PHYSICAL REVIEW D VOLUME 49, NUMBER 6 15 MARCH 1994

Pseudostable bubbles

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(Received 18 August 1993)

The evolution of spherically symmetric unstable scalar field configurations ("bubbles") is examined for both symmetric and asymmetric double-well potentials. Bubbles with initial static energies $E_0 \lesssim E_{\rm crit}$, where $E_{\rm crit}$ is some critical value, shrink in a time scale determined by their linear dimension or "radius." Bubbles with $E_0 \gtrsim E_{\rm crit}$ evolve into time-dependent, localized configurations which are *very* long-lived compared to characteristic time scales in the models examined. The stability of these configurations is investigated and possible applications are briefly discussed.

PACS number(s): 11.10.Lm, 98.80.Cq

A remarkable consequence of nonlinear field theories is the existence of localized, nonsingular solutions of the classical equations of motion which are nondissipative. In general, these solutions can be time dependent or static. As is well known, the existence and simplicity of static solutions is severely constrained by dimensionality [1]. For a single self-interacting real scalar field $\phi(\mathbf{x},t)$, such solutions are only possible in $1+1$ dimensions. More realistic $(3+1)$-dimensional static solutions must invoke more than one field, as in the case of the 't Hooft–Polyakov monopole [2].

Given their relevance to the study of nonperturbative effects in field theories, static, nondissipative solutions have been, for the last 20 years or so, the focus of most efforts in the study of nonlinear solutions in classical field theories. However, within the last decade, the possibility that spontaneous symmetry breaking occurred in the early universe has called for a better understanding of time-dependent phenomena in the context of relativistic field theories. For example, the dynamics of cosmological phase transitions [3] naturally invokes out-of-equilibrium conditions, with fields interacting with themselves and with a hot plasma in the background of an expanding universe [4].

In the present work the possibility that *time-dependent*, localized, nondissipative solutions exist in the context of simple $(3+1)$-dimensional scalar field theories is examined. In particular, the focus will be on models involving only a single real scalar field with self-interactions dictated by a double-well potential. Since we know that for a symmetric double-well potential (SDWP) all field configurations are unstable, it is possible to obtain the lifetime of a given spherically symmetric field configuration by numerically evolving the equation of motion. By adopting this procedure, it was shown in the mid-1970's that certain configurations evolved into a state which was considerably long lived (even though the lifetimes measured then were not very accurate). These time-dependent solutions were called "pulsons" [5]. However, not much has been done in order to further explore the properties of pulsons. A few exceptions, which are mainly related to the existence of these solutions for the sine-Gordon potential, different symmetries, and somewhat contrived stability studies, are listed in Ref. [6]. In fact, recent studies of bubble evolution in SDWP's overlooked the existence of pulsons [7]. Also, their existence has never been investigated for asymmetric double-well potentials (ADWP's).

Here it will be argued that the existence of pulsons is a very general feature of models with both symmetric *and* asymmetric potentials, depending only on the initial amplitude and energy of the configuration. It will always be assumed that the initial configuration interpolates between the two minima of the potential (bubbles with amplitude below a certain value cannot evolve into pulsons, as will be clear later) and that it is smooth enough, say, $\phi(r,t=0)\sim\exp[-r^2/R_0^2]$, or $\sim\tanh(r-R_0)$, with R_0 the initial "radius." (From now on these configurations will be called the Gaussian and tanh bubbles.) It is remarkable that during the nonlinear evolution of these configurations a regime of dynamical stability may be achieved in which energy is practically conserved within a localized region, due to some as yet unclear conspiracy between the many degrees of freedom involved.

The action for a real scalar field is

$$S[\phi]=\int d^4x\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-V_{S(A)}(\phi)\right] ,$$

where the subscripts S and A stand for the SDWP's and ADWP's, with

$$V_S(\phi)=\frac{\lambda}{4}\left[\phi^2-\frac{m^2}{\lambda}\right]^2$$

and

$$V_A(\phi)=\frac{m^2}{2}\phi^2-\frac{\alpha}{3}m\phi^3+\frac{\lambda}{4}\phi^4 .$$

A field configuration $\phi_0(\mathbf{x},t)$ is a solution of the equation of motion (an overdot denotes partial time derivative) $\ddot{\phi}-\nabla^2\phi=-\partial V(\phi)/\partial\phi$ and has an energy

$$E[\phi_0]=\int d^3x\left[\tfrac{1}{2}\dot{\phi}_0^2+\tfrac{1}{2}(\nabla\phi_0)^2+V(\phi_0)\right] . \quad (1)$$

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For a sufficiently well-localized spherically symmetric configuration $\phi_0(r,t)$ (the case of interest here) with linear "size" $\sim R_0$, the integral over all space can be restricted to a spherical volume containing the configuration

$$E[\phi_0](t)=E_K(t)+E_S(t)+E_V(t) , \quad (2)$$

where the kinetic, surface, and volume energies are defined, respectively, by ($\epsilon \gtrsim 2$)

$$E_K(t)=2\pi \int_0^{\epsilon R_0} dr\, r^2 \dot{\phi}_0^2 ,$$
$$E_S(t)=2\pi \int_0^{\epsilon R_0} dr\, r^2 (\phi_0')^2 ,$$
$$E_V(t)=4\pi \int_0^{\epsilon R_0} dr\, r^2 V(\phi_0) . \quad (3)$$

Introducing the dimensionless variables $\Phi(r,t) = \sqrt{\lambda}\phi(r,t)/m$, $\rho=mr$, and $\tau=mt$, the equation of motion becomes, for the SDWP (ADWP),

$$\frac{\partial^2\Phi}{\partial\tau^2}-\frac{\partial^2\Phi}{\partial\rho^2}-\frac{2}{\rho}\frac{\partial\Phi}{\partial\rho}=\Phi-\Phi^3(-\Phi+\tilde{\alpha}\Phi^2-\Phi^3) , \quad (4)$$

where $\tilde{\alpha}\equiv\alpha/\sqrt{\lambda}$. Note that for the ADWP a solution where $\Phi=0$ is the local minimum is possible only if $\tilde{\alpha}^2>\frac{9}{2}$. The equation above was solved numerically using a finite-differencing method fourth order accurate in space and second order accurate in time. Since the problem is two dimensional, a very fine grid could be used. By taking the lattice spacing to be $h=10^{-2}$ and the time step to be $\theta=5\times10^{-3}$, energy was conserved throughout the evolution to 1 part in 10^5.

Let us concentrate on the SDWP for now. In this case all bubbles are unstable, since there is no gain in volume energy in going from one vacuum to another. Take the vacuum to be at $\Phi=-1$ and consider configurations which interpolate between the two minima. Both thick- and thin-wall bubbles will be considered. In Fig. 1 the energy within a spherical shell [Eq. (2)] is shown for initial configurations $\Phi_0(\rho,0)=-\tanh(\rho-\rho_0)$, with $\rho_0=3$ and $\rho_0=10$, and for $\Phi_0(\rho,0)=2\exp[-\rho^2/\rho_0^2]-1$, with $\rho_0=4$ and $\rho_0=8$. Note the existence of an extended period of stability of duration $\sim(10^3-10^4)m^{-1}$, where practically no energy is radiated away. Thus a necessary condition (which was verified numerically) for the existence of this pseudostable behavior is, from Eq. (2),

$$\frac{dE_K}{dt}\simeq-\left[\frac{dE_S}{dt}+\frac{dE_V}{dt}\right] . \quad (5)$$

An interesting point is that the value of the energy at the plateau is fairly independent of the initial configuration. This suggests the existence of an attractorlike configuration in field space which is approached in the course of the bubble's evolution. An extensive (but not exhaustive) search indicates that only for configurations of initial energies $E_0\gtrsim 60m/\lambda\equiv E_{\rm crit}$ does this behavior occur. This corresponds to a Gaussian bubble of radius $R_0\sim2.4/m$. For smaller energies, the evolution is well fitted by the relation $E(t) = E_0\exp[-t/\tau_L(R_0)]$, where $\tau_L(R_0)$ is the bubble's lifetime. For Gaussian bubbles of radii $mR_0=1,2$, $m\tau_L(R_0)=3.5, 12$, respectively.

From Fig. 1 it is clear that the evolution of large enough bubbles can be divided into three stages. First, the bubble sheds its initial energy by quickly shrinking into a thick-wall bubble of energy roughly $E\sim50m/\lambda$. For thin-wall bubbles, such as the tanh bubble with $\rho_0\gg1$, this shrinking is well described by the relativistic motion of the bubble wall. (See, e.g., Ref. [7].) The field then settles into the pulson configuration. At any given time, the configuration is well approximated by a half-Gaussian, but with softer asymptotic behavior $\Phi(\rho\rightarrow\infty,\tau)\sim\exp(-\rho)$. The field is localized within a small volume with linear size $\sim3/m$, while its amplitude is rapidly oscillating about a value between the two minima. (In this author's opinion, a better name for these configurations would be "oscillons.") Finally, during the last stage of the evolution, the amplitude of oscillations decreases and the pulson quickly radiates its remaining energy away.

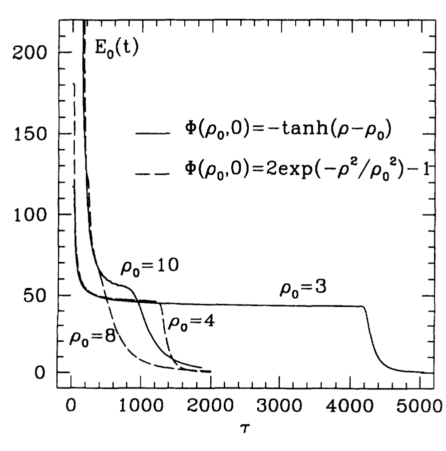

FIG. 1. Energy within a spherical shell surrounding several initial configurations as a function of time for the SDWP. Larger bubbles have shorter lifetimes.

In order to gain some insight into the properties of the pulson, consider the behavior of radial perturbations about $\Phi_0(\rho,\tau)$, defined as $\Phi(\rho,\tau)=\Phi_0(\rho,\tau)+\delta\Phi(\rho,\tau)$. Since $\Phi_0(\rho,\tau)$ satisfies the equation of motion, expanding $\delta\Phi(\rho,\tau)$ in normal modes, $\delta\Phi(\rho,\tau)={\rm Re}\sum_n\phi_n(\rho)\exp(i\omega_n\tau)$, it is found that the amplitudes $\phi_n(\rho)$ satisfy the radial Schrödinger equation (only the $l=0$ mode will be considered here)

$$-\frac{d^2\phi_n}{d\rho^2}-\frac{2}{\rho}\frac{d\phi_n}{d\rho}+V(\rho,\tau_0)\phi_n=\lambda_n\phi_n , \quad (6)$$

where $V(\rho,\tau_0)\equiv3\Phi_0^2(\rho,\tau_0)-3$ and $\lambda_n\equiv\omega_n^2-2$. Here λ_n is introduced to make sure that $V(\rho\rightarrow\infty)\rightarrow0$. The stability of a given configuration at time τ_0, $\Phi_0(\rho,\tau_0)$, is determined by the lowest eigenvalue being positive, $\omega_0^2>0$ or $\lambda_0>-2$.

Clearly the general problem is quite complicated, as the potential $V(\rho,\tau)$ is time dependent. A reasonable simplification is to solve this equation for a given time τ_0 for which the field configuration $\Phi_0(\rho,\tau_0)$ is obtained by evolving the equation of motion. This implicitly assumes that $\Phi_0(\rho,\tau)$ varies slower than $\delta\Phi(\rho,\tau)$. Otherwise, unstable modes would not have enough time to grow and destabilize the pulson. One can then find the eigenvalue for a succession of snapshots and thus investigate its time evolution. The Schrödinger equation was solved using the shooting method [8]. To make sure the method worked, the equation was solved for the Coulomb potential and for the "kink potential" $V(\rho)=3\cosh^{-2}(\rho/\sqrt{2})$, since in both cases the eigenvalues are known analytically [9]. [For the kink case, one must recall that the first eigenvalue in three dimensions (3D) corresponds to the second in 1D, $E_0(3D)=E_1(1D)=-\frac{1}{2}$, due to the boundary conditions at the origin.]

The results are shown in Fig. 2 during part of the pulson stage. Clearly, the pulson is stable against small radial perturbations. (No instability was detected during the whole pulson stage using the above method. An argument to explain the pulson's final disappearance is advanced shortly.)

Next, the existence of pseudostable behavior for ADWP's is investigated. The asymmetry is controlled by the dimensionless coupling $\tilde{\alpha}$. As two examples, consider the nearly degenerate case $\tilde{\alpha}=2.16$ and the nondegenerate case $\tilde{\alpha}=2.23$. In the context of first-order phase transitions (finite temperature), a state initially localized at $\Phi=0$ will decay by the nucleation of bubbles larger than the critical bubble, which is an extremum of the O(3)-invariant Euclidean action. The critical bubble is thus the energy barrier for vacuum decay. For a given value of $\tilde{\alpha}$ it is easy to obtain the critical bubble and its energy numerically by using the shooting method. (Now the method finds the value of the field in the bubble's core.) For the two values of the asymmetry above, the energies and radii of the critical bubble are, respectively, $E(\tilde{\alpha}=2.16)\simeq 6.12\times 10^2 m/\lambda$, $R(\tilde{\alpha}=2.16)\simeq 19.6/m$ and $E(\tilde{\alpha}=2.23)\simeq 1.17\times 10^2 m/\lambda$, $R(\tilde{\alpha}=2.23)\simeq 7.9/m$. The interest here is in investigating the evolution of *subcritical* bubbles to see if they display the pseudostable behavior observed for the SDWP. The results are shown in Fig. 3 where the energy within a spherical shell surrounding the initial configurations is shown as a function of time. For each value of $\tilde{\alpha}$, two tanh bubbles with initial radii $\rho_0=3$ and 4 were examined. Again, the existence of very long-lived pulsons is observed, with $E_{\rm crit}$ and lifetimes depending on the asymmetry. In Fig. 4 the phase-space portrait of the pulson's core ($\rho=0$) for $\tilde{\alpha}=2.16$ and $\rho_0=4$ is displayed for $\tau\geq 1000$. Note the similarity with the motion of a "damped" (an)harmonic oscillator. During the pulson stage, the motion is restricted to a band in phase space. As the pulson becomes unstable, it will spiral around the final stable point $\Phi=0$ and $\dot{\Phi}=0$; as energy is gradually radiated away, the maximum possible amplitude (that is, for $\dot{\Phi}\simeq 0$) is driven below a critical value for stability. Bubbles with initial amplitudes below this value, roughly about the maximum of $V(\Phi)$, *and* small kinetic energy will quickly shrink. Note that this is the value below which the potential is approximately parabolic, so that nonlinear effects become subdominant. (Of course, small-amplitude but high-velocity bubbles can still escape the attractive well centered at $\Phi=0$, as during the pseudostable pulson stage.)

The present work raises many questions for future investigation. Apart from a more detailed study of the pulson's properties and stability, it is still not clear why the initial configurations evolve into the pulson stage.

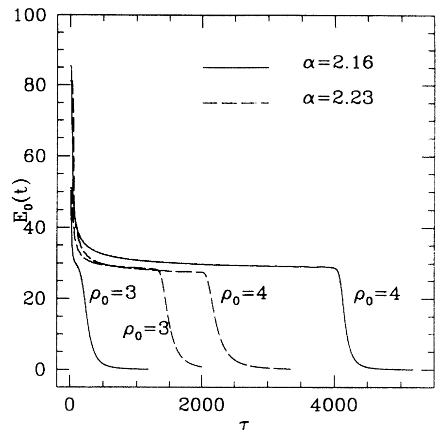

FIG. 3. Energy within a spherical shell surrounding several initial configurations as a function of time for the ADWP.

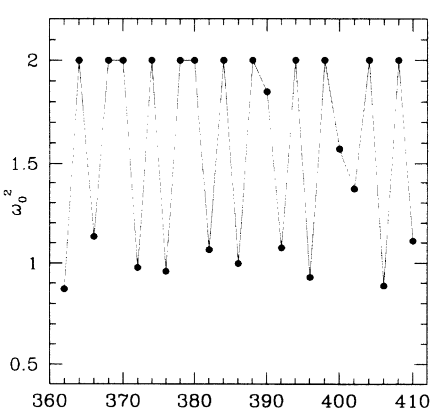

FIG. 2. Time evolution of lowest eigenvalue. For clarity, points with $\omega_0^2>2$ were displayed as $\omega_0^2=2$.

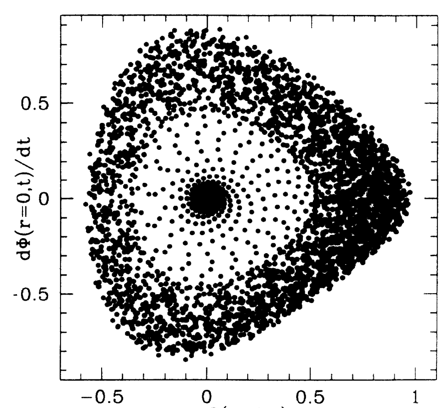

FIG. 4. Phase-space evolution of pulson's core for $\tilde{\alpha}=2.16$, $\rho_0=4$, and $\tau\geq 1000$.

The pulson's behavior is suggestive of some sort of nonlinear resonance effect occurring between the different modes of the field. It should be possible to separate the field into short and long wavelengths, with a cutoff around m^{-1}. The short-wavelength modes would act as a perturbation, which due to the nonlinear coupling might induce the observed behavior. Apart from their potential interest for, among other topics, nonlinear optics and long Josephson junctions [10], these solutions are also of interest in the context of cosmological phase transitions [3]. Consider two possible regimes of interest, defined by the ratio of time scales $\mathcal{R}=\tau_\phi/\tau_U\sim(\phi/\dot{\phi})/(m_{\rm Pl}/T^2)\sim T/m_{\rm Pl}$, where T is the temperature and $m_{\rm Pl}$ is the Planck mass. For $\mathcal{R}\sim 10^{-3}$, corresponding to the grand unified theory (GUT) scale, the extended lifetime of one subcritical bubble can be relevant to the dynamics of a first-order transition. The Universe cools off quite fast, and if the subcritical bubble lives long enough, it can *become* a critical bubble. For $\mathcal{R}\sim 10^{-16}$, corresponding to the electroweak scale, many subcritical bubbles can be present before a critical bubble nucleates [11]. Given their long lifetimes, they could serve as nucleation sites for critical bubbles, very much like impurities in ordinary phase transitions, speeding the process of vacuum decay considerably. Finally, in the context of late-time phase transitions, it is desirable to have bubbles live longer in order for matter accretion to be efficient [12]. These and related questions are currently under investigation.

I am grateful to J. D. Harris for many enlightening comments and to R. Ramos for useful discussions. I also thank M. Briggs for his numerical skills during the early stages of this investigation. This work was partially supported by a National Science Foundation Grant No. PHYS-9204726.

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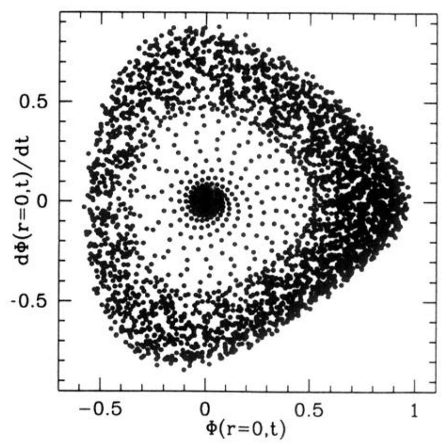

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