Small massless excitations against a nontrivial background

N. G. Khariton and V. B. Svetovoy*
Department of Theoretical Physics, Yaroslavl State University, Sovetskaya 14, 150000 Yaroslavl, Russia
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We propose a systematic approach for finding bosonic zero modes of nontrivial classical solutions in a gauge theory. The method allows us to find all the modes connected with the broken space-time and gauge symmetries. The ground state is supposed to be dependent on some space coordinates y^α and independent of the rest of the coordinates x^i. The main problem which is solved is how to construct the zero modes corresponding to the broken $x^i y^\alpha$ rotations in vacuum and which boundary conditions specify them. It is found that the rotational modes are typically singular at the origin or at infinity, but their energy remains finite. They behave as massless vector fields in x space. We analyze local and global symmetries affecting the zero modes. An algorithm for constructing the zero mode excitations is formulated. The main results are illustrated in the Abelian Higgs model with the string background.

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I. INTRODUCTION

Massless excitations (zero modes) against nontrivial classical backgrounds play an important role in many physical problems. In higher-dimensional theories [1,2] they are treated as observable fields in physical dimensions and are the main subject of investigation. For cosmic strings [3,4] these excitations induce such phenomena as cosmic-string superconductivity [5] and baryon number violation [6]. Zero modes were extensively discussed in connection with the quantization of classical solutions in field theory [7]. Domain walls in 1+1 [7], vortices in 1+2 [8], monopoles in 1+3 [9] dimensions and instantons in four-dimensional Euclidean space [10] give well-known examples of such solutions. Zero modes appearing in the excitation spectrum about these solutions are connected with the translational and internal symmetries breaking in vacuum. The rotations between time and space coordinates are also broken. Do they induce new zero modes? The answer is no. However, if one embeds any of these solutions into a space with additional dimensions, we will really find new massless excitations with very special properties. These modes will be the main subject for our study.

Consider, for example, a vortex solution in (1+3)-dimensional Minkowski space-time (string solution). Let the string background depend on $y^\alpha (\alpha = 1, 2)$ coordinates and let two additional coordinates be x^i $(i = 0, 1)$. Then the massless excitations of the string describe a two-dimensional theory of x-dependent fields. It includes the zero modes connected with the broken translations in y^α and those for the broken rotations in the $x^i y^\alpha$ planes. At x coordinates transformations they behave as scalar and vector fields, respectively. For the $\mathrm{U}(1) \times \tilde{\mathrm{U}}(1)$ model of cosmic string [5] the electromagnetic U(1) field is absent in vacuum and it reduces the x-vector modes to the gradient of some scalars. Such excitations can be treated as Goldstone bosons of the electromagnetic U(1) gauge group [11]. Their interpretation becomes unambiguous if the gauge field is nonzero in vacuum [12]. In this case two vector modes and a series of the scalar modes localized at the origin were found. A very interesting feature of the modes connected with the broken rotations is their singular behavior in y coordinates. They inevitably contain singularities at $|y| \to \infty$ or at $|y| \to 0$, but their energy remains finite.

The idea that we live inside a topological defect [13–15] gives another example of zero mode manifestation. Rubakov and Shaposhnikov [13] considered a domain wall in the fifth coordinate as a ground state in the Φ^4 model defined on a five-dimensional Minkowski space-time M^5. The only zero mode is confined inside a potential well in the fifth dimension and connected with the broken translations in this direction. The domain wall also breaks the rotational symmetry between the fifth and the rest of the four coordinates, but in scalar theory it does not give new massless excitations. This approach allows us to get massless fermion and scalar particles living in 1+3 physical dimensions.

To get Lorentz-vector massless excitations, we should choose a ground state with a nontrivial gauge field in the additional dimensions. Lorentz-vector zero modes are connected then with the broken xy rotations. The simplest system of this kind is the Abelian Higgs model in M^6. At low energies there are four-dimensional massless vector and scalar fields interacting with each other [15]. The masslessness of the vector fields guarantees some gauge symmetry of the lower-dimensional Lagrangian. An interesting feature of such a theory is spontaneous symmetry breaking on the scale which is naturally small in comparison with the mass scale in the original higher-dimensional theory [15]. One can expect a non-Abelian gauge symmetry in the low-energy limit for some clas-

*Electronic address: imras@iman.yaroslavl.su

sical solutions embedded in a higher-dimensional flat or curved space-time manifold (for example, an instanton in M^8). However, we will not discuss here the interaction between massless excitations leaving it a topic for future publications.

The $x^i y^\alpha$-rotational modes and the gauge modes (see below) have been studied in a few works [11,12,15] where very special features of the modes have been stressed. Different methods for the description of the modes have been used, but they were heuristic rather than systematic and so could be applied only to the particular problems which were solved. In this paper we propose a common approach to the description of small massless excitations against a nontrivial background. All the modes connected with the broken space-time or gauge symmetries are treated on the same basis. The paper is organized as follows. In Sec. II we investigate the coordinate transformations inducing zero modes for an arbitrary background solution in a Yang-Mills-Higgs theory. An analysis is given for some gauge group G in a D-dimensional Minkowski space-time. In Sec. III local and global symmetries affecting zero modes are discussed. In Sec. IV we analyze the xy-rotational modes and formulate boundary conditions for them. A common algorithm for constructing the zero mode excitations is also presented in this section. The explicit solutions are given in Sec. V for string backgrounds in the Abelian Higgs model. In Sec. VI we present our concluding remarks.

II. COORDINATE TRANSFORMATIONS INDUCING ZERO MODES

Consider a Yang-Mills-Higgs theory in a D-dimensional Minkowski space-time M^D. Let A^μ_a and Φ_k be the gauge fields and real scalar fields, respectively. The Lagrangian of the system which is locally gauge invariant under some compact Lie group G with structure constants C_{abc} is

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} (D_\mu \Phi)_k (D^\mu \Phi)_k - V(\Phi), \quad (2.1)$$

where $V(\Phi)$ is some G-invariant polynomial in Φ, and where

$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g C_{abc} A^\mu_b A^\nu_c, \quad (2.2)$$

$$(D_\mu \Phi)_k = \partial_\mu \Phi_k - ig\theta^a_{kl} \Phi_l A_{a\mu}. \quad (2.3)$$

The indices μ, ν run from 0 to $D-1$. The scalar fields belong to some, in general, reducible representation of G, with corresponding representation matrices of the generators θ^a_{kl}.

Consider a nontrivial vacuum state which obeys classical equations of motion:

$$(D_\mu F^{\mu\nu})_a - ig (D^\nu \Phi) \theta^a \Phi = 0, \quad (2.4)$$

$$(D_\mu D^\mu \Phi)_k + \partial V / \partial \Phi_k = 0. \quad (2.5)$$

A solution with finite energy is supposed to be dependent on y^α coordinates ($\alpha = 1, 2, \ldots, D-d$):

$$A^i_a = 0, \quad A^\alpha_a = A^\alpha_a(y), \quad \Phi_k = \Phi_k(y), \quad (2.6)$$

where M^D coordinates are $X^\mu = (x^i, y^\alpha)$, and $x^i (i = 0, 1, \ldots, d-1)$ will be the coordinates in a lower-dimensional theory. The boundary behavior of the solution (2.6) is not important at this stage.

Variation of the classical equations of motion gives the equations describing small excitations about the background (2.6):

$$[\bar{D}_\mu (\bar{D}^\mu V^\nu - \bar{D}^\nu V^\mu)]_a + g C_{abc} V_{b\mu} F^{\mu\nu}_c - ig (\bar{D}^\nu \phi) \theta^a \Phi$$
$$-ig (\bar{D}^\nu \Phi) \theta^a \phi + g^2 (\Phi \theta^b \theta^a \Phi) V^\nu_b = 0, \quad (2.7)$$

$$(\bar{D}_\mu \bar{D}^\mu \phi)_k - ig\theta^a_{kl} (\bar{D}_\mu \Phi)_l V^\mu_a - ig\bar{D}_\mu (\theta^a \Phi V^\mu_a)_k$$
$$+ (\partial^2 V / \partial \Phi_k \partial \Phi_l) \phi_l = 0, \quad (2.8)$$

where $\bar{D}_\mu$ is the background covariant derivative and V^μ and ϕ are the variations of the vector and scalar fields, respectively.

Among all the excitations there are a few zero modes connected with the broken transformations of the space-time symmetry. To extract these modes, one considers the coordinate transformation

$$X^\mu \to X^\mu + \xi^\mu(x, y) \quad (2.9)$$

with a small arbitrary vector $\xi^\mu(x, y)$. At such a variation the vector and scalar fields get the increase

$$\delta_\xi A^\mu = (\partial_\nu \xi^\mu) A^\nu - \xi^\nu \partial_\nu A^\mu, \quad (2.10)$$

$$\delta_\xi \Phi = -\xi^\nu \partial_\nu \Phi, \quad (2.11)$$

where δ_ξ is the Lie differential with respect to ξ^μ. We should find those functions $\xi^\mu(x, y)$ which generate the zero modes we are looking for. For this purpose one takes the variation of Eqs. (2.4) and (2.5) with respect to ξ^μ and demands the field increase to be an excitation of the ground state. Applying the δ_ξ operator to Eq. (2.5), one gets the left-hand side of Eq. (2.8) with $V^\mu = \delta_\xi A^\mu$ and $\phi = \delta_\xi \Phi$ plus some additional terms. These terms are connected with the metric and the spin connection variations:

$$\delta_\xi g^{\mu\nu} = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu, \quad (2.12)$$

$$\delta_\xi \Gamma^\rho_{\mu\nu} = -\partial_\mu \partial_\nu \xi^\rho. \quad (2.13)$$

Since we consider $\delta_\xi A^\mu$ and $\delta_\xi \Phi$ as one of the possible excitations, the additional terms must be zero:

$$\delta_\xi g^{\alpha\beta} [\partial_\alpha \partial_\beta \Phi + g^2 (\theta^a \theta^b \Phi) A^a_\alpha A^b_\beta]$$
$$+ig (\theta^a \Phi) A^a_\beta \partial^\beta (\partial_\alpha \xi^\alpha) + (\partial_\alpha \partial^\alpha \xi^\beta) \partial_\beta \Phi$$
$$-\partial_i \partial^i (\delta_\xi \Phi) + ig (\theta^a \Phi) \partial_i (\delta_\xi A^i) = 0. \quad (2.14)$$

Here we have taken into account that vacuum depends only on y^α coordinates.

Let $\delta_\xi A^i$ and $\delta_\xi \Phi$ be the massless excitations in the sense that they satisfy the equations

$$\partial_i \partial^i \left(\delta_\xi \Phi \right) = 0, \tag{2.15}$$

$$\partial_i \left[\partial^i \left(\delta_\xi A^j \right) - \partial^j \left(\delta_\xi A^i \right) \right] = 0. \tag{2.16}$$

Then, in the Lorentz gauge for $\delta_\xi A^i$, Eq. (2.14) can be treated as an equation for $\xi^\alpha(x,y)$. The most common solution is

$$\xi^\alpha(x,y) = C^\alpha(x) + C^{\alpha\beta}(x) y_\beta, \tag{2.17}$$

where $C^\alpha(x)$ and $C^{\alpha\beta}(x)$ are arbitrary functions of x^i and $C^{\alpha\beta} = -C^{\beta\alpha}$. According to Eqs. (2.11) and (2.15) the functions $C^\alpha(x)$ are massless scalar fields in the lower-dimensional space-time M^d. The same conclusion can be drawn for the functions $C^{\alpha\beta}(x)$ if there is no residual symmetry leading to $\delta_\xi \Phi = 0$ for some space transformations. Possible types of the residual symmetry will be discussed in the next section. The interpretation of the fields $C^\alpha(x)$ and $C^{\alpha\beta}(x)$ is obvious and well known. They can be considered as the parameters of the broken space transformations. $C^\alpha(x)$ correspond to the broken translations in y^α and $C^{\alpha\beta}(x)$ to the broken rotations in the $y^\alpha y^\beta$ planes. Their x dependence is connected with the fact that the ground state does not depend on x^i coordinates.

There are many ways for the description of the translational modes[1] and in this respect we did not say something new. Our goal is to apply the same approach to the modes arising as a result of breaking $x^i y^\alpha$ rotations. We can deal with the xy-rotational modes on the same basis as was done for the translational ones. To this end one takes δ_ξ from Eq. (2.4) and after some transformations one gets, for $\nu = i$ and $\nu = \alpha$,

$$\begin{aligned} &\bar{D}_\alpha \bar{D}^\alpha \left(\delta_\xi g^{i\beta} A_\beta \right)_a + g^2 \left(\Phi \theta^a \theta^b \Phi \right) \delta_\xi g^{i\beta} A^b_\beta \\ &-\partial^i \left(\partial_\alpha \xi_\beta \right) F^{\alpha\beta}_a - \bar{D}_\alpha \left[\partial^i \left(\delta_\xi g^{\alpha\beta} \right) A_\beta \right]_a \\ &+\partial_j \left[\partial^i \left(\delta_\xi A^j \right) - \partial^j \left(\delta_\xi A^i \right) \right]_a = 0, \end{aligned} \tag{2.18}$$

$$\partial_i \partial^i \left(\delta_\xi A^\alpha \right)_a + \bar{D}^\alpha \left(\partial_i \delta_\xi A^i \right)_a = 0, \tag{2.19}$$

respectively. Again, let $\delta_\xi A^i$ and $\delta_\xi A^\alpha$ be the massless excitations, i.e., $\delta_\xi A^i$ satisfy Eq. (2.16) and

$$\partial_i \partial^i \left(\delta_\xi A^\alpha \right)_a = 0. \tag{2.20}$$

In this case (2.19) is held true in the Lorentz gauge. In Eq. (2.18) the term $\bar{D}_\alpha \left[\partial^i \left(\delta_\xi g^{\alpha\beta} \right) A_\beta \right]$ must vanish and that implies

$$\delta_\xi g^{\alpha\beta} = 0 \tag{2.21}$$

if we are not interested in pure y-dependent ξ^α. It coincides with Eq. (2.17) for $\xi^\alpha(x,y)$. All the other terms give an equation for $\xi^i(x,y)$:

$$\bar{D}_\alpha \bar{D}^\alpha \left(\delta_\xi g^{i\beta} A_\beta \right)_a + g^2 \left(\Phi \theta^a \theta^b \Phi \right) \delta_\xi g^{i\beta} A^b_\beta = \partial^i \left(\partial_\alpha \xi_\beta \right) F^{\alpha\beta}_a. \tag{2.22}$$

Solutions of this equation allow us to construct the rotational zero modes, which appear in the lower-dimensional theory on M^d as vector fields. We will discuss the solutions and the boundary conditions for them, but first let us turn to the residual symmetry of the background.

[1]The $y^\alpha y^\beta$-rotational modes are quite similar to the translational ones and they have no special interest.

III. SYMMETRIES ON ZERO MODES

A. Residual symmetry

To specify the structure of the lower-dimensional theory, it is important to know the residual symmetry of the ground state. The d-dimensional fields will be the subjects of this symmetry. In the studied model the vacuum solution breaks all the translations in y^α directions, but it can admit some rotational symmetry. The maximal symmetry group is SO$(D-d)$.

Two kinds of residual symmetries can exist. We will refer to them as the direct and the combined symmetries. In the first case the vacuum is directly symmetric under some group $S \subseteq$ SO$(D-d)$. In the second case it is H symmetric up to a gauge transformation and so the group H must be contained in SO$(D-d) \supseteq H$ and in the gauge group $G \supseteq H$. Vortices, monopoles, and instantons give well-known examples of both cases [7].

The conditions of the direct symmetry are

$$\hat{S}_n \Phi(y) = 0, \quad \left(\hat{S}_n \right)^\alpha_\beta A^\beta(y) = 0. \tag{3.1}$$

Here $\hat{S}_n[n = 1, 2, \ldots, \dim(S)]$ are the generators of the group S. These generators can be explicitly constructed as linear combinations of the $y^\alpha y^\beta$ rotations:

$$\hat{S}_n = \frac{1}{2} a^{\alpha\beta}_n \hat{M}_{\alpha\beta}, \tag{3.2}$$

where $\hat{M}_{\alpha\beta}$ are the rotational generators in the corresponding representation (scalar or vector) and $a^{\alpha\beta}_n$ are the numerical coefficients forming a Lie algebra of the group S,

$$a^{\beta\alpha}_m a^{\alpha\gamma}_n - a^{\beta\alpha}_n a^{\alpha\gamma}_m = f_{mnk} a^{\beta\gamma}_k. \tag{3.3}$$

If vacuum is S symmetric, then not all of the rotational parameters $C^{\alpha\beta}(x)$ have to appear as massless fields. Some linear combinations of $C^{\alpha\beta}(x)$ should not satisfy the d'Alembert equation because of the symmetry constraints. Really, one considers transformations from the S subgroup of the group SO$(D-d)$ for the fields Φ and A^α:

$$\delta^S_\xi \Phi = \alpha_n(x) \hat{S}_n \Phi(y), \quad \delta^S_\xi A^\alpha = \alpha_n(x) \left(\hat{S}_n \right)^\alpha_\beta A^\beta(y), \tag{3.4}$$

where

$$\alpha_n(x) = a^{\alpha\beta}_n C_{\alpha\beta}(x). \tag{3.5}$$

The field variations vanish as a result of Eq. (3.1) and $\alpha_n(x)$ remain arbitrary functions. We can conclude that $\alpha_n(x)$ are the parameters of the local group S which affects massless fields in the d-dimensional theory.

The combined symmetry is defined by the conditions

$$\hat{H}_n \Phi_k(y) = g\,(\theta^a \Phi)_k\,\chi_n^a,$$

$$(\hat{H}_n)^\alpha_\beta A^\beta_a(y) = -i\,(\bar{D}^\alpha \chi_n)_a\,, \tag{3.6}$$

where the coefficients χ_n^a extract the H subgroup from the group G. The operators $\hat{H}_n$ are constructed in the same way as it was done for the direct symmetry [Eq. (3.2)]. H transformation for the fields is quite similar to that for the S group, but in the case of the combined symmetry the field variations are not zero in accordance with Eq. (3.6). Therefore, the group parameters $\alpha_n(x)$ must obey the equation $\partial_i \partial^i \alpha_n(x) = 0$ and H symmetry will act as a global group on massless fields. We could also conclude that $\delta^H_\xi \Phi$ and $\delta^H_\xi A^\alpha$ are the zero modes. However, it would be a hasty conclusion. These excitations really obey Eqs. (2.8) and (2.7) with $\mu = \alpha$, but Eq. (2.7) with $\mu = i$ knows nothing about the x dependence of $C_{\alpha\beta}$. Really, substituting $\delta^H_\xi \Phi$ and $\delta^H_\xi A^\alpha$ in it, one finds

$$\partial_i \alpha_n(x)\,[\,(\bar{D}_\alpha \bar{D}^\alpha \chi_n)_a + g^2\,(\Phi\theta^a\theta^b\Phi)\,\chi_n^b\,] = 0. \tag{3.7}$$

Generally speaking, it will hold true for $\alpha_n(x)$=const.[2] Indeed, one can obtain the same result from our basic equation (2.22). Although we consider it as an equation for $\xi^i(x,y)$, it can impose some restrictions on $C_{\alpha\beta}(x)$. In Sec. V, where an example is analyzed, we will see that the restriction can arise from the demand for $\xi^i(x,y)$ to be single valued in y^α.

Thus, not all of the rotational parameters appear as zero modes. The linear combinations of the parameters corresponding to the direct symmetry of vacuum remain arbitrary functions of x and describe the parameters of a local symmetry group instead of zero modes. Those for the combined symmetry are reduced to arbitrary constants which are the parameters of a global group. Massless excitations are subjects of these local or/and global groups.

B. Gauge transformations and gauge modes

If there are massless vector fields in a theory, there must be some gauge symmetry to guarantee their masslessness. Now we are going to discuss the origin of this symmetry. The residual group S is not a good candidate since the vector modes do exist even if the ground state does not possess the direct rotational invariance. Moreover, S symmetry cannot become apparent on small excitations because an excitation gets the increase of the second order under a small S transformation.

Local G symmetry of the original theory induces the gauge transformations for small excitations:

$$V^i \to V^i + \partial^i \chi,$$
$$V^\alpha \to V^\alpha + \bar{D}^\alpha \chi,$$
$$\phi \to \phi + ig\chi\Phi, \tag{3.8}$$

where $\chi(x,y)$ is a Lie-algebra-valued function. Consider the massless excitations connected with the broken space-time symmetry. To work with explicitly G-covariant expressions, it will be useful to make the transformation (3.8) with $\chi = \xi^\alpha A_\alpha$. Then the excitations can be written as

$$V^i = \delta_\xi g^{i\alpha} A_\alpha, \;\; V^\alpha = \xi_\beta F^{\alpha\beta}, \;\; \phi = -\xi_\beta \bar{D}^\beta \Phi. \tag{3.9}$$

Here $V^i(x,y)$ obeys Eq. (2.22). Let us note that any vector excitation of the form

$$V^i = \delta_\xi g^{i\alpha} A_\alpha + \partial^i \psi \tag{3.10}$$

with ψ solving the equation

$$(\bar{D}_\alpha \bar{D}^\alpha \psi)_a + g^2\,(\Phi\theta^a\theta^b\Phi)\,\psi^b = 0 \tag{3.11}$$

will also obey Eq. (2.22) and can be treated as a zero mode. Since $\partial^i \psi$ and $\delta_\xi g^{i\alpha} A_\alpha$ obey the same homogeneous equation, the gradient part can be absorbed into the rotational part of the excitation. However, one should remember that Eq. (2.22) is an equation for $\xi^i(x,y)$ and there may exist some solutions of Eq. (3.11) for which an appropriate ξ^i cannot be found. Only in such cases is $\partial^i \psi$ in (3.10) important.

Up to now we have discussed zero modes connected with the broken space-time symmetry. The ground state also breaks the gauge group G and induces additional modes. For the string solution one of these modes was discussed in connection with cosmic-string superconductivity [5], another mode was shown to play a specific role for the string excitations in the Abelian Higgs model [15]. Detailed analysis of the modes for the string solutions has been carried out in Ref. [11]. We will call them the gauge modes stressing their origin from gauge symmetry breaking. One can find the explicit form of the gauge modes transforming (3.9) with the gauge function $\chi(x,y) = \varphi(x,y)$ obeying Eq. (3.11). The gradient can be absorbed into the rotational part (or into $\partial^i \psi$ if $\delta_\xi g^{i\alpha} A_\alpha$ vanishes) and we find the zero mode:

$$V^i = 0, \;\; V^\alpha = \bar{D}^\alpha \varphi, \;\; \phi = ig\varphi\Phi. \tag{3.12}$$

Now one can write down the expansion of an arbitrary massless excitation in the complete set of zero modes:

$$V^i = \delta_\xi g^{i\alpha} A_\alpha + \partial^i \psi,$$
$$V^\alpha = \xi_\beta F^{\alpha\beta} + \bar{D}^\alpha \varphi,$$
$$\phi = -\xi_\beta \bar{D}^\beta \Phi + ig\varphi\Phi. \tag{3.13}$$

The gauge transformation (3.8), with $\chi(x,y)$ obeying Eq. (3.11), transforms massless excitations into themselves and, therefore, it will affect d-dimensional fields.

[2] We cannot exclude some exceptional cases (especially for curved space-time manifolds) when the term in the square brackets is zero, but such cases will not be discussed here.

That is just the symmetry we are looking for. Under it action d-dimensional vector fields get usual gradient type increase[3] and gauge modes get shift like the phase of some scalar. Translational and $y^\alpha y^\beta$-rotational modes do not change at all.

By a gauge transformation one can exclude the gauge modes from V^α and ϕ and present (3.13) in the alternative form

$$V^i = \delta_\xi g^{i\alpha} A_\alpha + \partial^i(\psi - \varphi),$$
$$V^\alpha = \xi_\beta F^{\alpha\beta},$$
$$\phi = -\xi_\beta \bar{D}^\beta \Phi. \tag{3.14}$$

Indeed, if $\delta_\xi g^{i\alpha} A_\alpha$ vanishes, the physical mode is $(\psi-\varphi)$. The reason why we kept both of the functions ψ and φ in the expansions (3.13) or (3.14) is the explicit gauge invariance of the mode $(\psi - \varphi)$.

IV. BOUNDARY CONDITIONS

We have described the method for constructing of the xy-rotational modes with the help of Eq. (2.22). Only in exceptional cases this equation contains nonsingular solutions. In a common case a solution is singular at the origin or at infinity. Singular behavior of an excitation is not the reason to reject it. Any zero mode can be represented as the product of an x-dependent field and a definite y-dependent function. The separation procedure is defined up to a numerical factor. It is convenient to choose this factor so that the x-dependent part will be normalized as a d-dimensional field. To be more precise, let us suppress the inner structure of a vector mode and present it in the form

$$V^i(x,y) = C^i(x) f(y). \tag{4.1}$$

The normalization factor Z one finds integrating over y the kinetic term $(\partial^i V^j - \partial^j V^i)^2$ in the quadratic in excitations part of the Lagrangian:

$$Z = \int d^{D-d}y f^2(y). \tag{4.2}$$

The corresponding d-dimensional physical field is

$$W^i(x) = Z^{1/2} C^i(x) \tag{4.3}$$

and the excitation can be written via this field as

$$V^i(x,y) = W^i(x) f(y) Z^{-1/2}. \tag{4.4}$$

If the function $f(y)$ is singular (in the sense that $Z \to \infty$) at the origin or at infinity, we will get special kinds of excitations which are confined at the origin like δ function or displaced to infinity.

What we really must care about is not the singularity itself but finiteness of the excitation energy. Consider the quadratic in the excitation part of the Lagrangian:

$$L^{(2)} = -\frac{1}{4}\left(\bar{D}_\mu V_\nu - \bar{D}_\nu V_\mu\right)_a^2 + \frac{1}{2}\left(\bar{D}_\mu \phi\right)^2 - \frac{g}{2} C_{abc} F^a_{\mu\nu} V^{b\mu} V^{c\nu}$$
$$-ig\left(\bar{D}^\mu \Phi \theta^a \phi + \bar{D}^\mu \phi \theta^a \Phi\right) V^a_\mu + \frac{g^2}{2}\left(\Phi \theta^a \theta^b \Phi\right) V^a_\mu V^{b\mu} + \left(\partial^2 V / \partial \Phi_k \partial \Phi_l\right) \phi_k \phi_l. \tag{4.5}$$

The energy will be finite if the second-order surface term

$$L^{(2)}_{\text{surf}} = \frac{1}{2}\partial_\mu[-(\bar{D}^\mu V^\nu - \bar{D}^\nu V^\mu)_a V^a_\nu + \phi \bar{D}^\mu \phi$$
$$-ig(\phi \theta^a \Phi) V^{a\mu}], \tag{4.6}$$

disappears after integration. Strictly speaking, we must integrate in all x and y coordinates. The integral is decomposed into two parts containing the integrals over an infinite surface in x and over a surface around the singularity in y. The integral over x^i bounds the x dependence of the excitation in the usual way. In the integral over y^α all the terms which do not include V^i vanish [in the gauge (3.14)] since they are regular. The only nontrivial restriction is

$$\int d^d x \int d^{D-d} y \partial_\alpha\left[\left(\bar{D}^\alpha V^i - \partial^i V^\alpha\right)_a V^a_i\right] = 0. \tag{4.7}$$

Here the excitations are supposed to be written via the normalized d-dimensional fields in a way similar to Eq. (4.4). It is expected that Eq. (4.7) bounds the y dependence of V^i, but we left in it the integral over the x volume since a special kind of excitation may exist for which Eq. (4.7) restricts the x dependence of the gauge modes at space-time infinity [12] (see also Sec. V).

Special analysis should be given for the excitations V^i which can be represented as gradient of some scalars (gradient modes). The difference from true vector modes is connected with the origin of the kinetic term. For the gradient excitations the kinetic energy appears from the quadratic in the V^i term which does not contain derivatives with respect to x^i, that is just the structure in $L^{(2)}_{\text{surf}}$ we have demanded to be zero. Thus, there is no need to impose condition (4.7) on the gradient modes. For these modes any singularity is permitted since the strongest singularities can be hidden in the normalization factors.

Above we have discussed the gauge transformations for small massless excitations assuming that the gauge functions $\chi(x,y)$ are arbitrary solutions of Eq. (3.11). It is not quite correct. The reason is that we consider only the quadratic in the excitation part of the Lagrangian (4.5). It is invariant under the gauge transformations (3.8) only up to a surface term:

[3] Of course, working with small fields and small gauge functions we cannot distinct Abelian and non-Abelian transformations. For this end it is necessary to consider the interaction between zero modes.

$$\delta_\chi L^{(2)}_{\rm surf} = \partial_\alpha \left[\frac{g}{2} C_{abc} F^{\alpha\beta}_a \chi_b \left(\bar{D}_\beta \chi \right)_c + g C_{abc} F^{\alpha\beta}_a \chi_b V_{c\beta} \right] - \partial_\alpha \left[\frac{g^2}{2} \left(\bar{D}^\alpha \Phi \theta^a \theta^b \Phi \right) \chi_a \chi_b + ig \left(\bar{D}^\alpha \Phi \theta^a \phi \right) \chi_a \right]. \quad (4.8)$$

Therefore, true gauge functions affecting small massless excitations must obey the condition

$$\int d^{D-d} y \delta_\chi L^{(2)}_{\rm surf} = 0. \quad (4.9)$$

It is assumed here that the gauge functions are normalized as vector zero modes to provide correct gauge transformations of the d-dimensional vector fields.

Now we are able to formulate the whole algorithm for constructing zero mode excitations. One starts from the basic equation

$$\left(\bar{D}_\alpha \bar{D}^\alpha X \right)_a + g^2 \left(\Phi \theta^a \theta^b \Phi \right) X_b = 0, \quad (4.10)$$

which describes the gauge functions $X = \chi(x,y)$, the gauge modes $X = \varphi(x,y)$, and homogeneous part of the vector modes $X^i = \delta_\xi g^{i\alpha} A_\alpha$. One finds a complete set of the solutions and normalizes every solution as was described schematically in the beginning of this section. All the solutions are divided into two classes. The first class includes the solutions which are singular at the origin or nonsingular at all. Together with the translational modes it contains the excitations localized (in y coordinates) inside the ground-field configuration. The second class consists of the solutions which are singular at infinity and therefore localized outside the ground state. Each class can be analyzed independently because they describe different physics. The next step is to find among the solutions those normalized gauge functions obeying (4.9). The allowed gauge modes are the same. The solutions which do not satisfy Eq. (4.9) should not be rejected. The corresponding vector excitations lose the gauge freedom and for this reason they must be reduced to the gradient of some scalars. Since the kinetic energy for such modes comes from another term in $L^{(2)}$, the excitations have to be renormalized. After that Eqs. (4.7) and (4.9) do not restrict the modes at all. The existence of a gauge function for a given vector excitation does not guarantee that it is not reduced to the gradient form. Real vector excitations must be represented as $X^i = \delta_\xi g^{i\alpha} A_\alpha$, where X^i is a solution of Eq. (4.10). It is considered as an equation for $\xi^i(x,y)$. The vector modes are those for which it can be solved. If the solution cannot be found, X^i is reduced to the gradient form.

The vector excitations which are not reduced to the gradient form must obey Eq. (4.7). Substituting $V^i(x,y)$ in the gauge (3.14), one can find those vector modes which survive. We would like to stress that careful analysis of Eq. (4.7) has to be done to not miss the solutions which satisfy the equation only after integration over the x volume. An example is analyzed in Sec. V.

V. ABELIAN STRING AS AN EXAMPLE

The above-described algorithm gives a frame for the constructing of zero mode excitations against a nontrivial background. Details depend on the classical solution which is considered and on the physical problem which is solved. It was not our goal to describe in this paper as many examples as possible, but it is important to illustrate the main features of the developed approach by a simple and at the same time nontrivial example. For this purpose we have chosen the Abelian Higgs model with a string solution as a ground state. Originally the most important aspects of massless excitations in this model have been studied in Refs. [15,12]. Since we did not specify any particular physical problem in this paper, all possible massless excitations will be described.

Consider the string solution in the model

$$A^i = 0, \quad A^\alpha = \epsilon^{\alpha\beta} y_\beta A(r), \quad \Phi_k = e^{(n)}_k(\vartheta) \Phi_0(r), \quad (5.1)$$

where $k = 1, 2$, $\epsilon^{\alpha\beta}$ is the two-dimensional antisymmetric tensor, r and ϑ are the polar coordinates in the $y_1 y_2$ plane, $e^{(n)}_k$ is the unit vector corresponding to the winding number n:

$$e^{(n)}(\vartheta) = \begin{bmatrix} \cos(n\vartheta) \\ \sin(n\vartheta) \end{bmatrix}. \quad (5.2)$$

The boundary conditions for the functions $A(r)$ and $\Phi_0(r)$ are

$$A(r) \to \text{const}, \quad \Phi_0(r) \to \text{const} \times r^{|n|} \quad \text{for } r \to 0,$$
$$A(r) \to n/gr^2, \quad \Phi_0(r) \to \Phi_0 = \text{const} \quad \text{for } r \to \infty. \quad (5.3)$$

The broken translations in the y^α directions generate the translational zero modes

$$V^i = 0, \quad V^\alpha = F^{\alpha\beta} C_\beta(x), \quad \phi = -\left(\bar{D}^\alpha \Phi \right) C_\alpha(x). \quad (5.4)$$

The ground state (5.1) also breaks the rotational symmetry in the $y_1 y_2$ plane if $n \neq 0$. The field increase can be compensated by a gauge transformation and we get the simplest example of the combined symmetry $H = \mathrm{U}(1)$. In Sec. III we concluded that this symmetry is the global one and does not induce the zero mode. Let us show it directly from Eq. (2.22), which has the following form in our case:

$$\partial_\alpha \partial^\alpha \left(\delta_\xi g^{i\beta} A_\beta \right) + g^2 \Phi_0^2(r) \left(\delta_\xi g^{i\beta} A_\beta \right) = \partial^i C(x) \epsilon_{\alpha\beta} F^{\alpha\beta}. \quad (5.5)$$

If one denotes the common solution of the homogeneous equation as $v^i(x,y)$, the solution of Eq. (5.5) will be

$$\delta_\xi g^{i\beta} A_\beta = v^i(x,y) + \partial^i C(x) \epsilon^{\alpha\beta} y_\beta \left(A_\alpha - n \partial_\alpha \vartheta \right). \quad (5.6)$$

It should be considered as an equation for $\xi^i(x,y)$ at given $\xi^\alpha(x,y)$ [see Eq. (2.17)]. A solution for ξ^i also can be divided into two parts corresponding to the common and partial solutions of Eq. (5.5): $\xi^i(x,y) = \xi^i_c(x,y) + \xi^i_p(x,y)$. As is easy to see, there is no single value in the ϑ

function $\xi_p^i(x,y)$ producing the second term on the right-hand side of (5.6). So, one can conclude that $C(x)$=const in accordance with the expectations.

A quite different conclusion can be drawn for $n = 0$. Indeed, such a ground state is unstable, but it is not important for investigation of zero modes. The case gives us an example of the direct symmetry $S = \mathrm{U}(1)$ in vacuum. Really, for $n = 0$, Eq. (5.6) is solved at $\xi_p^i(x,y) = -\partial^i C_\alpha(x) y^\alpha$ and $C(x)$ remains arbitrary describing a local group parameter.

To find the solution $v^i(x,y)$ of the homogeneous equation and together with it the gauge mode $\varphi(x,y)$ and the gauge function $\chi(x,y)$, one expands them in Fourier series:

$$v^i(x,y) = \sum_{m=0}^{\infty} C^i_{\alpha m}(x) e^{(m)}_\alpha(\vartheta) f_m(r),$$

$$\varphi(x,y) = \sum_{m=0}^{\infty} \varphi_{\alpha m}(x) e^{(m)}_\alpha(\vartheta) f_m(r),$$

$$\chi(x,y) = \sum_{m=0}^{\infty} \chi_{\alpha m}(x) e^{(m)}_\alpha(\vartheta) f_m(r). \tag{5.7}$$

The coefficients $C^i_{\alpha m}(x), \varphi_{\alpha m}(x)$, and $\chi_{\alpha m}(x)$ can be treated as d-dimensional vector fields, gauge modes, and gauge functions, respectively. The functions $f_m(r)$ satisfy the equation

$$f_m'' + \frac{1}{r} f_m' - [g^2\Phi_0^2(r) + m^2/r^2] f_m = 0. \tag{5.8}$$

As was explained, in the common case the solutions of Eq. (5.8) must be singular at the origin or at infinity where they behave as

$$f_m(r) \sim \begin{cases} \ln(r), & m = 0, \\ r^{-m}, & m > 0, \end{cases} \qquad r \to 0 \tag{5.9}$$

for the solutions regular at infinity and as

$$f_m(r) \sim \frac{\exp(g\Phi_0 r)}{\sqrt{r}}, \qquad r \to \infty \tag{5.10}$$

for those regular at the origin. The only exception exists for the $n = 0$ string. In this case there is a special solution proportional to the background function $A(r)$:

$$f_1(r) = rA(r), \tag{5.11}$$

which is regular at any r.

All the described solutions of Eq. (5.8) contribute to $\varphi(x,y)$ and $\chi(x,y)$ but not to $v^i(x,y)$. Let us see the reason. Since $\xi_c^i(x,y)$ is single valued, it can be expanded in a Fourier series. Calculating $\delta_\xi g^{i\alpha} A_\alpha$, one finds that the term with $m = 0$ does not appear in the expansion and we should choose $C^i_{\alpha 0}(x) = 0$. However, such a term appears in $V^i(x,y)$ from $\partial^i \psi$, as was explained in Sec. III, and $C^i_{\alpha 0}(x)$ may be represented as the gradient of some scalar.

Now one introduces normalized d-dimensional fields and gauge functions instead of $C^i_{\alpha m}(x), \varphi_{\alpha m}(x)$, and $\chi_{\alpha m}(x)$:

$$W^i_{\alpha m}(x) = Z_m^{1/2} C^i_{\alpha m}(x), \quad \bar\varphi_{\alpha m}(x) = Z_m^{1/2} \varphi_{\alpha m}(x),$$

$$\bar\chi_{\alpha m}(x) = Z_m^{1/2} \chi_{\alpha m}(x), \quad m \neq 0. \tag{5.12}$$

The normalization factors Z_m are calculated from the kinetic energy of the vector fields (see Sec. IV) and in our case they are $Z_1 \sim \ln(r_0), Z_m \sim r_0^{-2m+2}$ $(m > 1)$ for the excitations localized at the origin and $Z_m \sim \exp(2g\Phi_0 R)$ for those at infinity; r_0 and R are the corresponding cutoff parameters. With the normalized excitations we can examine Eq. (4.9) in order to find the permitted gauge functions. There are no restrictions on the gauge functions concentrated at infinity. The same conclusion is true for the functions localized at the origin if the winding number $n > 1$. For the most interesting case $n = 1$ (stable string) the allowed singularities in $\chi(x,y)$ are not stronger than $1/r$. Therefore, we expected all the coefficients $\bar\chi_{\alpha m}(x)$ to be zero for $m > 1$. The vector fields $W^i_{\alpha m}(x)$ which correspond to the forbidden gauge functions $\bar\chi_{\alpha m}(x)$ lose the gauge transformations and there is no reason for them to be zero modes. Hence, these vectors must be reduced to the gradient form. We can see how it happens by analyzing the boundary condition (4.7).

One substitutes in Eq. (4.7) the excitations in the gauge (3.14). It is easy to discover that the integral over y vanishes only for the regular mode (5.11). It seems quite natural that all the other zero modes ought to be reduced to the gradient form. That is really so for the modes which lost the gauge freedom, but we should be careful for those which did not. After integration over y one has, instead of (4.7),

$$B \int d^d x \, (W^i_{\alpha m} - \partial^i \bar\varphi_{\alpha m})^2 = 0, \tag{5.13}$$

where B can be finite or divergent depending on which excitation is considered. Equation (5.13) can be interpreted as a boundary condition for the gauge mode $\bar\varphi_{\alpha m}(x)$. It becomes obvious in the Lorentz gauge $\partial_i W^i_{\alpha m} = 0$ where Eq. (5.13) can be obeyed by means of choosing a singular component in $\bar\varphi_{\alpha m}(x)$ at space-time infinity. Such a singular behavior is allowed for the gauge modes. Really, under the gauge transformations these modes get a shift $\bar\varphi_{\alpha m} \to \bar\varphi_{\alpha m} + \bar\chi_{\alpha m}$ as the phase of some scalar field. They do not appear in the quadratic in the fields part of the Lagrangian as the phase does. Singular behavior at space-time infinity is not forbidden for such degrees of freedom. It should be stressed that (5.13) is an unusual boundary condition in field theory. It may be important for interacting fields [12].

VI. CONCLUSION

We have proposed the method for constructing small massless excitations about a nontrivial classical solution which depends on part of the space coordinates. These excitations can be important for different physical problems. In particular, they describe a lower-dimensional theory of massless fields, which can be considered as the low-energy limit of the initial theory. The results of this paper allow one to construct the subjects of such a theory but do not define their interaction. In this connection we would like to make some comments.

Interaction of the zero modes has been analyzed for the $n = 1$ string in the Abelian Higgs model in Refs. [15,12]. The lower-dimensional Lagrangian was found to possess the global O(2) symmetry, which comes from the combined symmetry of the string background, and the local U(1) symmetry connected with the gauge transformations in the initial theory. The direct symmetry of a ground state generates another local group as was shown in Sec. III. To understand how this symmetry becomes apparent in interaction, one can investigate the simplest model which is the $n = 0$ string in the Abelian Higgs model.

When we analyze the interaction of the zero modes which are singular in y, one can expect additional trouble with singularities. They do not appear in the Abelian Higgs model since the terms describing interaction are of second order in singular fields. The problems can appear if the initial model is non-Abelian. Then the terms responsible for the self-interaction of the vector modes are expected to give additional constraints on the vector excitations to guarantee their finite energy.

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