N-string vertices in string field theory

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We give the general form of the vertex corresponding to the interaction of an arbitrary number of strings. The technique employed relies on the "comma" representation of string field theory where string fields and interactions are represented as matrices and operations between them such as multiplication and trace. The general formulation presented here shows that the interaction vertex of N strings, for any arbitrary N, is given as a function of particular combinations of matrices corresponding to the change of representation between the full string and the half string degrees of freedom.

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I. INTRODUCTION

String field theory has provided a consistent picture for the treatment of open strings [1]. Recently, some advances have been made in the formulation of a field theory for closed strings [2]. This is welcomed from a phenomenological point of view since closed strings appear to give a suitable picture of string physics at low energy. A complete understanding of low energy string physics seems to require the treatment of strings in this framework.

The closed string field theory (CSFT) proposed in [2] has the particularity of requiring a nonpolynomial action in which at every step one has to include a term in the action corresponding to the interaction of an arbitrary number of strings over a world sheet given by the so-called restricted polyhedra. The edges of these polyhedra play the role of the modular parameters, and one restricts the region of integration over these in a prescribed way. In the theory, the interaction terms are interpreted as the overlapping of closed strings in a way which resembles the original theory for open strings due to Witten [1]. On the other hand, there is the suggestion in [3] that the overlapping of closed strings can be formulated, using the property of reparametrization invariance of the string amplitudes, as the overlapping of standard string segments which, following the example of the open strings, can be considered as half-strings. In turn, one interprets the string functionals as matrices and the interaction between strings as the product and trace of these matrices. This feature shows that, even in the case of closed strings, the half-string picture is relevant in the construction of the string interaction. However, the formulation from a Fock space approach seems to be a formidable task since in the absence of a compact formulation one should calculate separately every term in the action.

Hence our main motivation comes for the fact that, following [3], the N-faced polyhedra describing the interaction of N strings at the level of the action can be written as a reparametrization of the vertex in a contact interaction. Thus, as a first step, one finds it useful to work out the contact interaction vertex for an arbitrary number of strings.

The purpose of this work is then the study of the N-string contact interaction in a general form. This will form the basis of a compact formulation of the closed string vertex in the nonpolynomial theory. We perform the calculation of open strings since their formulation at the level of Fock space is firmly settled. On the other hand, they share many of the peculiarities of the operator formulation for closed strings [4]. The results obtained here are very similar, from a technical point of view, to those for the closed string, and so they can be adapted, with minor modifications, to the latter.

Related to this point, we will see that, apart from the fact that half-strings play a conceptual role in the formulation of closed string interactions, linking the CSFT [2] with the approach described in [3], they are revealed as a useful technical tool for the treatment of string amplitudes, both for open and closed strings.

In the theory of Witten [1] for string fields, the interaction between strings is defined by a product ($*$) and integration ($\int$) which are defined through the joining of half-strings. In particular, the interaction between two strings to give a third one is defined from the $*$ as

$$(\psi*\phi)[\mathbf{x}]=\int D\mathbf{y}\psi[\mathbf{x}_L;\mathbf{y}]\phi[\bar{\mathbf{y}}/\mathbf{x}_R] , \tag{1}$$

where ψ represents the string fields. The interaction takes place through the joining of half-strings. The operation of integration, which allows us to obtain invariant quantities, is defined as

$$\int\psi=\int D\mathbf{x}\psi[\mathbf{x};\bar{\mathbf{x}}] . \tag{2}$$

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Therefore the fundamental degree of freedom from the interaction point of view is the half-string. It is then justified to adopt an approach in which the half-string plays explicitly an important role. We will base our formalism on the "comma" representation of string field theory developed in [5].

In the "comma" formulation, we single out the midpoint and represent string fields as matrices. The string is divided into left and right parts which play the role of row and column indices of these matrices. Interaction takes place simply by multiplying [product (1)] and taking the trace [integration (2)] over the matrices. The advantage of this approach is that one can handle in a compact form the N-string contact interaction for any arbitrary N. In fact, the vertex for N strings is simply given by

$$V_N=\int d\mathbf{x}\left[\frac{\pi}{2}\right]\mathrm{Tr}[A_1A_2\cdots A_N]\ , \qquad (3)$$

where A_i are the matrices representing the string states.

We must point out that the former equation does not give the tree level N-string interaction, but only the contact term. In order to get the tree level amplitude, one could consider the reparametrization approach as described in [3] to get the correct moduli space of parameters. This, however, is beyond the scope of this paper, although work in this direction is under way in order to find a formulation suitable for the nonpolynomial CSFT.

To finish the Introduction, some comments are necessary about the ghost degrees of freedom. It is known that in Witten's theory the violation of the ghost number at the vertices and the ghost number of physical states fixes the value of N. In this sense, the vertex (3) vanishes unless $N=3$. This is of no relevance to us since our purpose is just the calculation of the N-string interaction vertex in order to get some insight of the structure of the terms in the theories of [2]. Hence in the following we will ignore the ghost degrees of freedom, although they certainly could be treated with the techniques presented here [6].

The plan of the paper is as follows. In Sec. II we present the formulation of the half-string degrees of freedom ("comma" representation). In Sec. III we discuss the construction of the string physical states, thereby settling the basis for the calculation of the N-string vertex. This calculation is performed in Sec. IV. In Sec. V we find the Fourier coefficients of the Neumann functions for the appropriated geometry of the vertex. A comparison with particular cases worked out previously is also given. Finally, we summarize our results in Sec. VI.

II. STRING AND "COMMA" COORDINATES

The first task we must face is the expression of the elements of string field theory, namely, the fields and interactions between them, in terms of matrices and operations such as trace and multiplication (3). To this end one has to introduce the degrees of freedom referring to the left and right halves of the string. Following our previous work [5], we define the functions

$$\begin{aligned}&\chi^{(1)}(\sigma,\tau)=\mathbf{x}(\sigma,\tau)-\mathbf{x}(\pi/2,\tau)\ ,\\&\chi^{(2)}(\sigma,\tau)=\mathbf{x}(\pi-\sigma,\tau)-\mathbf{x}(\pi/2,\tau)\ ,\\&\sigma\in[0,\pi/2]\ ,\end{aligned} \qquad (4)$$

where $\mathbf{x}(\sigma,\tau)$ are the string coordinates (space-time indices will be suppressed throughout). From their definition, it is clear that $\chi^{(1)}$ and $\chi^{(2)}$ are related to the left and right parts of the string, respectively (henceforth we will call them "comma" coordinates).

The boundary conditions as well as the constraint implied in the change of representation (see [5] for details) imply the Fourier expansion of the "comma" functions in terms of odd cosine modes with the explicit form

$$\chi^{(r)}(\sigma,\tau)=\sqrt{2}\sum_{n\geq 1}\chi_n^{(r)}(\tau)\cos(2n-1)\sigma \quad \text{where } r=1,2,\ \ \sigma\in[0,\pi/2]\ . \qquad (5)$$

The inverse relations are obtained by integration over one period:

$$\chi_n^{(r)}(\tau)=\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2}d\sigma\,\chi^{(r)}(\sigma,\tau)\cos(2n-1)\sigma \quad \text{where } r=1,2\ . \qquad (6)$$

Our purpose now is to find the relation between the "comma" modes and the conventional string ones. This will allow us to represent the physical states in terms of the "comma" degrees of freedom.

From definitions (4) and (6), using the standard open string mode expansion,

$$\mathbf{x}(\sigma,\tau)=x_0+p\tau+i\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in\tau}\cos n\sigma\ , \qquad (7)$$

one arrives at the relation

$$\chi_n^{(r)}(\tau)=\sum_{m\neq 0}\chi_{n,m}^{(r)}e^{-im\tau}\ ,$$

where the $\chi_{n,m}^{(r)}$ are time-independent coefficients given in terms of string oscillator modes by

$$\begin{aligned}&\chi_{n,2m}^{(r)}=-\sqrt{2}\frac{(-)^{n+m}}{\pi}\left[\frac{1}{2m+2n-1}-\frac{1}{2m-(2n-1)}-\frac{2}{2n-1}\right]\frac{i\alpha_{2m}}{2m}\ ,\\&\chi_{n,2m-1}^{(r)}=0\ ,\\&\text{for } n\neq m,-m \text{ and}\\&\chi_{n,n}^{(r)}=\frac{(-)^{r+1}}{\sqrt{2}}\frac{i\alpha_{2n-1}}{2n-1}\ ,\\&\chi_{n,-n}^{(r)}=\frac{(-)^{r}}{\sqrt{2}}\frac{i\alpha_{-(2n-1)}}{2n-1}\ .\end{aligned} \qquad (8)$$

Since we will mainly deal with string fields, namely, string wave functionals, we are only interested in the relations at fixed τ. Fixing $\tau=0$ in (8), we end up with a relation between the "comma" and the string oscillator modes, the latter defined according to

$$x_m=\frac{i}{\sqrt{2m}}(\alpha_m-\alpha_{-m}) ,$$

to give

$$\chi_n^{(r)}=(-1)^{r+1}x_{2n-1}+\sum_{m\geq 1}\left[\frac{2m}{2n-1}\right]^{1/2}[(M_1)_{m,n}+(M_2)_{m,n}]x_{2m} , \quad r=1,2 , \tag{9}$$

where the matrices M_1 and M_2 are

$$(M_1)_{n,m}=\frac{2}{\pi}\left[\frac{2n}{2m-1}\right]^{1/2}\frac{(-)^{n+m}}{2n-(2m-1)} ,$$
$$(M_2)_{n,m}=\frac{2}{\pi}\left[\frac{2n}{2m-1}\right]^{1/2}\frac{(-)^{n+m}}{2n+2m-1} . \tag{10}$$

It has been shown that the transformation (9) is nonsingular [5]; the inverse relation can be obtained from (6) to give

$$x_{2n-1}=\tfrac{1}{2}(\chi_n^{(1)}-\chi_n^{(2)}) ,$$
$$x_{2n}=\tfrac{1}{2}\sum_{m\geq 1}\left[\frac{2m-1}{2n}\right]^{1/2}[(M_1)_{m,n}-(M_2)_{m,n}]\times(\chi_m^{(1)}+\chi_m^{(2)}) . \tag{11}$$

In the decomposition of the string into left and right halves (4), we have singled out the midpoint coordinate. Therefore to complete the picture we need its expression in string coordinates, which, at $\tau=0$, reads

$$\mathbf{x}\left[\frac{\pi}{2}\right]=x_0+\sqrt{2}\sum_{n\geq 1}(-)^n x_{2n} ;$$

conversely, the center of mass in the "comma" representation is (again we take $\tau=0$)

$$x_0=\mathbf{x}\left[\frac{\pi}{2}\right]-\frac{\sqrt{2}}{\pi}\sum_{\substack{r=1,2\\ n\geq 1}}\frac{(-)^n}{2n-1}\chi_n^{(r)} .$$

These two relations together with the oscillator mode relations (9) and (11) complete the equivalence between the string oscillator modes $\{x_n\}_{n=0}^{\infty}$ and the position degrees of freedom describing the "comma" representation, which, in the transformation we have defined, are equivalent to the midpoint and the "comma" oscillator modes, i.e., $\{\chi_n^{(r)};\mathbf{x}(\pi/2)\}$, where $r=1,2$ and $n=1,\ldots,\infty$.

A. Conjugate momentum

For our purpose, we merely need the relations between the "comma" and string conjugate momenta; we can define the quantized momentum conjugate to $\chi_n^{(r)}$ and $\mathbf{x}(\pi/2)$ in the usual way:

$$\mathcal{P}_n^{(r)}=-i\frac{\partial}{\partial\chi_n^{(r)}} , \quad \mathcal{P}=-i\frac{\partial}{\partial\mathbf{x}(\pi/2)} .$$

Thus, using (9) and applying the chain rule, one can find the relation with the conventional string momenta (p_m). In summary, they are given by

$$\mathcal{P}_n^{(r)}=\tfrac{1}{2}p_{2n-1}+\tfrac{1}{2}\sum_{m\geq 1}\left[\frac{2n-1}{2m}\right]^{1/2}[(M_1)_{m,n}+(M_2)_{m,n}]p_{2m}-\frac{\sqrt{2}}{\pi}\frac{(-)^n}{2n-1}p_0 ,$$
$$\mathcal{P}=p_0 .$$

Also, the inverse relations read

$$p_{2n-1}=\mathcal{P}_n^{(1)}-\mathcal{P}_n^{(2)} ,$$
$$p_{2n}=\sum_{m\geq 1}\left[\frac{2n}{2m-1}\right]^{1/2}[(M_1)_{n,m}+(M_2)_{n,m}]\times(\mathcal{P}_m^{(1)}+\mathcal{P}_m^{(2)})+\sqrt{2}(-)^n\mathcal{P} . \tag{12}$$

Upon quantization, the commutation relations for the "comma" coordinates and momenta are the usual ones corresponding to a discrete set of conjugate variables: namely,

$$[\chi_n^{(r)},\mathcal{P}_m^{(s)}]=i\delta^{rs}\delta_{mn} ,$$
$$[\mathbf{x}(\pi/2),\mathcal{P}]=i ,$$

as can be explicitly checked from the previous relations.

From the one-to-one correspondence between the "comma" and string degrees of freedom as we have established in (9)–(13), we see that the identification

$$\mathcal{H}=\overline{\mathcal{H}_M\otimes\mathcal{H}_1\otimes\mathcal{H}_2}$$

is needed, where $\mathcal{H}$ stands for the string space, $\mathcal{H}_r$ are two copies of the half-string spaces, and $\mathcal{H}_M$ describes the midpoint. The overbar in the former expression stands for the completion of spaces; we need to take this completion in order to ensure a Hilbert space structure in both terms.

III. OPERATOR APPROACH TO THE COMMA REPRESENTATION

For practical purposes it is convenient to develop a formulation based on creation and annihilation operators and express the elements of the theory in terms of the tensor product of two copies of the Fock space representing the states of each half-string. This will allow us to overcome the ambiguities in the definition of the functional integral appearing in the product (1) and integration (2), since one ends up with a representation in which the states are just infinite matrices and the above-mentioned operations become products and traces of matrices, respectively.

In order to construct the Fock space of the "comma" states, let us define the creation and annihilation operators for the "comma" modes in the usual way:

$$b_n^{(r)}=\frac{-i}{\sqrt{2}}\left[\frac{2n-1}{2}\right]^{1/2}\left\{\chi_n^{(r)}+i\frac{2}{2n-1}\mathcal{P}_n^{(r)}\right\},$$

$$b_n^{(r)\dagger}=\frac{i}{\sqrt{2}}\left[\frac{2n-1}{2}\right]^{1/2}\left\{\chi_n^{(r)}-i\frac{2}{2n-1}\mathcal{P}_n^{(r)}\right\}$$

$$(n\geq 1)\ .$$

The degrees of freedom relative to the midpoint, namely, $\mathbf{x}(\pi/2)$ and $\mathcal{P}$, only appear in the Fock space states in the form of a plane wave. We can use directly those variables to generate the midpoint Hilbert space.

The meaning of the operators $b_n^{(r)\dagger}$ is clear: Acting on the "comma" vacuum state $|0\rangle_r$ creates a "comma" oscillator mode of half integral frequency $(n-\frac{1}{2})$; these vacua satisfy the relation $b_n^{(r)}|0\rangle_r=0$, $n=1,\ldots,\infty$.

Repeated action of $b_n^{(r)\dagger}$ on the vacuum gives the Fock space states corresponding to each half-string $(\mathcal{H}_r)$. Hence the complete space on which the "comma" states reside is given by the tensor product $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_M$ after folding in the piece corresponding to the midpoint motion $(\mathcal{H}_M)$.

Following similar steps as in the preceding section, we can find the relation between the "comma" creation and annihilation operators and the conventional ones. Using Eqs. (9) and (12), we find

$$b_n^{(r)}=\frac{\sqrt{2}}{\pi}\frac{(-)^{(n-1)}}{(2n-1)^{3/2}}p_0+\frac{1}{\sqrt{2}}(-)^{r+1}a_{2n-1}+\sum_{m=1}^{\infty}[(M_1)_{m,n}a_{2m}-(M_2)_{m,n}a_{2m}^\dagger] \tag{13}$$

and the corresponding relation for $b_n^{(r)\dagger}$ by changing $a_n \rightleftharpoons a_n^\dagger$.

Again, the inverse relations are given by

$$a_{2n-1}=b_n^{(-)}\ ,$$
$$a_{2n}=\frac{(-)^n}{\sqrt{2n}}\mathcal{P}+\sum_{m\geq 1}[(M_1)_{n,m}b_m^{(+)}-(M_2)_{n,m}b_m^{(+)\dagger}]\ . \tag{14}$$

We have defined the combinations $b_m^{(\pm)}=(1/\sqrt{2})(b_m^{(1)}\pm b_m^{(2)})$; the ones corresponding to the creation operators $b_m^{(r)\dagger}$ are defined in an analogous fashion. Finally, the relation for $a_n^\dagger$ is accordingly obtained.

With relations (13) and (14) in hand, we are able to obtain the string states in terms of the tensor product of "comma" Fock space states; in particular, we find that they belong to the completion of the space $\mathcal{H}$ defined above.

A. String states in the "comma" representation

It was already mentioned that string states, as gauge-invariant states and eigenstates of the string Hamiltonian [7], are defined from the conventional string creation and annihilation operators. In the following we use this definition and the relation with the "comma" modes (14) to write them in the "comma" representation.

First of all, we shall start with the string vacuum, which, in the string representation, is defined through the relations $a_n|0\rangle=0\forall n$. Then in view of relations (14) we can express the vacuum state as an exponential of a quadratic form in creation operators $b_n^{(r)\dagger}$ acting on the tensor product $|0\rangle_1|0\rangle_2$. Note that $a_{n=\text{odd}}|0\rangle=b_n^{(-)}|0\rangle=0$, and so only the combination $b_n^{(+)\dagger}$ appears in the exponential. Under these conditions the vacuum takes on a generic form similar to the BCS vacuum involving only the $b_n^{(+)\dagger}$ operators: namely,

$$\|0\rangle=\exp(-\tfrac{1}{2}b_n^{(+)\dagger}\phi_{n,m}b_m^{(+)\dagger})|0\rangle_1|0\rangle_2\ , \tag{15}$$

and the matrix ϕ is determined by acting with $a_{n=\text{even}}$. One finds

$$\phi=M_1^{-1}M_2\ .$$

Using the properties of the coefficients given in the Appendix and noting that in the particular case $p=2$ they are related to the combinatorial numbers $\binom{-1/2}{n}$, one arrives at the following expression for the elements of the matrix ϕ:

$$\phi_{n,m}=(2n-1)^{1/2}(2m-1)^{1/2}\frac{1}{2(n+m-1)}\times\begin{bmatrix}-\frac{1}{2}\\ n-1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ m-1\end{bmatrix}\ . \tag{16}$$

The tachyon state is immediately obtained just by inserting the plane wave of momentum p corresponding to the center-of-mass motion. Since the center-of-mass operator is expressible in terms of creation and annihilation operators as

$$x_0=\mathbf{x}\left[\frac{\pi}{2}\right]=i\frac{\sqrt{2}}{\pi}\sum_{n\geq 1}\frac{(-)^n}{(2n-1)^{3/2}}(b_n^{(+)}-b_n^{(+)\dagger})\ ,$$

one gets the tachyon in the "comma" representation

$$\|T\rangle=e^{ip\mathbf{x}(\pi/2)}\exp[-\tfrac{1}{2}p^2\vec{k}^T(I+\phi)\vec{k}-p\vec{k}^T(I+\phi)\vec{b}^{(+)\dagger}]\times\exp(-\tfrac{1}{2}\vec{b}^{(+)\dagger}\phi\vec{b}^{(+)\dagger})|0\rangle_1|0\rangle_2\ . \tag{17}$$

The vector $\vec{k}$ is given by

$$k_n=\frac{2}{\pi}\frac{(-)^n}{(2n-1)^{3/2}}\ ;$$

and the operators $b_n^{(-)\dagger},b_n^{(+)\dagger}$ are given above as a combination of the "comma" annihilation operators. As we can see, the momentum insertion unfolds into two pieces: the first one giving the midpoint motion, the second relating the two halves of the string.

B. String states

In the usual string picture, higher states are obtained by the action on (17) of the creation operators $a_n^\dagger$ (since $[a_n^\dagger,x_0]=0$, there is no ambiguity in this construction).

To get the general setting, it is useful to work out the coherent state bases of the string states defined by

$$\|\vec{\lambda},\vec{\lambda}';p)=e^{ipx_0}\|\vec{\lambda},\vec{\lambda}')$$

$$=e^{ipx_0}\exp\left[\sum_{n=1}^{\infty}(\lambda_n' a_{2n}^\dagger+\lambda_n a_{2n-1}^\dagger)\right]\|0\rangle \qquad (18)$$

in the "comma" representation. By taking the appropriate derivatives of these states, one can obtain string states with any occupation numbers. In general, for the string state with occupation numbers $\{n_i\}_{i-1}^{\infty}$ one has

$$\|\{n_i\}_{i=1}^{\infty}\rangle=\prod_{i=1}^{\infty}\frac{1}{\sqrt{n_i!}}(a_i^\dagger)^{n_i}\|0\rangle$$

$$=\prod_{i=1}^{\infty}\frac{1}{\sqrt{n_i!}}\frac{\partial^{n_{2i-1}}}{\partial\lambda_i^{n_{2i-1}}}\frac{\partial^{n_{2i}}}{\partial\lambda_i'^{n_{2i}}}\|\vec{\lambda},\vec{\lambda}')\ . \qquad (19)$$

These states are written in the "comma" representation making the appropriate substitution of the string oscillator creation operators in terms of the "comma" ones (14). Moving the annihilation operators to the right, one is left with the following expression for the coherent string state:

$$\|\vec{\lambda},\vec{\lambda}';p)=\mathbf{C}(p,\vec{\lambda}')e^{ipx(\pi/2)}\exp(\vec{\lambda}^T\vec{b}^{(-)\dagger}+\vec{\rho}^T\vec{b}^{(+)\dagger}-\frac{1}{2}\vec{b}^{(+)\dagger}\phi\vec{b}^{(+)\dagger})|0\rangle_1|0\rangle_2\ , \qquad (20)$$

where

$$\mathbf{C}(p,\vec{\lambda}')=\exp[-\tfrac{1}{2}p^2\vec{k}^T(I+\phi)\vec{k}+\tfrac{1}{2}\vec{\lambda}'^T(M_1^T)^{-1}M_2^T\vec{\lambda}'+p\vec{\lambda}'^T(M_1^T)^{-1}\vec{k}]\ ,$$

$$\vec{\rho}(p,\vec{\lambda})=-p(I+\phi)\vec{k}+(M_1^T)^{-1}\vec{\lambda}\ .$$

The matrix notation we mentioned in the Introduction can be now put forward. We can consider the string state as an operator acting on one copy of the Hilbert space corresponding to the half-string; its explicit form can be extracted from the states (20). Alternatively, we can define the matrix elements taking the scalar product with a generic string state with definite "comma" occupation numbers. The associated matrix is then defined as

$$[\vec{\lambda},\vec{\lambda}']_{\{n_i^{(2)}\}_{i=1}^{\infty}}^{\{n_i^{(1)}\}_{i=1}^{\infty}}=(-)^{\sum_{i=1}^{\infty}n_i^{(2)}}\langle\{n_i^{(1)}\};\{n_i^{(2)}\}\|\vec{\lambda},\vec{\lambda}')\ , \qquad (21)$$

where $|\{n_i^{(1)}\};\{n_i^{(2)}\}\rangle$ is the tensor product of two "comma" states: namely,

$$|\{n_i^{(r)}\}_{i=1}^{\infty}\rangle=\prod_{i=1}^{\infty}\frac{1}{\sqrt{n_i^{(r)}!}}(b_i^{(r)\dagger})^{n_i^{(r)}}|0\rangle_r\ . \qquad (22)$$

The factor $(-)^{\sum_{i=1}^{\infty}n_i^{(2)}}$ appears in the definition of (21) to conform with the standard convention in string field theory that the parametrization of the second half of the string is reversed.

By using standard techniques, the creation and annihilation operators can be dealt with to work out the explicit form of the matrix elements in (21). The final answer is

$$[\vec{\lambda},\vec{\lambda}']_{\{n_i^{(2)}\}_{i=1}^{\infty}}^{\{n_i^{(1)}\}_{i=1}^{\infty}}=e^{ipx(\pi/2)}\mathbf{C}(p,\vec{\lambda}')\prod_{i=1}^{\infty}\frac{1}{\sqrt{n_i^{(1)}!n_i^{(2)}!}}\left[-\frac{1}{\sqrt{2}}D_i^-\right]^{n_i^{(1)}}\left[\frac{1}{\sqrt{2}}D_i^+\right]^{n_i^{(2)}}e^{-z^T\phi z/2}\Big|_{\vec{z}=0}\ . \qquad (23)$$

The only new quantities are $D^\pm$ containing derivatives with respect to the auxiliary parameter $\vec{z}$. They are defined by

$$D_i^\pm=\frac{\partial}{\partial z_i}-pk_j(I+\phi)_{ji}+\lambda_j'(M_1^T)_{j,i}^{-1}\pm\lambda_i\ .$$

This expression gives the matrix form of the coherent state basis of the string representation of the physical string states. It can be used in equations such as (3) to express the star product as a product of matrices and integration as a trace. This will be described in detail in the next section.

Before going on, two comments are in order. First, note that the matrix elements defined in Eqs. (21) and (23) can be viewed as the ones corresponding to the change of basis between the representations given by the string states (19) and the "comma" states (22). If the Hilbert space were finite dimensional, the transformation would be automatically complete since, by construction, it relates two orthogonal bases of it. In our case, the string Hilbert space being of infinite dimension, one should actually prove this property explicitly by working out Parseval's identity both ways. This proof was carried out in [5], showing the equivalence between the two representations.

Second, in order to establish the validity of the "comma" representation to describe string field theory, we should be able to generate the physical state spectrum and the scattering amplitudes among the string states. To that end we have constructed in [5] the Virasoro algebra in the half-string representation, taking care of the possible ambiguities coming from normal ordering in both representations. It was shown that gauge and on-shell conditions are kept in the "comma" representation.

IV. N-STRING INTERACTION VERTEX

This section is devoted to the calculation of the generic N-string vertex. Once the matrix representation of the string states has been put forward, we can interpret the $*$ and integral (1),(2) in terms of these matrices. In fact, since the $*$ product involves the identification between left and right parts of contiguous strings, we immediately see that this translates into our picture as the product of matrices representing two contiguous strings. On the other hand, the integral, which in some sense closes the cycle, is just the trace of the product of these matrices.

Therefore, defining the string states through the coherent states $|\vec{\lambda}^{(k)},\vec{\lambda}^{(k)\prime}\rangle$ given in (18), the N-string interaction vertex, according to (3) and the comments in the paragraph above, reads

$$\mathcal{V}_N=\int d\mathbf{x}\left[\frac{\pi}{2}\right]\exp\left[\sum_{i=1}^{N}p^{(i)}\mathbf{x}\left[\frac{\pi}{2}\right]\right]\mathrm{Tr}([\vec{\lambda}^{(1)},\vec{\lambda}^{(1)\prime}]\cdots[\vec{\lambda}^{(N)},\vec{\lambda}^{(N)\prime}])\ ,$$

where the matrices $[\vec{\lambda}^{(k)},\vec{\lambda}^{(k)\prime}]$ were defined in (23). The midpoint coordinate has been explicitly separated. Integration over this coordinate can be performed in a straightforward way, giving a δ function of conservation of momentum. This proves that the midpoint plays the role of the translational mode of the string as one would guess from the definition of the "comma" coordinates. Thus the part in the vertex involving the trace of the product of matrices contains all the relevant information. It is given explicitly by

$$\begin{aligned}\mathrm{Tr}(&[\vec{\lambda}^{(1)},\vec{\lambda}^{(1)\prime}]\cdots[\vec{\lambda}^{(N)},\vec{\lambda}^{(N)\prime}])\\ &=\prod_{i=1}^{N}\mathbf{C}(p^{(i)},\vec{\lambda}^{(i)\prime})\sum_{\{n_i^{(r)}\}}\prod_{i=1}^{\infty}\frac{1}{n_i^{(1)}!\cdots n_i^{(N)}!}(-\tfrac{1}{2}D_i^{+(1)}D_i^{-(N)})^{n_i^{(1)}}(-\tfrac{1}{2}D_i^{+(2)}D_i^{-(1)})^{n_i^{(2)}}\\ &\qquad\times\ldots(-\tfrac{1}{2}D_i^{+(N)}D_i^{-(N-1)})^{n_i^{(3)}}\exp(-\tfrac{1}{2}\vec{z}\Phi\vec{z})|_{\vec{z}=0}\ ,\end{aligned}$$

where the upper index in $D_i^{(r)}$ refers to the rth string state, $\vec{z}=(\vec{z}^{(1)},\ldots,\vec{z}^{(N)})$, and $\mathbf{\Phi}=\phi\mathbf{I}_N$, $\mathbf{I}_N$ being the identity matrix in the N-dimensional space spanned by the N strings.

The sum over $n_i^{(r)}$ can be readily performed, giving a more manageable expression:

$$\begin{aligned}\mathcal{V}_N=\delta&\left[\sum_{r=1}^{N}p^{(r)}\right]\prod_{i=1}^{N}\mathbf{C}(p^{(1)},\vec{\lambda}^{(1)\prime})\ldots\mathbf{C}(p^{(N)},\vec{\lambda}^{(N)\prime})\\ &\times\exp(-\tfrac{1}{2}[\vec{D}^{-(1)}\vec{D}^{+(N)}+\vec{D}^{-(2)}\vec{D}^{+(1)}+\ldots+\vec{D}^{-(N)}\vec{D}^{-(N-1)}])\exp(-\tfrac{1}{2}\vec{z}\mathbf{\Phi}\vec{z})|_{\vec{z}=0}\ .\end{aligned}$$

The derivatives in the auxiliary variable $\vec{z}$ can be carried out by using standard techniques on quadratic forms, and after a tedious, although straightforward, calculation, one ends up with the expression for the vertex. Up to a global normalization factor, we have

$$\mathcal{V}_N=\delta\left[\sum_{r=1}^{N}p^{(r)}\right]\exp\left[b\sum_{r=1}^{N}p^{(r)2}\right]\exp(\vec{\Lambda}^T\mathbf{B}_1\vec{\Lambda}+\vec{\Lambda}'^T\mathbf{B}_1'\vec{\Lambda}')\exp(\vec{\Lambda}^T\mathbf{B}_2\vec{\mathbf{p}}+\vec{\Lambda}^{T\prime}\mathbf{B}_2'\mathbf{B}_2'\vec{\mathbf{p}}+\vec{\Lambda}^T\mathbf{B}\vec{\Lambda}')\ . \tag{24}$$

In (24), boldface characters refer to vectors and matrices in the N-space spanned by the N strings, namely, $\vec{\mathbf{p}}=(p^{(1)}\vec{k},\ldots,p^{(N)}\vec{k})$, $\Lambda=(\vec{\lambda}^{(1)},\ldots,\vec{\lambda}^{(N)})$, and $\mathbf{\Lambda}'=(\vec{\lambda}^{(1)\prime},\ldots,\vec{\lambda}^{(N)\prime})$; on the other hand, $\mathbf{B}$, $\mathbf{B}_i$, and $\mathbf{B}_i'$ are $(N\times N)$-dimensional matrices whose elements are again infinite-dimensional matrices.

These quantities have the following explicit expressions in terms of the matrices of change of representation M_1 and M_2:

$$\begin{aligned}b&=-\tfrac{1}{2}\vec{\mathbf{k}}^T(\mathbf{M}_1^T+\mathbf{M}_2^T)(1+\mathbf{S}_+)(\mathbf{M}_1^T-\mathbf{S}_+\mathbf{M}_2^T)^{-1}\vec{\mathbf{k}}\ ,\\ \mathbf{B}_1&=\tfrac{1}{2}[\mathbf{S}_+-\mathbf{S}_-^T(\mathbf{M}_1^T-\mathbf{S}_+\mathbf{M}_2^T)^{-1}\mathbf{M}_2\mathbf{S}_-]\ ,\\ \mathbf{B}_1'&=\tfrac{1}{2}(\mathbf{M}_1^T-\mathbf{S}_+\mathbf{M}_2^T)^{-1}(\mathbf{M}_2^T-\mathbf{S}_+\mathbf{M}_1^T)\ ,\\ \vec{\mathbf{B}}_2&=\mathbf{S}_-(\mathbf{M}_1^T-\mathbf{S}_+\mathbf{M}_2^T)^{-1}(\mathbf{M}_2+\mathbf{M}_1)\vec{\mathbf{k}}\ ,\\ \vec{\mathbf{B}}_2'&=-(\mathbf{M}_1^T-\mathbf{S}_+\mathbf{M}_2^T)^{-1}(1+\mathbf{S}_+)\vec{\mathbf{k}}\ ,\\ \mathbf{B}&=-\mathbf{S}_-(\mathbf{M}_1^T-\mathbf{S}_+M_2^T)^{-1}\end{aligned} \tag{25}$$

[the only quantities not yet defined are the matrices $(\mathbf{S}_\pm)_{ij}=\tfrac{1}{2}(\delta_{i+1,j}\pm\delta_{i-1,j})$, where the lower indices are defined mod$N$].

We first diagonalize the matrix $\mathbf{S}_+$. It is worthwhile going into some detail. The characteristic equation for $\mathbf{S}_+$ ($\det[\mathbf{S}_+-\lambda\mathbf{I}_N]=0$) can be written as

$$\det(\mathbf{S}_+-\lambda\mathbf{I}_N)=-\lambda\det\mathbf{M}_{N-1}-2\det\mathbf{M}_{N-2}-2(-)^N=0\ ,$$

where $\mathbf{M}_N$ is the N-dimensional matrix with elements

$$(\mathbf{M}_N)_{i,j}=\tfrac{1}{2}(\delta_{i,j+1}+\delta_{i,j-1})\ .$$

The determinant $\det\mathbf{M}_N$, and in turn the above equation, can be calculated in a recursive form in terms of the dimension N. We can write this in matrix form

$$\begin{bmatrix}\mathbf{M}_N\\ \mathbf{M}_{N-1}\end{bmatrix}=\begin{bmatrix}-2\lambda & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{M}_{N-1}\\ \mathbf{M}_{N-2}\end{bmatrix}\ , \tag{26}$$

with the subsidiary conditions $\det\mathbf{M}_1=-\lambda$ and $\det\mathbf{M}_0=1$. After diagonalization one gets the solution for $\det\mathbf{M}_N$:

$$\det\mathbf{M}_N=\frac{\mu_+^{N+1}-\mu_-^{N-1}}{\mu_+-\mu_-},$$

where $\mu_\pm=-\lambda\pm[\lambda^2-1]^{1/2}$. Hence the solutions of the characteristic equation satisfy

$$[-\lambda\pm(\lambda^2-1)^{1/2}]^N=(-)^N.$$

Solving this equation, we get the values of $\lambda=\cos(2k\pi/N)$ with $k=1,\ldots,N$, which are the eigenvalues of the matrix $\mathbf{S}_+$.

Now the matrix that performs the diagonalization of $\mathbf{S}_+$ [we use the notation $\mathbf{S}_+=\mathbf{R}^T\mathbf{D}\mathbf{R}$, with $\mathbf{D}_{i,j}=\delta_{i,j}\cos(2k\pi/N)$] is given by

$$\begin{aligned}
\mathbf{R}_{i,j}&=\left[\frac{2}{N}\right]^{1/2}\cos\frac{\pi i}{N}(2j-3)\quad \text{for } 1\le k\le\left[\frac{N-1}{2}\right],\\
\mathbf{R}_{N/2,j}&=\left[\frac{1}{N}\right]^{1/2}(-)^{j+1},\\
\mathbf{R}_{i,j}&=\left[\frac{2}{N}\right]^{1/2}\sin\frac{\pi i}{N}(2j-3)\quad \text{for } \left[\frac{N+1}{2}\right]<k<N,\\
\mathbf{R}_{N,j}&=\left[\frac{1}{N}\right]^{1/2}.
\end{aligned}\tag{27}$$

Substituting in (25) and using the symmetry properties of the trigonometric functions, one ends up with the following expressions for the terms appearing in the exponent of the vertex:

$$\begin{aligned}
b&=-\frac{4}{N}\sum_{k=1}^{[(N-1)/2]}\cos^2\frac{k\pi}{N}\cos\frac{2k\pi(i-j)}{N}\vec{k}^T(M_1^T+M_2^T)\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}\vec{k},\\
\mathbf{B}_1^{i,j}&=\tfrac{1}{2}(\delta_{i,j+1}+\delta_{i,j-1})I-\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\sin^2\frac{2k\pi}{N}\cos\frac{2k\pi(i-j)}{N}M_2^T\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1},\\
\mathbf{B}_1'^{i,j}&=\frac{1}{N}[(-)^{i+j}-1]I+\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\cos\frac{2k\pi(i-j)}{N}\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}\left[M_2^T-\cos\frac{2k\pi}{N}M_1^T\right],\\
\vec{B}_2^{i,j}&-\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\sin\frac{2k\pi}{N}\cos\frac{2k\pi(i-j)}{N}\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}(M_2+M_1)\vec{k},\\
\vec{B}_2'^{i,j}&=\frac{2}{N}(M_1+M_2)\vec{k}+\frac{4}{N}\sum_{k=1}^{[(N-1)/2]}\cos^2\frac{k\pi}{N}\cos\frac{2k\pi(i-j)}{N}\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}\vec{k},\\
\mathbf{B}^{i,j}&=\frac{1}{N}\sum_{k=1}^{[(N-1)/2]}\sin\frac{2k\pi}{N}\sin\frac{2k\pi(i-j)}{N}\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}.
\end{aligned}\tag{28}$$

This gives the final form of the vertex obtained in this approach. The important thing to note at this point is that this result is given in terms of the change of representation matrices M_1 and M_2 (10) that appear in the combination $[M_1^T-\cos(2k\pi/N)M_2^T]^{-1}$ with $k=1,\ldots,N$. Once the inverse of this matrix is obtained, we can directly identify the elements of the exponent of Eq. (24) with the Fourier components of the Neumann function corresponding to the N-string contact interaction. We postpone the calculation of this matrix and the identification of the Neumann functions until the next section.

Before concluding this section, let us make some comments on the symmetries of the coefficients appearing in (28). First note that regarding the diagonal terms ($i=j$), they are independent of the position. This corresponds to the fact that they describe the self-interaction of the strings and, in the general form worked here, they are equivalent. The second property to note is that, if A is any of the matrices in (28), they satisfy $A^{i,j}=A^{i+l,j+l}$ as it corresponds to the fact that they describe the interaction of the string in position i ($i+l$) with the one at position j ($j+l$). Finally, the diagonal terms in the matrices $\mathbf{B}^{i,j}$ vanish; this corresponds to the lack of connection between *odd* and *even* modes in the same string.

V. NEUMANN COEFFICIENTS FOR THE N-STRING VERTEX

In this section we finish the calculation of the vertex and proceed to the identification of the coefficients appearing in the exponent of the vertex in Eqs. (24) and (28), with the ones obtained via the conventional approach based on the path integral formulation of string amplitudes. We want to stress the fact that in this approach the Neumann coefficients are obtained in a compact form once the way to invert the matrix $[M_1^T-\cos(2k\pi/N)M_2^T]^{-1}$ is known.

Before going on let us outline briefly the calculation of the vertex based on the path integral formalism of the string σ model.

The N-string amplitude for a closed region by the path integral

$$\int DX(z)\exp\left\{-\frac{1}{4\pi}\int d^2z[\partial X(z)]^2 + \sum_{r=1}^{N}\int_0^{\pi}dz_r X(z_r)p^r(z_r)\right\} ,$$

where a general state is represented by the momentum distribution $p^r(z_r)$. The vertex is given by

$$\mathcal{V}_N=\exp\left\{-\frac{1}{2}\sum_{r,s=1}^{N}\int_0^{\pi}dz_r\int_0^{\pi}dz_s p^r(z_r)N(z_r,z_s)p^r(z_s)\right\} ,$$

where the Neumann function is defined as the solution of the differential equation

$$(\partial_\tau^2+\partial_\sigma^2)N(z,z')=2\pi\delta^2(z-z') ,$$

$$\partial_n N(z,z')=f(z) ,$$

where $\vec{n}$ is a vector normal to the boundary. The former equation can be cast in terms of the Fourier components of the Neumann function as

$$\mathcal{V}_N=\exp\left\{\frac{1}{2}\sum_{r,s=1}^{N}\sum_{n,m=1}^{\infty}p_n^r N_{n,m}p_m^r\right\} . \tag{29}$$

Now we can compare the vertices of Eqs. (24) and (29) and identify the quantities in (28) with the Fourier components of the Neumann functions.

To get an explicit expression, we proceed to invert the matrix:

$$\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1} .$$

Consider the generic combination

$$\alpha M_2^T-\beta M_1^T .$$

We propose the ansatz

$$\alpha'\frac{v_{2m}^{(1/p)}u_{2n-1}^{(1/p)}+v_{2n-1}^{(1/p)}u_{2m}^{(1/p)}}{2m-2n+1} +\beta'\frac{u_{2n-1}^{(1/p)}v_{2m}^{(1/p)}-v_{2n-1}^{(1/p)}u_{2m}^{(1/p)}}{2m+2n-1}$$

for the inverse. The coefficients $u_n^{(1/p)}$ and $v_n^{(1/p)}$ are the coefficients of the Taylor expansion of the rational functions

$$\left[\frac{1+x}{1-x}\right]^{1/p} , \quad \left[\frac{1+x}{1-x}\right]^{1-1/p} ,$$

respectively. On the other hand, α' and β' as well as p are free parameters to be determined.

Imposing the condition that our ansatz is the required inverse, we end up with the following equations that restrict the values of the free parameters α', β' and $1/p$:

$$\alpha\alpha'-\beta\beta'\cos\frac{\pi}{p}=0 ,$$

$$\alpha\beta-\beta\alpha'\cos\frac{\pi}{p}=0 ,$$

$$\frac{(-)^{n+m}}{2\sin(\pi/p)}(2m)^{-1/2}(2n)^{-1/2}=\beta\alpha' .$$

From these equations we fix the free parameters. The relations read

$$\alpha'=-\frac{1}{4}\frac{1}{\sin(\pi/p)\beta} ,$$

$$\beta'=\frac{1}{4}\frac{\cos(\pi/p)}{\sin(\pi/p)\beta} , \tag{30}$$

$$\cos^2\frac{\pi}{p}=\frac{\alpha^2}{\beta^2} .$$

In the particular case of interest to us, the values of the free parameters are

$$\alpha=\cos\frac{2k\pi}{N} , \quad \beta=1 , \quad \text{and} \quad \cos^2\frac{\pi}{p}=\cos^2\frac{2k\pi}{N} .$$

From them we obtain

$$\alpha'=\beta'=-\frac{1}{4\sin(2k\pi/N)} \quad \text{and} \quad p=\frac{N}{2k} \quad \text{for } k=1,\ldots,\left[\frac{N-1}{2}\right] ,$$

or

$$p=\frac{N}{2(N-k)} \quad \text{if } k=\left[\frac{N+1}{2}\right],\ldots,(N-1)$$

(the cases $k=N,N/2$ are trivially solved). These results complete the form of the inverse matrix. For instance, if $k<[(N-1)/2]$, the result reads

$$\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}\Bigg|_{m,n}=\frac{(-)^{n+m}(2m)^{1/2}(2n-1)^{1/2}}{2\sin(2k\pi/N)}\left[v_{2m}^{(2k/N)}u_{2n-1}^{(2k/N)}+v_{2n-1}^{(2k/N)}u_{2m}^{(2k/N)} + u_{2n-1}^{(2k/N)}v_{2m}^{(2k/N)}-v_{2n-1}^{(2k/N)}u_{2m}^{(2k/N)}\right] .$$

Once the inverse matrix is obtained, it is a straightforward matter to calculate the matrices appearing in (28) and the further identification with the Neumann functions. To illustrate the sort of calculation involved, let us consider the case of the quadratic momentum term. We want to compute the quantity

$$\vec{k}^T(M_1^T+M_2^T)\left[M_1^T-\cos\frac{2k\pi}{N}M_2^T\right]^{-1}\vec{k}\ ,$$

which can be written as

$$\frac{8}{\pi^3}\frac{1}{\sin(2k\pi/N)}\frac{1}{(2n-1)^2}\left[\frac{1}{2m+2n-1}+\frac{1}{2m-2n+1}\right]v_{2m}^{(2k/N)}\Sigma_0^u\ .$$

Then, using the properties of the sum given in the Appendix [(A3), (A5), and (A7) in particular] we end up with the result

$$\frac{2}{\pi\sin(2k\pi/N)}\tilde{\Sigma}_0^v\ ,$$

which leads to the value of the Neumann coefficient $N_{0,0}$. Proceeding in a similar fashion, we can obtain the remaining Neumann coefficients described in (28) and (29). The final result is

$$\begin{aligned}
N_{0,0}^{i,j}&=\frac{1}{N}\sum_{k=1}^{[(N-1)/2]}\cos\frac{2k\pi(i-j)}{N}\left[\psi\left[1-\frac{k}{N}\right]+\psi\left[\frac{k}{N}\right]-2\psi(1)+4\ln 2\right]\ ,\\
(-)^nN_{n,m}^{i,j}&=-\delta_{n,m}\frac{\delta_{i,j-1}+\delta_{i,j+1}}{2}+\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\cos\frac{2k\pi(i-j)}{N}\frac{(-)^{n+m}}{2}n^{1/2}m^{1/2}\\
&\quad\times\left[\frac{u_n^{(2k/N)}v_m^{(2k/N)}+v_n^{(2k/N)}u_m^{(2k/N)}}{n+m}+\frac{u_n^{(2k/N)}v_m^{(2k/N)}-u_m^{(2k/N)}v_n^{(2k/N)}}{n-m}\right]\quad\text{for } n+m \text{ even}\ ,\\
N_{n,m}^{i,j}&=-\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\sin\frac{2k\pi(i-j)}{N}\frac{(-)^{n+m}}{2}n^{1/2}m^{1/2}\\
&\quad\times\left[\frac{u_n^{(2k/N)}v_m^{(2k/N)}+v_n^{(2k/N)}u_m^{(2k/N)}}{n-m}-\frac{u_n^{(2k/N)}v_m^{(2k/N)}-u_m^{(2k/N)}v_n^{(2k/N)}}{n+m}\right]\quad\text{for } n+m \text{ odd}\ ,\\
N_{0,2n}^{i,j}&=\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\cos\frac{2k\pi(i-j)}{N}\frac{(-)^n}{(2n)^{1/2}}v_{2n}^{(2k/N)}\ ,\\
N_{0,2n-1}^{i,j}&=\frac{2}{N}\sum_{k=1}^{[(N-1)/2]}\sin\frac{2k\pi(i-j)}{N}\frac{(-)^n}{(2n-1)^{1/2}}v_{2n-1}^{(2k/N)}\ ,
\end{aligned}\tag{31}$$

which is only valid for $N\geq 2$ (when $N=2$, there are no sums). These equations do not include the case $N=1$, although it can be treated trivially in the half-string formulation; see, for example, [6]. This completes the calculation of the N-string interaction vertex (24) and represents our main result. The interaction between string Fock space states is readily obtained by taking derivatives with respect to the parameters λ,λ', as was indicated in (20).

We insist again on the fact that the Neumann functions are generated in an explicit way from the representation changing matrices M_1 and M_2. Therefore in this picture they appear as derived quantities. This assertion is true for every N-string contact interaction, thus providing a relation among the Neumann functions associated with the N-string vertices.

To illustrate these results, we can consider the case $N=3$, for which the sums in (31) only have the term corresponding to $k=1$. The result one obtains from (31) agrees with previous calculations of the cubic interaction vertex performed earlier [8,9].

VI. CONCLUSIONS

In this work we have calculated the vertex corresponding to the contact interaction of N strings. We have used the "comma" representation of string field theory in which the prominent role that the joining of the half-strings plays is apparent. This feature also appeared in the pioneering work of Witten [1].

This approach has the advantage of furnishing a compact treatment of the vertex, giving a result of general validity, independent of the number of strings. The final answer is always given in terms of a matrix which involves particular combinations between the matrices that change from the string representation (in which string physical states take on a simple form) to the "comma" representation in which interaction takes a trivial form.

The relevant matrix has been calculated and in general is given in terms of the Taylor coefficients of particular rational functions. Those coefficients, their sum rules, and most of their properties which are relevant to our work have been worked out in the Appendix. With this result one can readily identify the Fourier coefficients of the Neumann function for the N-string geometry. Finally, we checked our results against the simple case of the cubic interaction in agreement with the known results.

The task to face now is twofold. On the one hand, work is under way to implement this result to the case of closed strings. The required modifications are trivial and will be reported in the near future. On the other hand, the extension to the description of the restricted polyhedra describing the terms in the nonpolynomial action of closed strings will require the study of these vertices under reparametrizations. Work in this direction is under way. In particular, it is not difficult to prove [10] that the modular parameters appearing in the CSFT, namely, the length of the string overlaps, are related in a straightforward way to the reparametrization parameters of [3].

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APPENDIX

In this appendix we give the properties of the coefficients of the Taylor expansion of the functions

$$\left[\frac{1+x}{1-x}\right]^{1/p}=\sum_{n=1}^{\infty}u_n^{(1/p)}x^n ,$$

$$\left[\frac{1+x}{1-x}\right]^{1-1/p}=\sum_{n=1}^{\infty}v_n^{(1/p)}x^n .$$

We need them for the construction of the Fourier coefficients of the Neumann functions carried out in Sec. V.

Most of the results derived here are a generalization of the calculations performed in Refs. [8,9] where the case $p=3$ was analyzed in detail.

With the above definition of the coefficients $u_n^{(1/p)}$ and $v_n^{(1/p)}$, we can express them in an integral form

$$u_n^{(1/p)}=\frac{1}{2\pi i}\oint_0\frac{dx}{x^{n+1}}\left[\frac{1+x}{1-x}\right]^{1/p} .$$

The $v_n^{(1/p)}$'s are expressed in an analogous fashion with the substitution of $1/p$ by $1-1/p$. This expression is useful to find the recursion relation obeyed by the coefficients. Integration by parts of the derivative of the integrated function gives

$$\frac{2}{p}u_n^{(1/p)}=(n+1)u_{n+1}^{(1/p)}-(n-1)u_{n-1}^{(1/p)} ,$$

$$2\left[1-\frac{1}{p}\right]v_n^{(1/p)}=(n+1)v_{n+1}^{(1/p)}-(n-1)v_{n-1}^{(1/p)} . \tag{A1}$$

Also, making use of the same integral representation, one can relate both coefficients:

$$\frac{2}{p}v_n^{(1/p)}=(-)^n[(n+1)u_{n+1}^{(1/p)}-2nu_n^{(1/p)}+(n-1)u_{n-1}^{(1/p)}] ,$$

$$2\left[1-\frac{1}{p}\right]u_n^{(1/p)}=(-)^n[(n+1)v_{n+1}^{(1/p)}-2nv_n^{(1/p)}+(n-1)v_{n-1}^{(1/p)}] . \tag{A2}$$

It is also possible to find a closed expression for the coefficients, although it will not be necessary for our purposes.

In the text and, in particular, in Sec. V, one needs the evaluation of several infinite sums involving these coefficients. The simplest of these sums is

$$\Sigma_n^u=\sum_{n+m\ \mathrm{odd}}\frac{u_n^{(1/p)}}{n+m} .$$

It can be written in an integral form allowing its evaluation. For instance, for *odd* indices, we have

$$\sum_{m=0}\frac{u_{2m-1}^{(1/p)}}{2n+2m-1}=\frac{1}{\sin(\pi/p)}\frac{1}{8i}\times\oint_0\frac{dx}{x^{2n+1}}\left[\left[\frac{1+x}{1-x}\right]^{1/p}+\left[\frac{1+x}{1-x}\right]^{-1/p}\right] ,$$

the last integral being proportional to the coefficient $u_{2n}^{(1/p)}$. As a result, we find

$$\Sigma_n^u=\sum_{n+m\ \mathrm{odd}}\frac{u_m^{(1/p)}}{n+m}=\frac{\pi}{2}\frac{1}{\sin(\pi/p)}u_n^{(1/p)} . \tag{A3}$$

From this relation, it is then straightforward to deduce a recursion relation for these sums similar to the one found previously for the coefficients $u_n^{(1/p)}$ and $v_n^{(1/p)}$ (A1):

$$\frac{2}{p}\Sigma_n^u=(n+1)\Sigma_{n+1}^u-(n-1)\Sigma_{n-1}^u . \tag{A4}$$

We can extrapolate this result to find the sum for negative values of the index: namely,

$$\Sigma_{-n}^u=-\frac{\pi}{2}\cot\frac{\pi}{p}u_n^{(1/p)} . \tag{A5}$$

To find this result, we need a boundary condition that can be given by the sum Σ_0. We can proceed by direct integration:

$$\Sigma_0^u=\frac{1}{2}\int_1^{\infty}\frac{dx}{x}\left[\left[\frac{x+1}{x-1}\right]^{1/p}-\left[\frac{x-1}{x+1}\right]^{1/p}\right]$$

$$=\frac{1}{2}\left[\psi\left[\frac{1}{2}+\frac{p}{2}\right]-\psi\left[\frac{1}{2}-\frac{p}{2}\right]\right]=\frac{\pi}{2}\tan\frac{\pi}{2p} .$$

The change of variable one needs to perform the integration, namely, $y=[\cosh(\frac{1}{2}\ln x)]^{-2}$, was already suggested in [8].

Note that, in the sums Σ_n evaluated above, one is summing over the index m with parity opposed to n. The sums for indices with the same parity are more involved, and we will only discuss the properties which are more relevant to our work.

First, we discuss the sums involving quadratic denominators: namely,

$$\tilde{\Sigma}_n^u=\sum_{n+m\ \mathrm{odd}}\frac{u_n^{(1/p)}}{(n+m)^2} .$$

One can show, solely making use of the recursion relations given in Eq. (A1), the recursion relation

$$(n+1)\tilde{\Sigma}_{n+1}^u=\frac{2}{p}\tilde{\Sigma}_n^u+(n-1)\tilde{\Sigma}_{n-1}^u+\Sigma_{n+1}^u-\Sigma_{n-1}^u ,$$

which can be extended to negative values of the index n and by direct evaluation of the combination

$$\cos\frac{\pi}{p}\tilde{\Sigma}_1^u-\tilde{\Sigma}_{-1}^u=\frac{\pi}{2}\sin\frac{\pi}{p}\Sigma_0^u S_1^u ,$$

which we use as a boundary condition, we find the relation between the sums for positive and negative indices:

$$\tilde{\Sigma}_{-n}^u-\tilde{\Sigma}_n^u\cos\frac{\pi}{p}=\frac{\pi}{2}\sin\frac{\pi}{p}\Sigma_0^u S_n^u .$$

The sum

$$S_m=\sum_{n+m\ \mathrm{even}}\frac{u_n^{(1/p)}}{n+m} ,$$

which involves summing with indices of the same parity, appears here. It can directly be shown that they satisfy the same recursion relation as before (A4):

$$\frac{2}{p}S_n^u=(n+1)S_{n+1}^u-(n-1)S_{n-1}^u .$$

Note at this point that a recursion relation of this kind has two different solutions, one proportional to the coefficients $u_n^{(1/p)}$, which is the one given in (A3), and the other one, corresponding to the sums S_n which behaves as $1/n$ when $n\to 0$. The general form of the latter can be obtained by using the generating function

$$S(x)=\sum_{n=1}^{\infty}S_n x^n$$

and converting the recursion relation into a differential equation whose solution is given by

$$S(x)=\left[\frac{1+x}{1-x}\right]^{1/p}\int_0^x dy\left[\frac{1+y}{1-y}\right]^{1/p}\frac{u_0^{(1/p)}y+S_1}{1-y^2} .$$

This can be solved in terms of the original coefficients. We will skip the details; instead, we evaluate several combinations of these sums which are relevant in Sec. V.

Let us calculate the combination given by

$$u_{2n-1}^{(1/p)}S_{2n-1}^v+v_{2n-1}^{(1/p)}S_{2n-1}^u . \qquad (A6)$$

We start from the quantity

$$T_{n,m}=\frac{u_n^{(1/p)}v_m^{(1/p)}+u_m^{(1/p)}v_n^{(1/p)}}{n+m} ,$$

which, using the relations (A1) and (A2), can be shown to satisfy

$$(m+1)T_{m+1,n}-(m-1)T_{m-1,n}+(n+1)T_{m,n+1}-(n-1)T_{m,n-1}=0 ,$$

for $n+m$ odd. Now, taking $m\to 2m-1$, $n\to 2n$, and summing over the index m, one finds the recursion relation

$$(2n+1)[u_{2n+1}^{(1/p)}S_{2n+1}^v+v_{2n+1}^{(1/p)}S_{2n+1}^u]=(2n-1)[u_{2n-1}^{(1/p)}S_{2n-1}^v+v_{2n-1}^{(1/p)}S_{2n-1}^u] ,$$

which has a solution

$$u_{2n-1}^{(1/p)}S_{2n-1}^v+v_{2n-1}^{(1/p)}S_{2n-1}^u=\frac{S_1}{2n-1} .$$

To determine the value of S_1, we proceed directly writing it in its integral form to find $S_1=2$, a result that is independent of the value of $1/p$.

Following the same steps, one can easily evaluate the combinations

$$u_{2n}^{(1/p)}S_{2n}^v+v_{2n}^{(1/p)}S_{2n}^u=\frac{2}{2n} ,$$

$$u_{2n-1}^{(1/p)}S_{2n-1}^v+v_{2n-1}^{(1/p)}S_{2n-1}^u-[u_{2n-1}^{(1/p)}S_{-(2n-1)}^v-v_{2n-1}^{(1/p)}S_{-(2n-1)}^u]=-\frac{1}{2n-1} ,$$

which also appear in the calculation of the interaction vertex.

To take care of the indetermination appearing in the functions $N_{n,m}^{i,j}$ in the limit when $n\to m$, we can show that

$$\lim_{n\to m}\frac{u_n^{(1/p)}v_m^{(1/p)}-v_n^{(1/p)}u_m^{(1/p)}}{2(n-m)}=\frac{2}{\pi}\sin\frac{\pi}{p}[u_m^{(1/p)}\tilde{\Sigma}_m^v-v_m^{(1/p)}\tilde{\Sigma}_m^u] .$$

This result is obtained just by writing the left-hand side in an integral form.

Finally, in the evaluation of the Neumann coefficient N_{00}, one needs the sum

$$\tilde{\Sigma}_0^u=\sum_{n=1}^{\infty}\frac{u_{2n-1}^{(1/p)}}{(2n-1)^2}\ ,$$

which can be performed using its integral representation:

$$\tilde{\Sigma}_0^u=-\lim_{k\to 0}\frac{1}{2}\frac{d}{dk}\int_0^1\frac{dx}{x}x^k\left[\left[\frac{1+x}{1-x}\right]^{1/p}-\left[\frac{1-x}{1+x}\right]^{1/p}\right]$$
$$=-\frac{1}{4}\frac{\cos(\pi/2p)}{\sin(\pi/2p)}\left[\psi\left[\frac{1}{2}+\frac{1}{2p}\right]+\psi\left[\frac{1}{2}-\frac{1}{2p}\right]-2\psi(1)+4\ln 2\right]. \quad \text{(A7)}$$

To end this appendix, note that the discussion we have undertaken for the coefficients $u_n^{(1/p)}$ can be translated into $v_n^{(1/p)}$ with the only change of $1/p \rightleftharpoons 1-1/p$.

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