the $1/m_b$ expansion. Here $\hat{p} = p_\ell/m_b$ denotes he rescaled lepton momentum, and we define the expectation value of an operator $\mathcal{O}$ as

$$\langle\, \mathcal{O}\, \rangle = \frac{\langle B(v)|\, \bar{h}_v\, \mathcal{O}\, h_v\, |B(v)\rangle}{\langle B(v)|\, \bar{h}_v\, h_v\, |B(v)\rangle}\,. \tag{17}$$

Here, h_v is the velocity-dependent heavy quark field in HQET [2], and the states are the eigenstates of the corresponding effective Lagrangian. Details of the derivation of (16), as well as the extension to the case of $\bar{B} \to X_c\,\ell\,\bar{\nu}$ decays, will be given elsewhere [20].

Expanding our result in powers of $1/m_b$, we obtain

$$S(y) = \sum_{n=1}^{\infty} \frac{1}{n!}\,\delta^{(n-1)}(1 - y) \left(\frac{2}{m_b}\right)^n (v - \hat{p})_{\mu_1} \cdots (v - \hat{p})_{\mu_n}\, \langle\, iD^{\mu_1} \cdots iD^{\mu_n} \rangle + \text{ less singular terms.} \tag{18}$$

The forward scattering matrix elements between B mesons, or between any other hadronic states that are unpolarized, can be parametrized in the form

$$\langle\, iD^{\mu_1} \cdots iD^{\mu_n} \rangle = A_n\, v^{\mu_1} \cdots v^{\mu_n} + \text{terms with } g^{\mu_i \mu_j}. \tag{19}$$

Since $(v - \hat{p})^2 = (1 - y)$ vanishes at the end point, only the coefficients A_n contribute to the most singular terms in $S(y)$. For the same reason, we can replace factors $2v \cdot (v - \hat{p}) = (2 - y)$, which arise upon contraction of the indices in (18), by 1. This leads to the following expression for the shape function:

$$S(y) = \sum_{n=1}^{\infty} \frac{1}{n!}\,\frac{A_n}{m_b^n}\,\delta^{(n-1)}(1 - y) + \text{ less singular terms.} \tag{20}$$

From a comparison with (9) and (14), we obtain for the moments of $S(y)$:

$$M_n = \frac{1}{(n + 1)}\,\frac{A_{n+1}}{m_b^{n+1}}\,, \quad a_n = \frac{A_{n+1}}{n + 1}\,. \tag{21}$$

The first three coefficients A_n are given by $A_0 = 1$, $A_1 = 0$, and $A_2 = -\lambda_1/3$.

V. A MODEL CALCULATION

It is instructive to consider a simple model for the shape function $S(y)$. For this purpose, we evaluate the expectation value in (16) adopting a simplified version of the phenomenological approach of Altarelli *et al.* (ACM) [14]. We emphasize, however, that this is mainly meant as an illustrative example rather than a prediction of the physical shape function. In fact, we will see very clearly the limitations and shortcomings of this model.

In the ACM model, one assumes the validity of the parton model and incorporates bound state effects by assigning a momentum distribution $\phi(|\mathbf{p}_b|)$ to the heavy quark inside the B meson at rest.[4] It is then appropriate to replace the covariant derivative in (16) by the spatial components of the heavy quark momentum $\mathbf{p}_b$. The gluon field in the covariant derivative is neglected. Accordingly, in the rest frame of the B meson, one makes the replacement

$$\frac{2}{m_b}\,(v - \hat{p}) \cdot iD \to \frac{2\,\mathbf{p}_\ell \cdot \mathbf{p}_b}{m_b^2} = \frac{y\,|\mathbf{p}_b|}{m_b}\,\cos\vartheta\,, \tag{22}$$

where ϑ is the angle between the lepton and the heavy quark momentum. Since we are interested in the behavior in the end-point region, we can set $y = 1$. The matrix element in (16) is now replaced by an integral over the momentum distribution of the heavy quark:

$$S(y) = \int_0^\infty d|\mathbf{p}_b|\, |\mathbf{p}_b|^2\, \phi(|\mathbf{p}_b|) \int_{-1}^1 \frac{d\cos\vartheta}{2} \left\{\Theta\left[1 - y + \frac{|\mathbf{p}_b|}{m_b}\,\cos\vartheta\right] - \Theta(1 - y)\right\}. \tag{23}$$

It is straightforward to calculate the moments of this model shape function, and from (21) the corresponding predictions for the hadronic matrix elements A_n. We find that $A_{2n+1} = 0$, and

$$(2n + 1) A_{2n} = \langle\, |\mathbf{p}_b|^{2n} \rangle = \int_0^\infty d|\mathbf{p}_b|\, |\mathbf{p}_b|^{2(n+1)}\, \phi(|\mathbf{p}_b|)\,. \tag{24}$$

In the ACM model, one assumes a Gaussian distribution,

$$\phi(|\mathbf{p}_b|) = \frac{4}{\sqrt{\pi}\, p_F^3}\, \exp\left(-\,\frac{|\mathbf{p}_b|^2}{p_F^2}\right), \tag{25}$$

where p_F is the Fermi momentum. This leads to the shape function

$$S(y) = \left[\frac{1}{2} - \Theta(1 - y)\right] \Phi\left(\frac{m_b}{p_F}|y - 1|\right). \tag{26}$$

$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2}$ denotes the error function. For the coefficients A_{2n}, one obtains

$$A_{2n} = \frac{(2n - 1)!!}{2^n}\, p_F^{2n}\,. \tag{27}$$

[4] In addition, the heavy quark mass is treated as a momentum dependent parameter. For simplicity, we shall not consider this aspect of the model.

Comparing this to the general relation $A_2 = -\lambda_1/3$, we derive

$$-\lambda_1 = \frac{3}{2}\,p_F^2 \simeq 0.08 \text{ GeV}^2, \tag{28}$$

where we have used $p_F \simeq 230$ MeV as obtained from the most recent fit of the ACM model to experimental data [21]. Our model thus predicts a rather small value of $-\lambda_1$.

In Fig. 3, we show the shape function (26) for three different values of the Fermi momentum. We stress again that this simple model calculation is presented for pedagogical purposes only. In particular, note that the even moments of the shape function (corresponding to odd coefficients A_{2n+1}) vanish by rotational invariance. This is a consequence of the fact that one replaces the *operator* of the covariant derivative by a c-number momentum vector. Whereas in QCD the commutator of two covariant derivatives gives the gluon field strength tensor, in the model this commutator vanishes. However, exactly those terms involving the gluon field are responsible for an asymmetry in the shape function around $y = 1$. To see this, consider the matrix element of three covariant derivatives. Using the equations of motion of HQET, it is easy to show that

$$\langle\, iD^\mu\, iD^\nu\, iD^\alpha \,\rangle = A_3\,(v^\mu v^\alpha - g^{\mu\alpha})\,v^\nu\,, \tag{29}$$

and taking the antisymmetric combination in μ and ν, we find that A_3 is related to a matrix element involving the gluon field strength tensor:

$$\langle\, ig_s\, G^{\mu\nu}\, iD^\alpha \,\rangle = A_3\,(v^\mu g^{\nu\alpha} - v^\nu g^{\mu\alpha})\,. \tag{30}$$

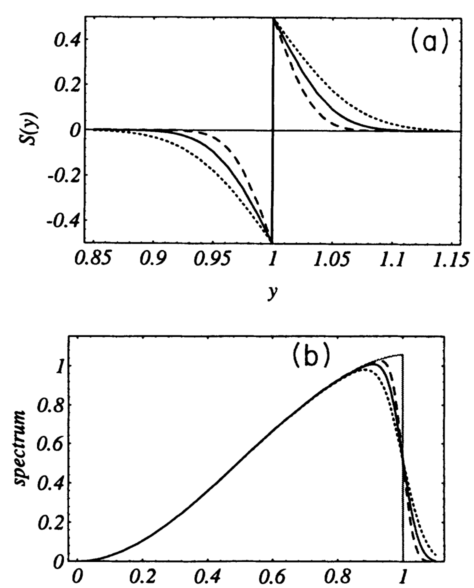

FIG. 3. The shape function of the toy model (a) and the corresponding lepton spectrum (b) for $-\lambda_1 = 0.05$ GeV^2 (dashed), 0.1 GeV^2 (solid), and 0.2 GeV^2 (dotted). The corresponding values of the Fermi momentum are $p_F \simeq 180$ MeV, 260 MeV, and 365 MeV, respectively. In (b), the parton model spectrum is shown as a grey line.

In QCD, there is no reason why such a matrix element should vanish. Hence, we expect an asymmetry of the physical shape function. The above example is instructive since it shows that a measurement of the moments of the shape function can provide quite fundamental information about the dynamical properties of the theory of strong interactions.

VI. SUMMARY AND CONCLUSIONS

We have presented a QCD-based approach to the inclusive lepton energy spectrum in $\bar{B} \to X_u\,\ell\,\bar{\nu}$ decays. We have introduced the concept of a shape function, which is a genuinely nonperturbative object that describes the rapid falloff of the spectrum in the end-point region. The moments of this function obey a very simple relation to forward scattering matrix elements of local, higher-dimension operators. QCD predicts that the leading contribution to the first moment vanishes. The second moment, which is a measure of the size of the end-point region, is proportional to the expectation value of the kinetic energy of the heavy quark inside the hadron.

Our approach goes beyond previous work on the lepton spectrum [7–9], which was applicable only for lepton energies not too close to the end point. It aims at a systematic use of the rich source of information contained in the end-point region. Whereas the main part of the spectrum is determined by kinematics and only receives small nonperturbative corrections, the behavior close to the end point is characterized by an infinite set of hadronic matrix elements. It is worth noting that, although there are no first-order power corrections to the spectrum and decay rate at small lepton energies, there exists a small region where the true spectrum is very different from the theoretical prediction of Refs. [7–9].

There are obvious improvements of the analysis presented here. Before confronting our results with data, it is necessary to include radiative corrections. For the case of $\bar{B} \to X_u\,\ell\,\bar{\nu}$ decays, they are known to affect the parton model spectrum in a significant way [13, 14]. Such corrections will affect the form of the shape function, too. We expect perturbative corrections of order $\alpha_s(\Lambda m_b)$ to the (appropriately defined) moments of the shape function. Moreover, radiative corrections will wash out the step in the shape function at $y = 1$, resulting in a rapidly varying, but not discontinuous, behavior. Another important generalization of our approach is that to the case of $\bar{B} \to X_c\,\ell\,\bar{\nu}$ decays. The nonvanishing mass of the charm quark will lead to technical complications, but conceptually there is no problem in defining a shape function $S(y,\rho)$ for a nonzero value of $\rho = m_c^2/m_b^2$. The (appropriately defined) moments of this generalized shape function are related to the same hadronic matrix elements as in the case of a massless final state quark. Work on these issues is in progress and will be reported elsewhere [20].

We believe that our approach will eventually lead to a better understanding of the nonperturbative aspects of inclusive decay spectra, with implications for the measurement of some of the fundamental parameters of the standard model, such as the heavy quark masses and the elements V_{cb} and V_{ub} of the Kobayashi-Maskawa matrix. Currently, the most promising applications of the method seem to be the following.

In B decays into charmless final states, an understanding of the end-point region is crucial for a reliable determination of V_{ub}. The approach of Refs. [7–9] cannot be used for this purpose, since current measurements are limited to a small energy range $2.3 \le E_\ell \le 2.6$ GeV (corresponding to $0.96 \le y \le 1.08$ for $m_b = 4.8$ GeV), which is too close to the end point. What is needed is some insight into the nonperturbative effects relevant to the shape of the spectrum in the end-point region. An expansion in powers of $1/m_b$ is not suitable for such a situation. The relevant physics is encoded in the moments of the shape function, which are related to forward scattering matrix elements of local, higher-dimension operators. Such matrix elements can be addressed using nonperturbative techniques such as lattice gauge theory or QCD sum rules. It may even be possible in these approaches to attempt a direct calculation of the shape function from its definition in (16).

For $\bar{B} \to X_c\,\ell\,\bar{\nu}$ transitions, the situation is very different. Already, there exist very accurate measurements of the lepton spectrum in this case. For a determination of V_{cb} and of the quark masses m_b and m_c, an understanding of the end-point region is thus not a necessary requirement. However, these decays offer the exciting possibility to extract the shape function from the data, by subtracting the (corrected) parton model spectrum from the measured distribution. One could then compute the first few moments of the shape function and extract some of the coefficients A_n, which encode fundamental dynamical properties of QCD. The fact that QCD predicts the size of the first moment M_0 in (13) provides an important constraint, which can help to obtain a precise determination of the b-quark mass.

We conclude with a speculation about yet another possibility, namely to combine the analyses of $\bar{B} \to X_c\,\ell\,\bar{\nu}$ and $\bar{B} \to X_u\,\ell\,\bar{\nu}$ decays. One can imagine measuring the shape function in B decays into charmed particles, and then predict the shape function for charmless transitions. This avenue may be a promising one with respect to a precise extraction of V_{ub}.

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