Nonrelativistic effective Lagrangians

H. Leutwyler

Institut für Theoretische Physik der Universität Bern, Sidlerstr. 5, CH-3012 Bern, Switzerland

Chiral perturbation theory is extended to nonrelativistic systems with spontaneously broken symmetry. In the effective Lagrangian, order parameters associated with the generators of the group manifest themselves as effective coupling constants of a topological term, which is gauge invariant only up to a total derivative. In the case of the ferromagnet, a term connected with the Brouwer degree dominates the derivative expansion. The general analysis includes antiferromagnetic magnons and phonons, while the effective field theory of fluids or gases is beyond the scope of the method.

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I. INTRODUCTION

The various low energy phenomena considered in the present paper are very well explored, at a level which goes much beyond the general discussion given below. The aim of the paper is not to contribute to the detailed physical understanding of the many different systems known to exhibit spontaneous symmetry breakdown, but to analyze their low energy structure from a unified point of view, relying on the method of effective Lagrangians. This method is widely used in condensed matter physics [1], but, as far as I know, a general analysis is not available. In particular, an effective Lagrangian describing the behavior of a ferromagnet at large wavelengths does not appear to exist in the literature. The main result of the present work is an expression for the general effective Lagrangian. As it turns out, the expression contains a term of rather remarkable structure, which distinguishes ferromagnets from other systems.

The analysis is based on general considerations, applicable to any system, for which the Goldstone modes represent the only excitations without an energy gap. It amounts to an extension of the effective theories used in particle physics [2–6] to the nonrelativistic domain. This extension is by no means trivial. The relativistic situation is considerably simpler, because Lorentz invariance imposes strong constraints on the low energy structure of the theory and, e.g., prevents the charge densities from picking up an expectation value in the ground state. These constraints do not apply to condensed matter, where the center of mass distinguishes a preferred frame of reference. Moreover, the lattice structure of a solid singles out preferred directions, such that the effective Lagrangian is not invariant under rotations either. In the case of a cubic lattice, the anisotropy, however, only shows up in the higher orders of the derivative expansion. As the following discussion mainly concerns the leading contributions, I disregard this complication and assume that, at large distances, the correlation functions are invariant, both under translations and rotations.

I consider a spontaneously broken exact symmetry in $d = 3 + 1$ dimensions (spontaneous breakdown of symmetries only occurs for $d > 2$ — the low energy behavior of the two-dimensional nonlinear σ-model, e.g., cannot be analyzed in terms of an effective Lagrangian [7]). The Hamiltonian is symmetric with respect to a Lie group G with generators Q_i,

$$[Q_i, H] = 0, \qquad [Q_i, Q_j] = if_{ij}^k Q_k \ , \tag{1}$$

but the ground state $|0\rangle$ is invariant only under a subgroup $H \subset G$. For Lorentz invariant theories, the Goldstone theorem [8] states that the spontaneous symmetry breakdown gives rise to $\dim(G) - \dim(H)$ massless particles. In the nonrelativistic regime, the occurrence of order parameters also implies that there are modes of excitation, for which the frequency ω disappears when the wave vector $\mathbf{k}$ tends to zero, but the number of independent such states and their dispersion law depend on the properties of the system [1, 9].

The generators Q_i of G are space integrals over the corresponding charge densities:

$$Q_i = \int d^3x J_i^0(x) \ . \tag{2}$$

Identifying the zeroth component of the coordinate vector with the time ($x^0 = t$, no factor of c), charge conservation takes the local form

$$\partial_\mu J_i^\mu(x) \equiv \partial_0 J_i^0(x) + \partial_r J_i^r(x) = 0 \ . \tag{3}$$

I am using the external field method and study the response of the system to the perturbations generated by a slowly varying external field. In the framework of effective field theories, the method was introduced in connection with the low-energy structure of the strong interactions [6]. In the context of condensed matter physics, the power of the external field method is demonstrated in a remarkable series of papers by Fröhlich and collaborators [10].

Denote the ground state of the Hamiltonian by $|0\rangle$ and consider the perturbations caused by external fields $f_0^i(x)$ and $f_r^i(x)$, coupled to the charge densities $J_i^0(x)$ and currents $J_i^r(x)$, respectively, $H \rightarrow H - \int d^3x f_\mu^i J_i^\mu$. The generating functional $\Gamma\{f\}$ describes the transitions generated by the perturbation: $e^{i\Gamma\{f\}}$ is the probability amplitude for the system to remain in the ground state for $t \rightarrow +\infty$, if it was there at $t \rightarrow -\infty$. This amplitude is determined by the correlation functions of the unperturbed system,

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$$e^{i\Gamma\{f\}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n f_{\mu_1}^{i_1}(x_1) \cdots f_{\mu_n}^{i_n}(x_n)\langle 0|T\{J_{i_1}^{\mu_1}(x_1) \cdots J_{i_n}^{\mu_n}(x_n)\}|0\rangle. \tag{4}$$

The generating functional thus collects the time-ordered correlation functions of the charge densities and currents.

The correlation functions play a central role in the analysis of the low-energy structure. The construction of the effective theory relies on the Ward identities, which express the symmetry properties of the system in terms of these quantities. If the theory does not contain anomalies, the Ward identities are equivalent to the statement that the generating functional is invariant under gauge transformations of the external field:

$$\delta f_\mu^i(x) = D_\mu g^i(x)$$

$$\equiv \partial_\mu g^i(x) + f_{jk}^i f_\mu^j(x) g^k(x) \rightarrow \delta\Gamma\{f\} = 0 \ . \tag{5}$$

The gauge functions $g^1(x), g^2(x), \ldots$ are arbitrary infinitesimal quantities. They may be viewed as coordinates of a space-time-dependent group element $g(x) \in G$ in the infinitesimal neighborhood of unity.

The low energy analysis concerns the behavior of the correlation functions at distances large compared to the intrinsic scales of the theory. In particular, the distances under consideration are assumed to be large compared to the lattice spacing a — the effective theory does not resolve the microscopic structure of the system, i.e., refers to the continuum limit. In the language of the generating functional, the effective theory concerns slowly varying external fields, such that $\partial f/\partial x \ll f/a$.

The Fourier transforms of the various correlation functions contain singularities at low energies and momenta, due to the propagation of Goldstone excitations. The singularities arise from processes involving the emission of Goldstone bosons, which travel over a long distance before being absorbed. In particular, one-particle-reducible contributions generate poles, while the simultaneous exchange of several Goldstone modes produces cuts.

II. EFFECTIVE LAGRANGIAN

The following discussion exclusively deals with the contributions due to the Goldstone excitations. As witnessed by superconductivity or by the Higgs sector of the standard model, the presence of additional degrees of freedom without an energy gap may change the low energy structure, even qualitatively: Gauge fields may absorb the Goldstone bosons. In the following, it is assumed that, at low frequencies and large wavelengths, the spectrum exclusively contains Goldstone excitations. More precisely, the discussion relies on the PCAC (partial conservation of axial-vector current) hypothesis, according to which the poles generated by the Goldstone bosons dominate the low energy behavior of the correlation functions.

As is well known, the singularities due to the exchange of Goldstone bosons may be described in terms of an effective field theory [2, 3]. I refer to the variables of the effective theory as "pion" fields, using the symbol $\pi^a(x)$ (in the applications to be discussed below, the "pi-

ons" represent magnons or phonons). Unlike the number of Goldstone *particles*, which depends on the form of the dispersion law, the number of *fields* needed to describe them is universal: The effective theory involves $\dim(G) - \dim(H)$ real fields. If the dispersion law is of the form $\omega(\mathbf{k}) = v|\mathbf{k}| + O(k^2)$, as is the case for Lorentz invariant theories, the number of independent one-particle states of momentum $\mathbf{k}$ is the same as the number of fields. For a dispersion law of the type $\omega(\mathbf{k}) = \gamma\mathbf{k}^2 + O(k^4)$, on the other hand, the number of states is given by $\frac{1}{2}\{\dim(G) - \dim(H)\}$. The difference is related to the order of the corresponding wave equations. In the first case, the wave equation is of second order in the time derivatives. The Fourier decomposition then contains both positive and negative frequencies and a real field suffices to describe a particle. In the second case, the wave equation takes the form of the Schrödinger equation, such that only positive frequencies occur and a complex field is needed per particle.

In the language of the effective field theory, the one-particle-reducible contributions responsible for the poles are represented by the tree graphs. The pole terms arise from pion field propagators, whose form is specified by the kinetic part of the effective Lagrangian, i.e., by the part which is quadratic in the pion field. The interaction terms of the effective Lagrangian are in one-to-one correspondence with the amplitudes for emission, absorption, and scattering. In addition to the purely pionic vertices, describing the interaction of the Goldstone bosons among themselves, the Lagrangian also contains vertices involving the external field, which describe the transitions generated by the perturbation $f_\mu^i J_i^\mu$. The matrix element $\langle 0|f_\mu^i J_i^\mu|\pi\rangle$, e.g., which represents the probability amplitude for the external field to excite one of the Goldstone states, is represented in the effective Lagrangian through a term linear in the fields $f_\mu^i(x)$, $\pi^a(x)$.

The low energy analysis relies on an expansion of the vertices in powers of the momenta. In the language of effective field theory, this corresponds to a derivative expansion of the Lagrangian, $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(\pi, \partial_\mu\pi, \partial_\mu\partial_\nu\pi, \ldots; f, \partial_\mu f, \ldots)$. The generic term occurring therein contains P pion fields, E external fields, and, altogether, D derivatives, some acting on $\pi^a(x)$, some on $f_\mu^i(x)$. It is convenient to count the external field on the same footing as the derivatives, $f_\mu^i \propto \partial_\mu$, but to distinguish between the time and space components of these quantities. The derivative expansion then consists of a double series of the form

$$\mathcal{L}_{\text{eff}} = \sum_{s, t} \mathcal{L}_{\text{eff}}^{(s, t)} \ . \tag{6}$$

The term $\mathcal{L}_{\text{eff}}^{(0,0)}$ exclusively contains the pion field and does not involve derivatives, $\mathcal{L}_{\text{eff}}^{(0,1)}$ collects the purely pionic vertices with one time derivative, as well as those involving one factor of f_0^i, but no derivatives, etc. Note that the number of Goldstone bosons entering the ver-

tices is not specified — the various terms occurring in the derivative expansion represent functions of the pion field.

The tree graphs yield the leading term in the low energy expansion of the generating functional, loops only generating corrections of nonleading order [5]. The tree graphs of a quantum field theory represent the corresponding classical field theory. More precisely, the tree graph contributions to the generating functional are given by the classical action

$$\Gamma\{f\}\big|_{\text{tree}} = S_{\text{eff}}\{\pi, f\},$$

$$S_{\text{eff}}\{\pi, f\} \equiv \int d^4x \mathcal{L}_{\text{eff}}(\pi, \partial\pi, \ldots; f, \partial f, \ldots) \ . \quad (7)$$

The action is to be evaluated at the extremum, where the pion field obeys the classical equation of motion

$$\frac{\delta S_{\text{eff}}\{\pi, f\}}{\delta \pi^a(x)} = 0 \ . \quad (8)$$

The Ward identities are obeyed if and only if the generating functional is gauge invariant. For this to be the case at leading order of the low energy expansion, the value of the classical action at the extremum must be gauge invariant:

$$D_\mu \frac{\delta S_{\text{eff}}\{\pi, f\}}{\delta f^i_\mu(x)} = 0 \ . \quad (9)$$

The pion field thus simultaneously obeys the two differential equations (8) and (9). While the first one is the standard equation of motion, the second incorporates the Ward identities connected with the hidden symmetry and very strongly constrains the form of the Lagrangian. In fact, this constraint determines the leading terms of the derivative expansion up to a few constants, which play the role of effective coupling constants (a detailed analysis of the same two differential equations for the case of a Lorentz invariant effective theory is given in Ref. [11]).

III. LEADING ORDERS OF DERIVATIVE EXPANSION

For the framework to be internally consistent, the form of the two differential equations (8) and (9) must be compatible with the derivative expansion. The leading term occurring in that expansion, $\mathcal{L}_{\text{eff}}^{(0,0)}$, does not contain derivatives of the pion field. To leading order, the "equation of motion" then reduces to a purely algebraic condition on this field, $\partial \mathcal{L}_{\text{eff}}^{(0,0)}(\pi)/\partial \pi^a = 0$. It is evident that the loop expansion does not make sense if the kinetic term only occurs among the higher order corrections: If this were so, the pions would not propagate at all, the "propagator" taking the form of a δ function. Indeed, it is well known that the hidden symmetry not only protects the Goldstone bosons from acquiring mass, but also suppresses their mutual interactions at low energies. Current conservation implies that all of the vertices disappear if the momenta become small, such that purely pionic vertices without derivatives do not occur, $\mathcal{L}_{\text{eff}}^{(0,0)} = 0$.

The derivative expansion of the effective Lagrangian

thus starts with $\mathcal{L}_{\text{eff}}^{(0,1)}$. Invariance under space rotations permits two contributions of this order:[1]

$$\mathcal{L}_{\text{eff}}^{(0,1)} = c_a(\pi)\dot{\pi}^a + e_i(\pi)f^i_0 \ . \quad (10)$$

The space derivatives of the pion field only show up at the next order of the expansion, where the general form of the Lagrangian consistent with rotation symmetry reads

$$\mathcal{L}_{\text{eff}}^{(2,0)} = -\tfrac{1}{2}g_{ab}(\pi)\,\partial_r\pi^a\partial_r\pi^b + h_{ai}(\pi)f^i_r\partial_r\pi^a$$
$$\quad\quad -\tfrac{1}{2}k_{ik}(\pi)f^i_sf^k_s \ ,$$

$$\mathcal{L}_{\text{eff}}^{(0,2)} = \tfrac{1}{2}\bar{g}_{ab}(\pi)\,\dot{\pi}^a\dot{\pi}^b - \bar{h}_{ai}(\pi)f^i_0\,\dot{\pi}^a + \tfrac{1}{2}\bar{k}_{ik}(\pi)f^i_0f^k_0 \ . \quad (11)$$

Note that terms involving second derivatives of the pion field or first derivatives of the external field may be removed by adding a suitable term of the form $\partial_\mu\omega^\mu$, which does not contribute to the action.

The term $\mathcal{L}_{\text{eff}}^{(0,1)}$ does not occur in Lorentz invariant effective theories — it represents the main novelty in the extension of these to nonrelativistic systems [in addition, Lorentz invariance implies that the functions $\bar{g}_{ab}(\pi), \bar{h}_{ai}(\pi), \bar{k}_{ik}(\pi)$ coincide with the corresponding unbarred quantities, up to a factor of c^2]. The value of $e_i(\pi)$ at $\pi = 0$ yields a term in the effective Lagrangian, which is linear in the external field and hence determines the one-point-function:

$$\langle 0|J^0_i(x)|0\rangle = e_i(0) \ . \quad (12)$$

For non-Abelian symmetries, the charge densities transform in a nontrivial manner under G, such that their expectation values represent order parameters. The ground state of a *ferromagnet*, e.g., singles out a direction of the magnetization, given by the expectation value of the spin density. The corresponding "charges" generate the group $G = O(3)$ of spin rotations. The direction of the magnetization need not be correlated with the orientation of the lattice. For the Heisenberg model, e.g., the spin rotations play the role of an internal symmetry; at long wavelengths, the Green functions of this model are indeed invariant under Euclidean transformations of three-dimensional space, as it is assumed here. The case of the *antiferromagnet* shows, however, that the expectation values of the charge densities are not necessarily different from zero. The constants $e_i(0)$ represent coupling constants of the effective Lagrangian; symmetry alone does not tell what values these constants take.

IV. SYMMETRY PROPERTIES OF THE LAGRANGIAN

As discussed above, the pion field must simultaneously obey the equation of motion (8) and the Ward identity

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[1]Notation: $i, j, k = 1, \ldots, \dim(G)$ label the generators of the group, $a, b, c = 1, \ldots, \dim(G) - \dim(H)$ denote the components of the effective field, and $r, s, t = 1, 2, 3$ refer to the spatial coordinates. Repeated indices are summed over.

(9). In general, the leading term in the derivative expansion of the equation of motion is of first order in the time derivative, while the space derivatives only enter at second order, through a term from $\mathcal{L}_{\mathrm{eff}}^{(2,0)}$, proportional to $\triangle\pi$. The equation of motion thus takes the form of a Schrödinger equation, leading to a dispersion law of the type $\omega \propto \mathbf{k}^2$. It is convenient to organize the bookkeeping accordingly, counting energies like two powers of momenta. The terms $\mathcal{L}_{\mathrm{eff}}^{(0,1)}$ and $\mathcal{L}_{\mathrm{eff}}^{(2,0)}$ then represent expressions of the same order k^2, while the remainder of the derivative expansion is of order k^3 or higher. The Ward identity is of the same form as the equation of motion, also relating $\dot{\pi}$ to $\triangle\pi$. The two equations are consistent with one another only if they are linearly dependent. Solving the equation of motion for $\dot{\pi}$ and inserting the result in the Ward identity, one obtains a relation which only involves the pion field, its spatial derivatives, and the external field. Since these are independent from one another, the condition is obeyed only if the coefficients occurring therein are equal to zero. This subjects the functions $c_a(\pi), e_i(\pi), g_{ab}(\pi), h_{ai}(\pi), k_{ik}(\pi)$, which specify the vertices of the effective Lagrangian, to the conditions:

(a) $\quad d_i h^a_{\ j} - d_j h^a_{\ i} = f^k_{\ ij} h^a_{\ k}$,

(b) $\quad \nabla_a h_{bi} + \nabla_b h_{ai} = 0$,

(c) $\quad k_{ik} = g^{ab} h_{ai} h_{bk}$,

(d) $\quad d_i e_j = f^k_{\ ij} e_k$,

(e) $\quad h^b_{\ i}(\partial_b c_a - \partial_a c_b) = \partial_a e_i$. $\qquad$ (13)

To simplify these formulas, I have used the following notation: The matrix $g_{ab}(\pi)$ plays the role of a metric on the manifold of pion field variables. Indices are lowered and raised with this metric and its inverse, $g^{ab}(\pi)$, e.g., $h^a_{\ i} = g^{ab} h_{bi}$. The symbol ∇_a is the corresponding covariant derivative, formed with the Christoffel symbol, $\nabla_a h_{bi} = \partial_a h_{bi} - \Gamma^c_{ab} h_{ci}$, and d_i stands for the differential operator $d_i = h^a_{\ i}(\pi)\partial_a$. Note that the above relations exclusively involve derivatives with respect to the pion field variables, which represent the arguments of the functions occurring in the effective Lagrangian, $\partial_a \equiv \partial/\partial\pi^a$.

The first three relations are identical with those relevant in the relativistic case, where $c_a(\pi) = e_i(\pi) = 0$. They state that the metric $g_{ab}(\pi)$ describes a symmetric space with isometry group G. The functions $h^a_{\ i}(\pi)$ represent the corresponding Killing vectors, which specify the shift in the pion field generated by infinitesimal group motions:

$$\delta\pi^a = h^a_{\ i}(\pi)\,g^i \ . \qquad (14)$$

The geometry of the groups G and H fixes the functions $h^a_{\ i}(\pi)$, except for the choice of field variables. The symmetry also very strongly constrains the form of the metric. In particular, if the Goldstone bosons transform irreducibly under H, the metric is fixed up to an effective coupling constant F: Denoting the intrinsic metric of the quotient space G/H by $\hat{g}_{ab}(\pi)$, the metric relevant for the effective Lagrangian is given by $g_{ab}(\pi) = F^2 \hat{g}_{ab}(\pi)$. A detailed discussion of these statements is given in Ref. [11], where it is also shown that the conditions (a), (b),

and (c) ensure invariance of $\mathcal{L}_{\mathrm{eff}}^{(2,0)}$ under a simultaneous gauge transformation of the fields $f^i_\mu(x)$ and $\pi^a(x)$. For the case of an Abelian symmetry, the coordinates may be chosen such that both the Killing vectors and the metric are constants, $h^a_{\ i}(\pi) = h^a_{\ i}(0)$, $g_{ab}(\pi) = g_{ab}(0)$.

The new couplings $e_i(\pi)$ and $c_a(\pi)$ only occur in the conditions (d) and (e). The first one of these states that, under the transformation (14) of the pion field, the vector $e_i(\pi)$ transforms according to the adjoint representation $D^i_{\ j}(g) = \delta^i_{\ j} + f^i_{\ jk} g^k + \cdots$ of G. Since the action of the group is transitive on G/H, this property fully determines the function $e_i(\pi)$ in terms of its values for $\pi = 0$, i.e., in terms of the magnetization. The relation (e) then specifies the rotation of $c_a(\pi)$ and thus fixes the function itself up to a gradient. The reason why $c_a(\pi)$ is not fully determined is that one may modify the Lagrangian by a total derivative without changing the generating functional: The operation $c_a(\pi) \to c_a(\pi) + \partial_a\omega(\pi)$ is equivalent to $\mathcal{L}_{\mathrm{eff}} \to \mathcal{L}_{\mathrm{eff}} + \frac{\partial}{\partial t}\omega(\pi)$. Except for this ambiguity, which is without physical significance, the effective coupling constants of the new vertices are fully determined by the order parameters $\langle 0|J^0_i|0\rangle$.

In the case of an Abelian symmetry, $f^k_{\ ij} = 0$, the relation (d) shows that $e_i(\pi)$ is a constant and the condition (e) then implies that $c_a(\pi)$ is a pure gradient and may thus be removed, $c_a(\pi) = 0$. Accordingly, the term $\mathcal{L}_{\mathrm{eff}}^{(0,1)}$ takes the form $e_i(0)f^i_0$. Since this expression does not involve the pion field, it leads a life of its own, exclusively generating an expectation value for the charge densities. For Abelian symmetries, the equation of motion is therefore of second order in the time derivative, such that the dispersion law takes the form $\omega \propto |\mathbf{k}|$.

From a methodical point of view, the most remarkable property of the new couplings is that the corresponding contribution to the effective Lagrangian in general fails to be gauge invariant. Subjecting the fields f^i_μ and π^a to the infinitesimal gauge transformations (5) and (14) and using the relations (d) and (e), one finds that the effective Lagrangian picks up a total derivative:

$$\delta\mathcal{L}_{\mathrm{eff}}^{(0,1)} = \frac{\partial}{\partial t}\{g^i\left[c_a(\pi)h^a_{\ i}(\pi) + e_i(\pi)\right]\} \ . \qquad (15)$$

Now, this may merely be due to a bad convention. If the expression in square brackets is of the form $h^a_{\ i}(\pi)\partial_a\omega(\pi)$, it suffices to modify the Lagrangian by a total derivative, to make it gauge invariant [$\mathcal{L}_{\mathrm{eff}} \to \mathcal{L}_{\mathrm{eff}} + \frac{\partial}{\partial t}\omega(\pi)$]. So, if the function $e_i(\pi)$ is of the form $e_i(\pi) = h^a_{\ i}(\pi)\bar{e}_a(\pi)$, with $\bar{e}_a = -(c_a + \partial_a\omega)$, there is no problem with gauge invariance. The condition, in particular, requires the vector $e_i(0)$ to be contained in the subspace spanned by the Killing vectors at $\pi = 0$.

Denote the Lie algebras of G and H by G and H, respectively, and set G = H + K. The Killing vectors span the subspace K. Hence, for the Lagrangian to be gauge invariant, the vector $e_i(0)$ must be contained in this subspace.

The Lie algebra G transforms with the adjoint representation of G. The corresponding representation of the subgroup H maps the two subspaces H and K onto themselves; in particular, K carries a representation of H. Since the order parameters $e_i(0)$ are invariant under

H, this representation must contain a one-dimensional invariant subspace. Unless this is the case, the charge densities can receive nonzero expectation values only if the Lagrangian violates gauge invariance. For the ferromagnet, e.g., the magnetization $e_i(0)$ belongs to H rather than K — the corresponding effective Lagrangian necessarily breaks gauge invariance.

V. FERROMAGNET

I now discuss the case of the ferromagnet in some detail, i.e., consider the groups $G = \mathrm{O}(3)$, $H = \mathrm{O}(2)$. The corresponding structure constants are given by $f^i_{jk} = \varepsilon_{ijk}$; there are three conserved currents, $i = 1, 2, 3$, and two pion fields, $a = 1, 2$. In the Heisenberg model, e.g., the magnetic moment of the lattice sites may be represented as $\mu\mathbf{s}_n$, where $\mathbf{s}_n$ is the spin of the site. In the notation used here, the interaction with a constant magnetic field, $\mu\sum_n \mathbf{s}_n \cdot \mathbf{H}$, corresponds to the term $\int d^3x f^i_0 J^0_i$. The spin rotations are generated by the total angular momentum, such that $\sum_n \mathbf{s}_n = \int d^3x \mathbf{J}^0$. Accordingly, the time components of the external field are related to the magnetic field by $f^i_0 = \mu H^i$.

It is convenient to use a covariant representation for the pion field, replacing the two variables π^1, π^2 by a three-dimensional unit vector $\mathbf{U} = (U^1, U^2, U^3)$, which transforms with the fundamental representation of $\mathrm{O}(3)$. The nonlinear transformation law (14) then takes the linear form $\delta U^i = \varepsilon_{ijk}U^j g^k$. In this notation, the term $\mathcal{L}^{(2,0)}_{\text{eff}}$ is proportional to the square of the covariant derivative of $\mathbf{U}$,

$$\mathcal{L}^{(2,0)}_{\text{eff}} = -\tfrac{1}{2}F^2 D_r U^i D_r U^i,$$
$$D_r U^i = \partial_r U^i + \varepsilon_{ijk}f^j_r U^k \ . \tag{16}$$

As mentioned above, symmetry determines the form of this part of the Lagrangian, up to one effective coupling constant F. The corresponding explicit expressions for the metric and for the Killing vectors are $g_{ab} = F^2\partial_a U^i \partial_b U^i$, $h_{ai} = F^2\varepsilon_{ijk}\partial_a U^j U^k$.

The analogous representation of the function $e_i(\pi)$ immediately follows from the completeness of the spherical harmonics on the two-sphere: There is only one set of three functions of the pion field transforming according to the fundamental representation of $\mathrm{O}(3)$. Hence the vectors $e_i(\pi)$ and $U^i(\pi)$ are proportional to one another, $e_i = \Sigma U^i$. The constant of proportionality Σ is the magnitude of the magnetization.

The expression for the function $c_a(\pi)$ is more complicated. Using the completeness relation for the Killing vectors, $\sum_i h_{ai}h_{bi} = F^2 g_{ab}$, the condition (e) may be rewritten in the form

$$\partial_a c_b - \partial_b c_a = -\Sigma\,\varepsilon_{ijk}\,\partial_a U^i\,\partial_b U^j\,U^k \ . \tag{17}$$

The right hand side is reminiscent of a topological invariant: Up to a factor of $4\pi\Sigma$, the integral over the sphere is the Brouwer degree of the map $\mathbf{U}(\pi)$.

The differential equation (17) may be integrated with the technique used in the construction of the Wess-Zumino term. Consider a point π on the sphere and join it smoothly to $\pi = 0$, along the path $\sigma^a[\pi, \lambda]$, $0 \le \lambda \le 1$,

with $\sigma^a[\pi, 0] = 0$, $\sigma^a[\pi, 1] = \pi^a$. Define the function $c_a(\pi)$ as the integral

$$c_a(\pi) = \Sigma \int_0^1 d\lambda\,\varepsilon_{ijk}\,\partial_a U^i\,\partial_\lambda U^j\,U^k \ , \tag{18}$$

with $U^i = U^i(\sigma[\pi, \lambda])$. The vectors $\partial_a\mathbf{U}$ and $\partial_\lambda\mathbf{U}$, which denote the derivatives with respect to π^a at constant λ and vice versa, are orthogonal to $\mathbf{U}$. Since the tangent plane only contains two linearly independent directions, the quantity $\varepsilon_{ijk}\,\partial_a U^i\,\partial_b U^j\,\partial_\lambda U^k$ is equal to zero. Using this property, one readily checks that the function defined in (18) indeed obeys the differential equation (17). As noted above, any other solution differs from this one by an irrelevant gradient.

Together with the contribution involving the external field, the Lagrangian thus becomes

$$\mathcal{L}^{(0,1)}_{\text{eff}} = \Sigma \int_0^1 d\lambda\,\varepsilon_{ijk}\,\partial_0 U^i\,\partial_\lambda U^j\,U^k + \Sigma f^i_0 U^i \ . \tag{19}$$

The form of the path $\sigma[\pi, \lambda]$ affects the result only through a total derivative. For the particular choice $U^i(\sigma[\pi, \lambda]) = \lambda U^i(\pi)$, $i = 1, 2$, the derivatives of the interpolating field may be expressed in terms of those of $U^i(\pi)$ and the integral may then be performed explicitly, with the result

$$\mathcal{L}^{(0,1)}_{\text{eff}} = \Sigma\,(1 + U^3)^{-1}(\partial_0 U^1 U^2 - \partial_0 U^2 U^1) + \Sigma f^i_0 U^i \ . \tag{20}$$

Visibly, the expression violates gauge invariance.

The corresponding equation of motion is obtained by evaluating the change in the action generated by a deformation of the pion field. Using the representation (19) for $\mathcal{L}^{(0,1)}_{\text{eff}}$, the calculation yields

$$\Sigma\,\varepsilon_{ijk}U^j\dot{U}^k + \Sigma f^i_0 + F^2\Delta U^i = \alpha U^i \ , \tag{21}$$

where $\Delta = D_r D_r$ is the covariant Laplacian and α is a Lagrange multiplier, arising from the constraint $\delta\mathbf{U}\cdot\mathbf{U} = 0$. The result may be rewritten in the vectorial form

$$\Sigma\,\dot{\mathbf{U}} + \Sigma\,\mathbf{f}_0\times\mathbf{U} + F^2\Delta\mathbf{U}\times\mathbf{U} = 0 \ . \tag{22}$$

Indeed, this equation is known to describe the spin waves of a ferromagnet — it is referred to as the Landau-Lifshitz equation [1,12]. The above discussion merely identifies a known model within the present framework: The Landau-Lifshitz equation is the equation of motion associated with the leading terms in the derivative expansion of the general effective Lagrangian, for $G=\mathrm{O}(3)$, $H = \mathrm{O}(2)$. The Lagrangian contains a term related to the Brouwer degree of the map $U^i(\pi)$; the corresponding effective coupling constant is the expectation value of the charge density.

The dispersion law of the spin waves may be worked out by considering the fluctuations of the field in the vicinity of the ground state $\mathbf{U}_0 = \text{const}$. Taking the magnetization to point along the third axis, $\mathbf{U}_0 = (0, 0, 1)$, the linearized equation of motion only involves the particular combination

$$f^a = f_0^a + \gamma \, \varepsilon_{ab3} \partial_r f_r^b \; , \qquad \gamma \equiv \frac{F^2}{\Sigma} \tag{23}$$

of external fields. Collecting the two transverse components of $\mathbf{U}$ in a complex field $u = U^1 + iU^2$, the equation of motion reduces to

$$-\,i\dot{u} - \gamma \, \triangle \, u = f \; , \tag{24}$$

with $f = f^1 + if^2$. So the dispersion law of the magnons takes the form

$$\omega(\mathbf{k}) = \gamma \, \mathbf{k}^2 + O(k^4) \; , \tag{25}$$

where γ is fixed by the two effective coupling constants F and Σ, according to (23).

A constant magnetic field, $f_0^i = \mu H(0,0,1)$, explicitly breaks the symmetry and generates a magnon "mass term," $\Sigma f_0^i U^i = \Sigma \mu H (1 - \frac{1}{2} u^\star u + \cdots)$, much like the quark masses explicitly break the chiral symmetry of QCD, providing the pions with mass. In the present case, the perturbation merely lifts the energy of all lattice sites by μH, such that the dispersion law remains the same, except for an overall shift, $\omega(\mathbf{k}) = \gamma \, \mathbf{k}^2 + \mu H$.

VI. CORRELATION FUNCTIONS OF A FERROMAGNET

The same coupling constants also determine the low energy behavior of the correlation functions of the charge densities and currents. The corresponding two-point functions are given by the part of the generating functional which is quadratic in the external field. At leading order of the low energy expansion, the generating functional is the classical action of $\mathcal{L}_{\mathrm{eff}}^{(0,1)} + \mathcal{L}_{\mathrm{eff}}^{(2,0)}$, evaluated at the solution of the equation of motion. Since the functional collects the *time-ordered* correlation functions, Feynman boundary conditions are relevant: The solution $U^i(x)$ is to contain only positive (negative) frequencies as $t \to +\infty \, (-\infty)$. For the combination $u = U^1 + iU^2$, the solution is given by

$$u(x) = \int d^4y \, G(x-y) f(y),$$
$$G(x) = \int \frac{d^3k \, d\omega}{(2\pi)^4} \frac{e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}}{\gamma \, \mathbf{k}^2 - \omega - i\epsilon}$$
$$= i \, \theta(t) \int \frac{d^3k}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{x} - i\gamma\mathbf{k}^2 t} \; . \tag{26}$$

Note that the Feynman solution is complex: The expression for the combination $\bar{u} = U^1 - iU^2$ does not coincide with the complex conjugate of the solution u, but is determined by the complex conjugate of f, according to $\bar{u}(x) = \int d^4y \, G(y-x) f^\star(y)$. The resulting expression for the pion field is of the form

$$U^a(x) = \int d^4y \, G_{ab}(x-y) \, f^b(y) \; , \tag{27}$$

where $G_{ab}(x)$ is the relevant Feynman propagator:

$$G_{ab}(x) = G_{ba}(-x)$$
$$= \tfrac{1}{2}\delta_{ab}\{G(x) + G(-x)\}$$
$$\quad + \tfrac{1}{2}i\varepsilon_{ab3}\{G(x) - G(-x)\} \; . \tag{28}$$

Note that $G_{ab}(x)$ describes the propagation of a single particle — although the effective theory contains two pion fields, there is only one magnon of a given momentum. The propagator may be written in the form

$$G_{ab}(x) = \varepsilon_a \varepsilon_b^\star \, G(x) + \varepsilon_b \varepsilon_a^\star \, G(-x) \quad , \quad \varepsilon_a = \tfrac{1}{\sqrt{2}}(1, -i) \; , \tag{29}$$

which explicitly shows the degeneracy of the propagation matrix.

Inserting the solution (27) in the expression for the action, one finally obtains

$$\Gamma\{f\} = \int d^4x \, \Sigma \, f_0^3 - \tfrac{1}{2}F^2 \int d^4x \, f_s^a f_s^a$$
$$+ \tfrac{1}{2}\Sigma \int d^4x \, d^4y \, f^a(x) \, G_{ab}(x-y) \, f^b(y) + \cdots, \tag{30}$$

with $f^a = f_0^a + \gamma \, \varepsilon_{ab3} \partial_r f_r^b$. The term linear in f_0^3 represents the one-point function $\langle 0|J_i^0|0\rangle = \delta_i^3 \Sigma$. The coefficient of the contribution which is quadratic in f_0^a is the leading term in the low energy expansion for the correlation function of the transverse charge densities,

$$\langle 0|T\{J_a^0(x)J_b^0(0)\}|0\rangle = (-i)\Sigma \, G_{ab}(x) + \cdots \; . \tag{31}$$

The Fourier transform thereof contains a pole, whose residue represents the square of the transition matrix element $\langle 0|J_a^0|\pi(\mathbf{k})\rangle$ between the ground state and a magnon of momentum $\mathbf{k}$. Using the nonrelativistic normalization

$$\langle \pi(\mathbf{k}')|\pi(\mathbf{k})\rangle = (2\pi)^3 \delta^3(\mathbf{k}' - \mathbf{k}) \; , \tag{32}$$

the result for the matrix element reads

$$\langle 0|J_a^0|\pi(\mathbf{k})\rangle = \varepsilon_a \, \sqrt{\Sigma} \; . \tag{33}$$

Euclidean invariance requires the corresponding matrix element of the currents to be proportional to the vector $\mathbf{k}$. Current conservation then shows that the coefficient of proportionality is given by

$$\langle 0|J_a^s|\pi(\mathbf{k})\rangle = k^s \varepsilon_a \, \gamma \, \sqrt{\Sigma} = k^s \varepsilon_a \, F^2/\sqrt{\Sigma} \; . \tag{34}$$

The corresponding expression for the correlation function of the currents is obtained by extracting the part of the generating functional which is quadratic in $f_r^a(x)$. The result reads

$$\langle 0|T\{J_a^r(x)J_b^s(0)\}|0\rangle = iF^2\gamma \, \partial_r \partial_s G_{ab}(x)$$
$$+ iF^2\delta_{rs}\delta_{ab}\delta^4(x) + \cdots \; . \tag{35}$$

The contact contribution $\propto \delta^4(x)$ is required by the Ward identities; it arises from the second term on the right hand side of (30).

With the above explicit form of the effective Lagrangian, it is a matter of straightforward calculation to work out magnon-magnon scattering amplitudes and to establish a low energy theorem analogous to Weinberg's prediction for the scattering lengths of $\pi\pi$ scattering [2]. Likewise, the expansion of the magnetization in

powers of the temperature may be evaluated by repeating the analogous calculation for the quark condensate [13], where the expansion has been worked out to order T^6. In that work, the explicit symmetry breaking due to the quark masses is taken into account, indicating that the same methods also allow one to study the perturbations generated by a weak, constant magnetic field.

Both the physics of magnon scattering and the structure of the low temperature expansion for the magnetization are well understood since the pioneering work of Dyson [14]. What a reanalysis of the same phenomena by means of an effective Lagrangian may add is a better understanding of the fact that many of the low energy properties of the system are immediate consequences of the hidden symmetry, while the microscopic structure of the system only manifests itself in the numerical values of a few effective coupling constants. Also, the method may prove to be more efficient, allowing one to carry the low energy expansion to higher orders. Work on applications of the effective Lagrangian constructed in the present paper is in progress [15].

VII. ANTIFERROMAGNET

Symmetry does not prevent the charge densities from picking up an expectation value, but does not ensure this to happen either. The antiferromagnet is a well-known system where the expectation value of the spin density vanishes. The corresponding effective field theory is discussed extensively in the recent literature [16]. The work goes beyond the leading terms of the low energy expansion and also includes an analysis of the anisotropies generated by the lattice. The present section does not add anything to what is known about antiferromagnetic systems. I merely wish to identify these within the general framework of effective field theory and to compare their low energy structure with the one of the ferromagnet.

In the language of the effective Lagrangian, antiferromagnets represent the special case where the effective coupling constants $e_i(0)$ happen to be zero. As discussed above, the conditions (d) and (e) then imply that the functions $e_i(\pi)$ and $c_a(\pi)$ vanish altogether, such that the derivative expansion of the effective Lagrangian only starts at second order, with the contributions listed in (11). The form of the functions $g_{ab}(\pi), h_i^a(\pi), k_{ik}(\pi), \bar{g}_{ab}(\pi), \bar{h}_i^a(\pi), \bar{k}_{ik}(\pi)$ occurring therein may be worked out along the same lines as before. In the absence of the term $\mathcal{L}_{\text{eff}}^{(0,1)}$, the standard power counting used in particle physics, which treats energies and momenta as quantities of the same algebraic order, is more appropriate than the one introduced above. The comparison of the two differential equations (8), (9) then again leads to a set of conditions, which these functions need to satisfy for the effective Lagrangian to give rise to a gauge invariant generating functional. In fact, the conditions for $g_{ab}(\pi), h_i^a(\pi), k_{ik}(\pi)$ are identical with those found previously: These quantities are subject to the conditions (a), (b), and (c) of Eq. (13). Moreover, the barred quantities must obey precisely the same constraints. The solution of these conditions was discussed in Sec. IV. As mentioned there, the resulting

expression for the effective Lagrangian is gauge invariant — a topological term only arises if the charge densities do pick up an expectation value.

I again specialize to the groups $G=O(3)$, $H=O(2)$, where the expectation values of the charge densities are given by $e_i(0) = \delta_i^3 \Sigma$; the present discussion thus concerns the special case $\Sigma = 0$. As mentioned above, the functions $g_{ab}(\pi), h_i^a(\pi), k_{ik}(\pi)$ are also fixed up to a constant. Since the barred quantities obey identical constraints, the same is true for these. The Lagrangian thus contains two copies of the same expression:

$$\mathcal{L}_{\text{eff}}^{(2)} = \tfrac{1}{2} F_1^2 D_0 U^i D_0 U^i - \tfrac{1}{2} F_2^2 D_s U^i D_s U^i,$$
$$D_\mu U^i = \partial_\mu U^i + \varepsilon_{ijk} f_\mu^j U^k . \tag{36}$$

At leading order in the derivative expansion, the Lagrangian involves two effective coupling constants F_1 and F_2. Except for the number of components of the vector $\mathbf{U}$ and for the magnitude of the constants F_1 and F_2, the effective Lagrangian is the same as for QCD with two quark flavors or for the Higgs sector of the standard model. There, the two coupling constants are related by the velocity of light, $F_2 = cF_1$. This shows that, for the antiferromagnetic systems under discussion here, Euclidean invariance implies Lorentz invariance, except that (i) the velocity of light is to be replaced by $v \equiv F_2/F_1$ and (ii) the statement only holds at leading order of the low energy expansion. In particular, the dispersion law corresponds to a massless particle moving with velocity v:

$$\omega(\mathbf{k}) = v|\mathbf{k}| + O(k^2) . \tag{37}$$

The known results of the low energy analysis for the strong interactions may be taken over as they are, merely replacing the velocity of light by v and adapting the number of components of $\mathbf{U}$. There are now two magnons because the equation of motion for the effective field $\mathbf{U}$ happens to be of second order with respect to time. In the nonrelativistic normalization of the states used in the preceding sections, the transition matrix elements of the charge and current densities are given by

$$\langle 0|J_a^0|\pi^b(\mathbf{k})\rangle = i\,\delta_a^b|\mathbf{k}|F_2/\sqrt{2\omega},$$
$$\tag{38}$$
$$\langle 0|J_a^r|\pi^b(\mathbf{k})\rangle = i\,\delta_a^b\,k^r v F_2/\sqrt{2\omega} .$$

In the case of the antiferromagnet, the transition elements of charge density and current are of the same order in the momentum and tend to zero for $\mathbf{k} \to 0$, while in the ferromagnetic case, they are of different magnitude, the charge density generating transitions even at infinite wavelength.

VIII. PHONONS IN SOLIDS

Historically, the phenomena associated with the propagation of sound were among the very first to be analyzed in terms of an effective field theory. For a solid, the relevant effective fields are the components of the vector $\xi(x) = (\xi^1(x), \xi^2(x), \xi^3(x))$, which specifies the displacement of the material from the position in the ground

state. The corresponding equation of motion follows from the conservation of momentum:

$$\partial_\mu \theta^{\mu r}(x) = \partial_0 \theta^{0r}(x) + \partial_s \theta^{sr}(x) = 0 \ . \tag{39}$$

The quantity θ^{0r} is the momentum per unit volume, while θ^{rs} is the stress tensor describing the momentum flow per unit area and time. To first order in the amplitude of the deformation, the momentum density is proportional to the mass density ρ of the solid and to the velocity field:

$$\theta^{0r} = \rho \dot{\xi}^r \ . \tag{40}$$

For simplicity, I consider a cubic lattice. Symmetry under reflections then implies that, to first order in the derivative expansion, the stress tensor is invariant under rotations:

$$\theta^{rs} = -\mu \xi^{rs} - K \delta^{rs} \, \boldsymbol{\partial} \cdot \boldsymbol{\xi}, $$
$$\xi^{rs} \equiv \partial_r \xi^s + \partial_s \xi^r - \tfrac{2}{3}\delta^{rs} \, \boldsymbol{\partial} \cdot \boldsymbol{\xi} \ . \tag{41}$$

The constants μ and K are referred to as torsion and compression modules, respectively [17].

The conservation law (39) shows that, at large wavelengths, the sound waves of a solid obey the wave equation

$$\rho \ddot{\boldsymbol{\xi}} - \mu \triangle \boldsymbol{\xi} - (K + \tfrac{1}{3}\mu) \, \boldsymbol{\partial} \, (\boldsymbol{\partial} \cdot \boldsymbol{\xi}) = 0 \ . \tag{42}$$

The corresponding dispersion law is of the form $\omega(\mathbf{k}) = v \mid \mathbf{k} \mid + O(k^2)$. For transverse vibrations ($\boldsymbol{\xi} \perp \mathbf{k}$), the velocity of sound is determined by the torsion module, $v_\perp = \sqrt{\mu/\rho}$, while longitudinal waves propagate with $v_\| = \sqrt{(K + \tfrac{4}{3}\mu)/\rho}$.

The energy density also admits an expansion in powers of the effective field and its derivatives. The leading contribution arises from the rest energy of the material and is of first order in $\boldsymbol{\xi}$, while the energy of the wave itself only shows up at second order. The leading term is readily obtained from the energy conservation law

$$\partial_0 \theta^{00} + \partial_r \theta^{r0} = 0 \ . \tag{43}$$

In view of the symmetry $\theta^{\mu\nu} = \theta^{\nu\mu}$, this yields

$$\theta^{00} = -\rho \, (\boldsymbol{\partial} \cdot \boldsymbol{\xi}) \ . \tag{44}$$

In the notation used here, the energy density is given by $c^2 \theta^{00}$, such that the undeformed solid corresponds to $\theta^{00} = \rho$. The expression (44) represents the change in the density generated by the deformation, $\rho \to \rho(1 - \boldsymbol{\partial} \cdot \boldsymbol{\xi})$. Note that the contribution from the ground state itself is dropped, in θ^{00} as well as in θ^{rs}.

The sound waves may be viewed as Goldstone excitations of spontaneously broken space-time symmetry: G is the Poincaré group and H is the group of time translations (in the nonrelativistic domain of interest here, G may equally well be identified with the Galilei group). The elements of the quotient G/H are parametrized by a rotation matrix R, a velocity $\mathbf{v}$, and a space translation $\mathbf{a}$; the corresponding generators are the angular momentum $\mathbf{J}$, the boost $\mathbf{K}$, and the momentum $\mathbf{P}$, respectively. For spontaneously broken *internal* symme-

tries, the effective theory involves as many pion fields as there are coordinates in G/H. Accordingly, one might expect that the effective field theory requires a matrix field $R(x)$ as well as two vector fields $\mathbf{v}(x), \mathbf{a}(x)$. The standard analysis sketched above, however, only involves a single vector field $\boldsymbol{\xi}(x)$. Indeed, the fields $R(x)$ and $\mathbf{v}(x)$ are redundant: The transformation law $\mathbf{x} \to R \, \mathbf{x} + \mathbf{v}t + \mathbf{a}$ shows that space-time-dependent translations also cover boosts and rotations. The local form of the symmetry group G is the set of general coordinate transformations; infinitesimally, these are described by space-time-dependent translations, $x^\mu \to x^\mu + a^\mu(x)$. The spontaneously broken part thereof consists of the translations in space. The state $\exp[i \, \mathbf{a}(x) \cdot \mathbf{P}]|0\rangle$ represents a deformed ground state, the point $\mathbf{x}$ being shifted into $\mathbf{x} + \mathbf{a}(x)$. Hence the field $\mathbf{a}(x)$ coincides with the effective field $\boldsymbol{\xi}(x)$ introduced above. One may thus view the phonons of a solid as Goldstone bosons associated with spontaneously broken translation invariance: At long wavelength $|\mathbf{k}| \to 0$, the frequency of the sound waves tends to zero, because there is no restoring force for displacements of the solid as a whole. The fact that the solid also breaks invariance under rotations and boosts does not give rise to additional Goldstone bosons.

The wave equation (42) is the equation of motion of the Lagrangian

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}\rho \, \dot{\xi}^r \dot{\xi}^r - \tfrac{1}{4}\mu \, \xi^{rs}\xi^{rs} - \tfrac{1}{2}K(\boldsymbol{\partial} \cdot \boldsymbol{\xi})^2 \ , \tag{45}$$

which is of the same structure as the effective Lagrangian relevant for the spin waves of an antiferromagnet. The mass density plays the role of the effective coupling constant F_1^2, while μ and K are the analogues of F_2^2. In the present case (i) the relevant symmetry group G is the Abelian group formed by the space translations rather than the group $O(3)$ considered in the preceding section and (ii) the symmetry is now fully broken, such that there are three Goldstone bosons rather than two — the subgroup H only contains the unit element.

IX. LOCAL FORM
OF THE TRANSLATION GROUP

As noted above, the relevant part of the space-time symmetry which characterizes a solid is the group G of space translations, spontaneously broken to $H = \{e\}$. The generators of G are the three components of the total momentum. Accordingly, the charge densities J_i^0 coincide with the components θ^{0i} of the energy-momentum tensor, while the currents J_i^r are represented by θ^{ri}. Their correlation functions may again be obtained by exposing the system to an external field. In the present case, where the sources of interest are the components of $\theta^{\mu\nu}(x)$, the relevant external field is the gravitational field $g_{\mu\nu}(x)$. The Ward identities obeyed by the correlation functions of the energy-momentum tensor are equivalent to the statement that the generating functional is invariant under general coordinate transformations.

The change in the Lagrangian due to the external field may be worked out as follows. Denote the metric of Minkowski space by $\eta_{\mu\nu}$ and set $g_{\mu\nu}(x) = \eta_{\mu\nu} + f_{\mu\nu}(x)$. Since the energy-momentum tensor is the variational

derivative of the action with respect to the metric, the modification of the Lagrangian is given by $\mathcal{L} - \frac{1}{2} f_{\mu\nu} \theta^{\mu\nu} + O(f^2)$. The effective Lagrangian picks up the analogous term linear in $f_{\mu\nu}$, involving the above representations of the energy-momentum tensor within the effective theory. In the presence of an external gravitational field, the effective Lagrangian thus becomes

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\rho \dot{\xi}^r \dot{\xi}^r - \frac{1}{4}\mu \xi^{rs} \xi^{rs} - \frac{1}{2}K(\boldsymbol{\partial} \cdot \boldsymbol{\xi})^2$$
$$+ \frac{1}{2}f_{00}\,\rho\,\boldsymbol{\partial} \cdot \boldsymbol{\xi} - f_{0r}\,\rho\,\dot{\xi}^r + \frac{1}{2}f_{rs}\,\mu\,\xi^{rs}$$
$$+ \frac{1}{2}f_{rr}\,K\,\boldsymbol{\partial} \cdot \boldsymbol{\xi} + \cdots. \qquad (46)$$

The conservation laws ensure that the corresponding action is invariant under the transformation

$$f_{\mu\nu} \rightarrow f_{\mu\nu} + \partial_\mu a_\nu + \partial_\nu a_\mu \qquad (47)$$

of the gravitational field — the linearized form of a general coordinate transformation on Minkowski space amounts to an Abelian gauge transformation. If the field $\boldsymbol{\xi}(x)$ solves the wave equation for $f_{\mu\nu}(x)$, then the solution belonging to the transformed gravitational field is given by $\boldsymbol{\xi}(x) + \mathbf{a}(x)$. Under a gauge transformation of the external field, the effective field thus transforms according to

$$\boldsymbol{\xi} \rightarrow \boldsymbol{\xi} + \mathbf{a} . \qquad (48)$$

Although the time component a^0 of the coordinate transformation changes the Lagrangian by a total derivative, it does not affect the solution at all. The essential part of the symmetry is contained in the spacial components $\mathbf{a}$, which represent the gauge transformations associated with the symmetry group G. Under these, the quantities $\dot{\xi}^r - f_{0r}$ and $\partial_r \xi^s + \partial_s \xi^r - f_{rs}$ are gauge invariant. Completing the squares in (46), one thus arrives at a gauge invariant effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\rho\, D_0\xi^r D_0\xi^r - \frac{1}{4}\mu\, \Xi^{rs}\Xi^{rs} - \frac{1}{2}K(D_r\xi^r)^2$$
$$+ \frac{1}{2}f_{00}\,\rho\, D_r\xi^r + \cdots , \qquad (49)$$

$$D_0\xi^r = \dot{\xi}^r - f_{0r}, \quad D_r\xi^r = \partial_r\xi^r - \frac{1}{2}f_{rr},$$

$$\Xi^{rs} = \xi^{rs} - f_{rs} + \frac{1}{3}\delta^{rs}f_{tt} .$$

We could just as well have applied the machinery of the preceding sections to the case of the translation group. Introducing external fields coupled to the charge densities θ^{0i} and currents θ^{ri} and imposing gauge invariance, the result would have been the same (except for the additional term involving the external field f_{00}, which is not related to the charge densities and currents of the group G). The point is that the Lagrangian describing the phonons of a solid emerges from the above general discussion as the special case which corresponds to the Abelian symmetry of the translation group. In particular, sound waves illustrate the remark made in Sec. IV, according to which the Goldstone bosons generated by the spontaneous breakdown of an Abelian group obey a dispersion law of the form $\omega \propto |\mathbf{k}|$. The transition matrix elements of the charge densities and currents between the ground state and a Goldstone boson may also be calculated in the same manner as before, with the result

$$\langle 0|\theta^{00}|\pi(\mathbf{k})\rangle = -i\sqrt{\rho}\,\mathbf{k}\cdot\boldsymbol{\varepsilon}/\sqrt{2\omega} ;$$
$$\langle 0|\theta^{0r}|\pi(\mathbf{k})\rangle = -i\sqrt{\rho}\,\omega\,\varepsilon^r/\sqrt{2\omega} ;$$
$$\langle 0|\theta^{rs}|\pi(\mathbf{k})\rangle = -i\sqrt{\rho}\,\{v_\perp^2\,(k^r\varepsilon^s + k^s\varepsilon^r - 2\delta^{rs}\mathbf{k}\cdot\boldsymbol{\varepsilon})$$
$$+ v_\parallel^2\delta^{rs}\mathbf{k}\cdot\boldsymbol{\varepsilon}\}/\sqrt{2\omega} , \qquad (50)$$

where $\boldsymbol{\varepsilon}$ is the polarization vector of the phonon.

Despite these evident similarities with the spontaneously broken *internal* symmetries discussed in the preceding sections, the fact that the translation group acts on space-time gives rise to some peculiarities. I add two remarks regarding the difference between phonons and Goldstone bosons of an internal symmetry.

The first point concerns the transformation properties of the generators under space rotations. While the charges considered in the preceding sections were assumed to be invariant, the generators of the translations transform with the vector representation of the rotation group. Euclidean invariance then prevents the charge densities from acquiring expectation values, $\langle 0|\theta^{0r}|0\rangle = 0$, while those of the currents may be different from zero, $\langle 0|\theta^{rs}|0\rangle = \delta^{rs}p$ (p is the pressure in the ground state). Apart from this modification, the general discussion of Sec. IV, however, applies. As pointed out there, the order parameters associated with the charges of an Abelian group lead a life of their own and do not manifest themselves in the dynamics of the Goldstone bosons. Indeed, in the above equations, the contributions to the energy-momentum tensor from the ground state were simply dropped.

The second remark is more significant. The form of the Ward identities is controlled by the *local* version of the symmetry group. The local form of the translation group is the set of general coordinate transformations and is not Abelian — it reduces to the set of Abelian gauge transformations (47) only at the linearized level. The intrinsic difference between the global and the local structure of the group shows up in the commutation rules: While the generators P^r of the translation group commute among themselves, they do not commute with the corresponding charge densities and currents, but obey a commutation rule of the form $[P^r, \theta^{\mu\nu}(x)] = i\hbar\partial_r\theta^{\mu\nu}(x)$. The phenomenon is related to the fact that an Abelian group admits different local versions. For the deformations of a solid, the one which matters is the group of coordinate transformations, while for the groups associated with the U(1) charges of particle physics, the local form relevant for the Ward identities is the set of Abelian gauge transformations. The full effective Lagrangian describing the deformations of a solid is gauge invariant under the transformation (47) only at the linearized level considered above. When imposing the symmetry on the higher order terms, the expansion of the coordinate transformation is needed to higher accuracy, such that the transformation laws of the fields $f_{\mu\nu}$ and ξ^r then involve additional terms.

X. PHONONS IN FLUIDS AND GASES

Finally, I briefly comment on sound waves in fluids or gases. Since the corresponding ground state is in-

variant under rotations as well as translations, the generators $\mathbf{J}$ and $\mathbf{P}$ now belong to the subgroup H. The spontaneously broken part of the space-time symmetry, G/H, is generated by the boost operators $\mathbf{K}$. Accordingly, the effective field is the field associated with space-time-dependent boosts, $\mathbf{v} = \mathbf{v}(x)$. To lowest order in this field, the leading terms in the derivative expansion of the energy-momentum tensor now take the form

$$\theta^{00} = \rho, \quad \theta^{0r} = \rho\, v^r, \quad \theta^{rs} = p\, \delta^{rs}\,, \tag{51}$$

and the conservation laws for energy and momentum become

$$\dot{\rho} + \boldsymbol{\partial} \cdot (\rho\,\mathbf{v}) = 0, \quad (\rho\,\mathbf{v})\dot{} + \boldsymbol{\partial} p = 0\,. \tag{52}$$

In general, the local configuration of the system depends on several variables, which, in principle are independent of one another: In addition to the temperature, the chemical potentials of the various particle species also need to be specified. To the extent that the sound waves represent adiabatic deformations, the change in the pressure is, however, determined by the one in the density, $\delta p = \kappa\,\delta\rho$. The coefficient of proportionality is the adiabatic compression module per unit mass, $\kappa = (\partial p/\partial\rho)_s$ (a detailed discussion, in particular also of the adiabatic approximation, may be found in Ref. [17]). Eliminating $\dot{p}$ in favor of $\dot{\rho}$ and retaining only terms linear in the velocity field, the time derivative of the momentum conservation law may be rewritten as

$$\ddot{\mathbf{v}} - \kappa\,\boldsymbol{\partial}\,(\boldsymbol{\partial}\cdot\mathbf{v}) = 0\,. \tag{53}$$

The phonons thus obey a wave equation which is similar to the one valid in solids ($\mathbf{v} \leftrightarrow \dot{\boldsymbol{\xi}}$). The term proportional to κ is the analogue of the one involving the compression module, $K = \rho\,\kappa$. A torsion term, on the other hand, does not occur here: In fluids or gases, torsion does not generate stress. A divergence-free velocity field obeys $\ddot{\mathbf{v}} = 0$, indicating that transverse modes do not oscillate. According to Eq. (53), layers perpendicular to the wave vector $\mathbf{k}$ glide along one another without transfer of energy or momentum.

In reality, the energy contained in the transverse modes dissipates. The attenuation rate is determined by the viscosity of the material, which manifests itself in the stress tensor, at the next order of the derivative expansion [17]:

$$\theta^{rs} = \delta^{rs} p - \eta\{\partial_r \mathbf{v}^s + \partial_s \mathbf{v}^r - \tfrac{2}{3}\delta^{rs}\,\boldsymbol{\partial}\cdot\mathbf{v}\} - \zeta\delta^{rs}\boldsymbol{\partial}\cdot\mathbf{v}\,. \tag{54}$$

Instead of a wave equation, the transverse modes obey a diffusion equation,

$$\rho\,\dot{\mathbf{v}} = \eta\,\triangle\mathbf{v}\,. \tag{55}$$

One may thus conclude that, in the case of fluids or gases, there is only one Goldstone particle. The two other degrees of freedom of the group G/H are dissipative and do not propagate like particles with real momenta and energies. Instead, the corresponding "dispersion law" corresponds to a pole in the complex plane, occurring at $\omega(\mathbf{k}) = -i\mathbf{k}^2\eta/\rho$.

For the effective Lagrangian method, this, unfortu-

nately, is the end. In the presence of phenomenological dissipative forces, the equation of motion cannot be formulated in terms of a Lagrangian. This does not mean that effective field theory is unable to cope with the motion of fluids or gases — quite to the contrary, the Navier-Stokes equations describe this motion perfectly well. They do represent an effective field theory, for which the velocity field $\mathbf{v}(x)$ is the relevant dynamical variable. That theory, however, cannot be represented in terms of an effective Lagrangian. The systematic expansion in powers of the derivatives provided by the effective Lagrangian method is not available here. In this expansion, the contributions arising at higher orders, from simultaneous exchange of several Goldstone bosons, are accounted for by the loop graphs, i.e., by the quantum fluctuations of the effective field. If the effective field theory does not admit a Lagrangian formulation, it is entirely unclear how to set up the corresponding quantum theory. Presumably, in the presence of phenomenological dissipative terms, it is impossible to extend the low energy analysis beyond leading order.

XI. SUMMARY AND CONCLUSION

(1) The paper deals with the effective field theory relevant for the low energy analysis of spontaneously broken symmetries in the nonrelativistic domain. The discussion applies to any system for which the only excitations without an energy gap are the Goldstone modes.

(2) The analysis is based on the Ward identities obeyed by the correlation functions of the charge densities and currents. The discussion assumes that the Ward identities are anomaly free and exploits the fact that the generating functional is then invariant under gauge transformations, i.e., under a local form of the symmetry group.

(3) The number of effective fields needed turns out to be universal. Denoting the symmetry groups of the Hamiltonian and of the ground state by G and H, respectively, the number of effective *fields* required to describe the properties of the system for large wavelengths is given by $\dim(G) - \dim(H)$. While, for relativistically invariant theories, the number of Goldstone *particles* coincides with the number of effective fields, this is not in general the case for nonrelativistic systems, where the above number only represents an upper bound.

(4) Nonrelativistic kinematics does not prevent the generators of the group from having expectation values in the ground state, representing order parameters of the spontaneously broken symmetry. The main result of the present paper is the statement that the phenomenon manifests itself through a term in the effective Lagrangian, which is of topological nature and does not occur in the effective field theories relevant for particle physics. The relevant term, in particular, violates gauge invariance of the effective Lagrangian.

(5) The form of the leading contributions in the derivative expansion of the general Lagrangian is discussed in detail. In the case of $G = O(3)$, $H = O(2)$, the two Goldstone fields may be described in terms of a three-component vector $U^i(x)$ of unit length, $U^i U^i = 1$. The derivative expansion of the effective Lagrangian then

starts with

$$\mathcal{L}_{\text{eff}} = \Sigma \int_0^1 d\lambda \, \varepsilon_{ijk} \, \partial_0 U^i \, \partial_\lambda U^j \, U^k + \tfrac{1}{2} F_1^2 \partial_0 U^i \partial_0 U^i$$
$$- \tfrac{1}{2} F_2^2 \partial_r U^i \partial_r U^i + \cdots . \tag{56}$$

The first term is the topological object mentioned above. The corresponding effective coupling constant Σ is the order parameter associated with the charge densities. In the case of a magnet, Σ is the magnetization of the ground state. The other two effective coupling constants F_1, F_2 are determined by the one-particle matrix elements of the charge densities and currents.

(6) The above expression for the effective Lagrangian implies that the dispersion law of the Goldstone bosons is of the form

$$\Sigma \omega + F_1^2 \omega^2 - F_2^2 \mathbf{k}^2 + \cdots = 0 . \tag{57}$$

(i) If the charge density acquires a nonzero expectation value — as it is the case with the ferromagnet — the first term is different from zero. At low frequencies, it then dominates over the second, such that the dispersion law is quadratic in $\mathbf{k}$:

$$\omega(\mathbf{k}) = (F_2^2/\Sigma) \, \mathbf{k}^2 + \cdots , \qquad \Sigma \neq 0 . \tag{58}$$

The corresponding wave equation takes the form of a Schrödinger equation. The wave function is complex and incorporates both of the two real Goldstone fields. The spectrum only contains one Goldstone particle of a given momentum.

(ii) The antiferromagnet corresponds to the case where the charge density does not acquire an expectation value. The dispersion law then takes the form

$$\omega(\mathbf{k}) = (F_2/F_1)|\mathbf{k}| + \cdots , \qquad \Sigma = 0 . \tag{59}$$

In this case, the wave equation is of second order in the time derivative, such that there are two Goldstone particles.

(7) The phonons of a solid represent a peculiar case, as they are associated with a spontaneously broken space symmetry, translation invariance. The relevant gauge group is the set of coordinate transformations. Accordingly, the Ward identities for the correlation functions of the energy-momentum tensor play a central role in the corresponding effective theory.

(8) While the effective Lagrangian method is perfectly suited for the low energy analysis of the deformations of a solid, the method fails for fluids or gases. There, the low energy behavior of two of the three effective fields is dominated by dissipative forces, which cannot be described in terms of a Lagrangian.

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