PHYSICAL REVIEW D VOLUME 49, NUMBER 6 15 MARCH 1994

Antisymmetric tensor coupling and conformal invariance in σ models corresponding to gauged WZNW theories

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(Received 2 November 1993)

String backgrounds associated with gauged G/H WZNW models generically depend on α' or $1/k$. The exact expressions for the corresponding metric $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$, and dilaton ϕ can be obtained by eliminating the two-dimensional gauge field from the local part of the effective action of the gauged WZNW model. We show that there exists a manifestly gauge-invariant prescription for the derivation of the antisymmetric tensor coupling and discuss some subtleties involved. When the subgroup H is one dimensional and G is simple the antisymmetric tensor is given by the semiclassical (α'-independent) expression. We consider in detail the simplest nontrivial example with $B_{\mu\nu}\neq 0$, the $D=3$ σ model corresponding to the $[\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW theory ("charged black string"), and show that the exact expressions for $G_{\mu\nu}$, $B_{\mu\nu}$, and ϕ solve the Weyl invariance conditions in the two-loop approximation. A similar conclusion is reached for the closely related $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ chiral gauged WZNW model. We find that there exists a scheme in which the semiclassical background is also a solution of the two-loop conformal invariance equations (but the tachyon equation takes a noncanonical form). We discuss in detail the role of field redefinitions (scheme dependence) in establishing a correspondence between the σ model and conformal field theory results.

PACS number(s): 11.25.Hf, 02.40.Dr

I. INTRODUCTION

Gauged Wess-Zumino-Novikov-Witten (WZNW) theories provided first examples of string solutions which depend nontrivially on α'. This fact should have important implications in the context of establishing closer relations between field theoretic (or σ model) and conformal field theory approaches. The derivation of the exact metric $G_{\mu\nu}$ and dilation ϕ corresponding to the $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ WZNW model was originally given [1] in the operator approach which is based on interpreting the Hamiltonian $L_0+\bar{L}_0$ of the coset conformal field theory as a Klein-Gordon-type operator in a background. This approach was systematized for establishing the exact form of $G_{\mu\nu}$ and ϕ in general G/H coset models in [2–5].

As the group space backgrounds of WZNW models, the geometries associated with gauged WZNW theories in general have a nontrivial antisymmetric tensor field $B_{\mu\nu}$ (which is crucial for conformal invariance of the corresponding σ models). This was found in simple $D=3$ and $D=4$ examples in the leading order ("semiclassical") approximation [6–10].[1] To be able to study the exact properties of these geometries one needs to know the expression for $B_{\mu\nu}$ to all orders in $1/k$.

The problem of establishing the exact form of the antisymmetric tensor background turns out to be quite nontrivial. The operator approach is not well suited for derivation of the expression for $B_{\mu\nu}$ since the antisymmetric tensor does not appear in the zero mode part of the L_0 operator. In principle, one is to consider the L_0 acting on states ψ of the first excited level and to try to deduce the value of $H_{\mu\nu\lambda}=3\partial_{[\mu}B_{\nu\lambda]}$ by identifying $L_0\psi$ with the "anomalous dimension operator," i.e., the derivative $(\partial\bar{\beta}^i/\partial\varphi^j)_{\varphi^*}$ of the σ-model Weyl anomaly coefficients parametrized by the values of $G_{\mu\nu},B_{\mu\nu},\phi$ at the conformal point [5]. This procedure looks rather indirect and complicated. Moreover, it may not work at all beyond the leading order in α' since the general functional expressions for the Weyl anomaly $\bar{\beta}$ functions are not known explicitly and so one is unable to deduce the exact expression for $B_{\mu\nu}(\alpha')$ from the comparison of L_0 with $(\partial\bar{\beta}^i/\partial\varphi^j)_{\varphi^*}$ unless some extra considerations (e.g., implying that for some reason there should exist a scheme in which $B_{\mu\nu}$ does not receive α' corrections at all) are invoked.

An alternative is provided by the effective action approach [15,16,5] which, in principle, offers a direct derivation of the whole σ-model action (i.e., $B_{\mu\nu}$ along with $G_{\mu\nu}$) at the field-theoretic level. The key point is to replace the classical gauged WZNW action $I(g,A)$ by the quantum effective one $\Gamma(g,A)$, solve for the two-dimensional ($2d$) gauge field and identify the local second-derivative part of the result with the σ-model ac-

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[1]There are of course examples of gauged WZNW models (based on non-Abelian groups) with vanishing antisymmetric tensor in the leading order approximation [11–14].

tion. It was proved in [15,16,5] that the operator and effective action approaches give identical expressions for the metric and dilaton. As for the antisymmetric tensor, there is a subtlety in its derivation from the effective action. It turns out that the procedure of omitting the nonlocal terms in $\Gamma(g,A)$ on the way to a σ-model action *a priori* is not completely unambiguous in what concerns the resulting expression for $B_{\mu\nu}$. Below we shall consider two natural prescriptions for the derivation of $B_{\mu\nu}$ that preserve the gauge invariance [present in the full nonlocal functional $\Gamma(g,A)$] at the level of the local part of the effective action. The gauge invariance is necessary in order to be able to reduce (gauge fix) the σ model from the group manifold of G to G/H as a configuration space. According to the first prescription, $B_{\mu\nu}$ in general depends on α', while $B_{\mu\nu}$ computed using the second prescription remains semiclassical.

We shall see that in certain cases (when the group G is simple and the H part of the matrix C_{AB} defining the adjoint representation of G is symmetric, or, equivalently, when $\dim H=1$) the exact expression for $B_{\mu\nu}$ found using the first prescription reduces to the semiclassical one (even though the metric and dilaton still depend on α'). For generic simple G and H the quantum corrections to $B_{\mu\nu}$ start with the "two-loop" $O(1/k^2)$ terms. We shall present the derivation of the σ-model action using the most general expression for the effective action $\Gamma(g,A)$ which includes as particular cases the effective actions for the gauged WZNW model as well as for the bosonic [17] and heterotic [18] (bosonic part of (1,0) supersymmetric) "chiral gauged" [19] WZNW models.

As was shown explicitly in the three-loop [20] and four-loop [21] approximations in the σ-model perturbation theory, the exact $D=2$ "black hole" metric-dilaton background of [1] is, in fact, a solution of the σ-model Weyl invariance conditions or the string effective equations. Below we shall conduct a similar check in the simplest possible case with a nontrivial $B_{\mu\nu}$: the three-dimensional $[\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ ("charged black string") model [6]. The exact metric and dilaton of this model were found in [3]. As for the antisymmetric tensor, its derivation turns out to be complicated by the fact that G is not simple here so that one needs to fix the total derivative ambiguity in the effective action in a specific way.

The Weyl anomaly coefficients ($\bar\beta$ functions) or the corresponding string effective action are not unambiguous, i.e., are scheme (field redefinition) dependent [22–24]. The "two-loop" $O(\alpha')$ term in the string effective action depends on a number of free parameters which change under field redefinitions. We shall show that there exists such a scheme (i.e., a choice of the parameters) in which the exact "black string" background is indeed a solution of the string equations in the two-loop approximation. We shall consider in detail several limits of the $D=3$ geometry, in particular, the $\mathrm{SL}(2,\mathbb{R})$ group space and the direct product of $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ model and $\mathbb{R}$ ("neutral black string"). We shall discuss the role of the coupling constant redefinitions or scheme dependence in establishing a correspondence between the conformal field theory and σ-model results.

A particular limit of the $[\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW model was shown [18] to be equivalent to the $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ chiral gauged WZNW model [25,17,18].[2] Our derivation of the exact form of the σ-model couplings in the two models gives equivalent results not only for the metric and dilaton but also for the antisymmetric tensor. We shall thus check explicitly that the $D=3$ σ model associated with the $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ chiral gauged model is also conformally invariant at the two-loop level.

In Sec. II we shall find the general expressions for the exact backgrounds starting from the effective action for the gauged WZNW model. We shall consider two natural ("corrected" and "semiclassical") prescriptions for extracting the σ-model couplings which give different expressions for the antisymmetric tensor coupling. In Sec. III we shall repeat the derivation using a more general form of the effective action which includes as particular cases the effective actions of the gauged WZNW and chiral gauged WZNW models. The explicit formulas for the σ-model couplings in Secs. II and III will be given in the case when the group G is simple and a gauged subgroup H is a vector one.

In Sec. IV we shall describe the $D=3$ σ model originating from the $[\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW model and show (in the α'^2 approximation) that the corresponding exact background solves the conformal invariance equations $\bar\beta^i=0$ in a proper scheme. Moreover, we shall find (in Sec. V) that there exists a scheme in which the semiclassical limit of this background is also a solution of the two-loop $\bar\beta^i=0$ equations and will discuss implications of this fact.

In Appendix A we shall prove that for a general model of Sec. III the "measure factor" $\exp(-2\phi)\sqrt{\det G}$ does not receive nontrivial quantum corrections. In Appendix B we shall discuss the derivation of the exact couplings in a theory with a nonsimple group G—the $[\mathrm{SL}(2,\mathbb{R}\times\mathbb{R}]/\mathbb{R}$ model. We shall show how to resolve the ambiguity in the resulting expression for the antisymmetric tensor coupling in a way that turns out to be consistent with conformal invariance of the σ model. In Appendix C we shall give the expressions for some geometrical quantities used in the computations of Sec. IV.

II. EXACT σ MODEL CORRESPONDING TO GAUGED WZNW MODEL

Let us start by recalling the basic steps one is to follow in order to derive the exact σ-model action corresponding to a gauged WZNW or coset model. If one solves the classical equations for the $2d$ gauge field A, substitutes the solution back into the classical WZNW action $S(g,A)$ [26] and fixes the gauge symmetry, one obtains a "semiclassical" σ model with the configuration space G/H which (with the dilaton coupling included [27]) is conformally invariant in the one-loop approximation. To obtain a σ model which is conformally invariant to all or-

[2]It can be shown in general [18] that for any Abelian subgroup H of a group G the G/H chiral gauged WZNW model is equivalent to a $(G\times H)/H$ gauged WZNW model in the axial gauging and special embedding of H into $G\times H$.

ders in the loop expansion one needs (in a "standard" scheme, see Sec. V) to modify the σ-model couplings by α'-dependent terms. By definition of the model, the gauge field A is to be treated as an auxiliary field (for which one does not introduce a source term in the path integral) but it is not obvious how to integrate it out completely (while keeping the group variable g classical) in a way that preserves conformal invariance to all orders. An approach that preserves conformal invariance is based on first determining the quantum effective action $\Gamma(g,A)$ for both g and A (i.e., treating g and A on an equal footing in the path integral) and then solving for A and eliminating it from the effective action. The resulting gauge invariant functional $\Gamma'(g)$ (restricted to G/H) is to be identified with the quantum *effective* action of the corresponding σ model. That means that the *local* part of $\Gamma'(g)$ should be equal (after gauge fixing to G/H) to the *classical* action $S(x)$ of the exact σ model one is looking for. It is clear that in deriving $S(x)$ from $\Gamma(g,A)$ one is free to ignore various nonlocal terms but at the same time one must be careful to preserve in the process the gauge invariance that makes possible to finally restrict (by the usual procedure of gauge fixing) the configuration space of the σ model to G/H.

The classical action of a gauged WZNW model

$$I(g,A)=I(g)+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(A\bar\partial g g^{-1}-\bar A g^{-1}\partial g+g^{-1}Ag\bar A-A\bar A)\ ,$$
$$I\equiv\frac{1}{2\pi}\int d^2z\,\mathrm{Tr}(\partial g^{-1}\bar\partial g)+\frac{i}{12\pi}\int d^3z\,\mathrm{Tr}(g^{-1}dg)^3\ , \tag{2.1}$$

is invariant under the standard vector H-gauge transformations ($A,\bar A$ take values in the algebra of H)

$$g\to u^{-1}gu,\quad A\to u^{-1}(A-\partial)u\ ,$$
$$\bar A\to u^{-1}(\bar A-\bar\partial)u,\quad u=u(z,\bar z)\in H\ . \tag{2.2}$$

In this section we shall assume that the group G and the subgroup H are simple. Parametrizing A and $\bar A$ in terms of subgroup elements $h,\bar h\in H$ which transform as $h\to u^{-1}h$, $\bar h\to u^{-1}\bar h$,

$$A=\partial h h^{-1},\quad \bar A=\bar\partial\bar h\bar h^{-1}\ , \tag{2.3}$$

one can use the Polyakov-Wiegman identity [28] to represent the action (2.1) in terms of the two WZNW actions corresponding to the group G and to the subgroup H,

$$I(g,A)=I(\tilde g)-I(\tilde h)\ ,$$
$$\tilde g=h^{-1}g\bar h,\quad \tilde h=h^{-1}\bar h\ . \tag{2.4}$$

The corresponding quantum effective action [15,16] has a simple form in terms of $\tilde g,\tilde h$ (we ignore extra nonlocalities introduced by field renormalizations because they give rise to nonlocal terms in the σ model, see [5])

$$\Gamma(g,A)=(k+g_G)I(h^{-1}g\bar h)-(k+g_H)I(h^{-1}\bar h)$$
$$=(k+g_G)[I(g,A)-b\Omega(A)]\ , \tag{2.5}$$

where g_G,g_H are the dual Coxeter numbers for the group G and the subgroup H, respectively, and $b\equiv-(g_G-g_H)/(k+g_G)$. $\Omega(A)$ is the nonlocal gauge invariant functional of $A,\bar A$,

$$\Omega(A)\equiv I(h^{-1}\bar h)$$
$$=\omega(A)+\bar\omega(\bar A)+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(A\bar A)\ , \tag{2.6}$$

where the nonlocal functionals $\omega(A)$ and $\bar\omega(\bar A)$ are defined as

$$\omega(A)\equiv I(h^{-1})=-\frac{1}{\pi}\int d^2z\,\mathrm{Tr}\left[\frac12 A\frac{\bar\partial}{\partial}A+\frac13 A\left[\frac1\partial A,\frac{\bar\partial}{\partial}A\right]+O(A^4)\right]\ ,$$
$$\bar\omega(\bar A)\equiv I(\bar h)=-\frac{1}{\pi}\int d^2z\,\mathrm{Tr}\left[\frac12 \bar A\frac{\partial}{\bar\partial}\bar A-\frac13 \bar A\left[\frac1{\bar\partial}\bar A,\frac{\partial}{\bar\partial}\bar A\right]+O(\bar A^4)\right]\ . \tag{2.7}$$

Before solving the equations for A and $\bar A$, which follow from (2.5), let us first drop the nonlocal cubic and higher order in $A,\bar A$ terms in (2.7). As explained above, the nonlocal terms should not contribute to the σ-model action which is our final goal. The truncated action has the form

$$\Gamma_{\rm tr}(g,A)=(k+g_G)[I(g,A)-b\Omega_{\rm tr}(A)]\ , \tag{2.8}$$

where

$$\Omega_{\rm tr}(A)\equiv\frac{1}{2\pi}\int d^2z\,\mathrm{Tr}(A-\tilde{\bar A})(\bar A-\tilde A)\ , \tag{2.9}$$

or

$$\Omega_{\rm tr}(A)=\frac{1}{2\pi}\int \mathrm{Tr}F_0\frac{1}{\partial\bar\partial}F_0,\quad F_0\equiv\bar\partial A-\partial\bar A\ .$$

The fields $\tilde A,\tilde{\bar A}$ in (2.8) are defined by

$$\tilde A\equiv\frac{\bar\partial}{\partial}A=\bar\partial h h^{-1}+\cdots,\quad \tilde{\bar A}\equiv\frac{\partial}{\bar\partial}\bar A=\partial\bar h\bar h^{-1}+\cdots\ , \tag{2.10}$$

and transform under the gauge transformations as

$$\tilde A\to u^{-1}(\tilde A-\bar\partial)u+\cdots\ ,$$
$$\tilde{\bar A}\to u^{-1}(\tilde{\bar A}-\partial)u+\cdots\ . \tag{2.11}$$

Here (and in similar equations below) ellipses stand for contributions of higher order nonlocal terms. Note that in (2.6) the Polyakov-Wiegmann formula was used, which presumes an integration by parts. In (2.8) we have undone this integration by parts, thus restoring the manifestly gauge invariant form of the quantum term under the transformation of the gauge fields (2.2) and (2.11). One may question whether the truncation of (2.5) to (2.8) is legitimate since $O(A^n, \bar{A}^m)$ terms with $n,m \geq 3$ in (2.5) are necessary for gauge invariance. In fact, (2.9) is invariant only under the "Abelian" part of the gauge transformations. The point, however, is that since the non-Abelian gauge invariance is violated only by *nonlocal* terms (or, equivalently, is preserved up to *nonlocal* terms) it should be present in a consistently extracted local part of (2.8).

To proceed, let us first establish our notation. $T_A=(T_a,T_i)$ are the generators of G; T_a are the generators of H; $A=1,\ldots,D_G$; $a=1,\ldots,D_H$; $i=1,\ldots,D$, $D=D_{G/H}$; η_{AB} is negative definite in the compact case, and

$$\begin{gathered}
A=A^aT_a,\quad C_{AB}\equiv\mathrm{Tr}(T_AgT_Bg^{-1}),\quad \mathrm{Tr}(T_AT_B)=\eta_{AB}\,,\\
J_A=\mathrm{Tr}(T_Ag^{-1}\partial g)=E_{AM}(x)\partial x^M,\quad \bar{J}_A=-\mathrm{Tr}(T_A\bar{\partial}gg^{-1})=\tilde{E}_{AM}(x)\bar{\partial}x^M\,,\\
\tilde{J}_A=\mathrm{Tr}(T_Ag^{-1}\bar{\partial}g)=E_{AM}(x)\bar{\partial}x^M,\quad \tilde{\bar{J}}_A=-\mathrm{Tr}(T_A\partial gg^{-1})=\tilde{E}_{AM}(x)\partial x^M\,,\\
\tilde{E}_{AM}=-C_{AB}E^B_M,\quad C^{AD}C_{BD}=\delta^A_B,\quad C^A{}_B=\eta^{AD}C_{DB}\,,\\
C_{AB}=\begin{bmatrix}C_{ab} & C_{ib}\\ C_{aj} & C_{ij}\end{bmatrix},\quad \tilde{E}_{aM}=-C_{ab}E^b_M-C_{ai}E^i_M\,.
\end{gathered}\tag{2.12}$$

In the following we shall also use[3]

$$\begin{gathered}
A=(A^a),\quad J=(J_a)\,,\\
M_{ab}\equiv C_{ab}-\eta_{ab}\,,\\
N_{ab}\equiv M_{ab}-b\eta_{ab}\,.
\end{gathered}\tag{2.13}$$

Then the truncated action (2.8) takes the form

$$\begin{gathered}
\Gamma_{\rm tr}(g,A)=(k+g_G)[I(g)+\Delta I(g,A)]\,,\\
\Delta I(g,A)\equiv\frac{1}{\pi}\int d^2z[-A\bar{J}-\bar{A}J+AM\bar{A}\\
-\tfrac{1}{2}b(A-\tilde{\bar{A}})(\bar{A}-\tilde{A})]\,.
\end{gathered}\tag{2.14}$$

Eliminating $A,\bar{A}$ from this action one finds a nonlocal expression which is bilinear in $J,\bar{J}$

$$\begin{gathered}
\Delta I(g)=-\frac{1}{2\pi}\int d^2z[J(QN^TQ^{-1}N-b^2I)^{-1}(QN^TQ^{-1}\bar{J}-bQJ)+\bar{J}(Q^{-1}NQN^T-b^2I)^{-1}(Q^{-1}NQJ-bQ^{-1}\bar{J})]\,,\\
Q\equiv\frac{\bar{\partial}}{\partial},\quad Q^{-1}\equiv\frac{\partial}{\bar{\partial}}\,.
\end{gathered}\tag{2.14$'$}$$

One is now to take a local part of the functional $I(g)+\Delta I(g)$ and identify it with the σ-model action. To determine the metric term in the resulting σ-model action it is sufficient just to set $Q=1$ or to take the $d=1$ limit of the action [16,5]. As for the parity-odd (antisymmetric tensor) part in the σ-model action its derivation is more subtle. It is not clear *a priori* how to extract the relevant local part of (2.14′). One natural suggestion [16,5] is to replace QN^TQ^{-1} and $Q^{-1}NQ$ by N^T and N and QJ and $Q^{-1}\bar{J}$ by $\tilde{J}$ and $\tilde{\bar{J}}$. The resulting local action does not, however, have gauge invariance in its parity-odd part. As we shall show below, there exist a procedure of extracting a local part from (2.14′) that not only preserves the Abelian gauge invariance of (2.14) and (2.14′) but also restores the full non-Abelian invariance which was present in the original action (2.5). The basic idea is to omit the nonlocal terms already at the level of the solution of the equations for $A,\bar{A}$ that follow from (2.14) and only then substitute the resulting local expressions for A, $\bar{A}$ *and* $\tilde{A},\tilde{\bar{A}}$ into (2.14).

The equations for $A^a,\bar{A}^a$ we obtain from (2.14) are

$$-\bar{J}+N\bar{A}+b\tilde{A}=0,\quad -J+N^TA+b\tilde{\bar{A}}=0\,.\tag{2.15}$$

[3] Our present notation differ from the notation of Ref. [5] where A, $\bar{A}$, $\bar{J}$, and $\tilde{E}$ have the opposite signs. The correspondence with the notation of Ref. [16] is the following: $A_-\to A$, $A_+\to\bar{A}$, $(L^A_M,R^A_M)\to(E^A_M,\tilde{E}^A_M)$.

Let us note that (2.15) take the form of the exact equations that follow from the full untruncated effective action (2.5) if one sets there [cf. Eq. (2.10)] $\tilde{A}=\bar{\partial}hh^{-1}$, $\tilde{\bar{A}}=\partial\bar{h}\bar{h}^{-1}$. Equations (2.15) imply

$$-\tilde{\bar{J}}+N\tilde{\bar{A}}+bA+\cdots=0\ ,\qquad -\tilde{J}+N^T\tilde{A}+b\bar{A}+\cdots=0\ . \tag{2.16}$$

Up to nonlocal terms the solution of (2.15) and (2.16) is (we shall use the star to denote the local part of the solution)

$$A_*=V^{-1}(NJ-b\tilde{\bar{J}}),\quad \bar{A}_*=\tilde{V}^{-1}(N^T\tilde{\bar{J}}-b\tilde{J})\ ,\qquad \tilde{A}_*=V^{-1}(N\tilde{J}-b\bar{J}),\quad \tilde{\bar{A}}_*=\tilde{V}^{-1}(N^T\bar{J}-bJ)\ , \tag{2.17}$$

where we have defined the matrices

$$V\equiv NN^T-b^2I,\quad \tilde{V}\equiv N^TN-b^2I\ ,\qquad V^T=V,\quad \tilde{V}^T=\tilde{V},\quad \tilde{V}^{-1}N^T=N^TV^{-1}\ . \tag{2.18}$$

The local expressions in (2.17) have the correct Lorentz structure. Moreover, A_*, $\bar{A}_*$ and $\tilde{A}_*,\tilde{\bar{A}}_*$ transform properly (as gauge fields) under the *non-Abelian* gauge transformation of g. In fact, the gauge transformation $g\to u^{-1}gu$ induces the following transformations on the H components of the currents and tensors:

$$J'=UJ-M'^T\epsilon,\ \bar{J}'=U\bar{J}-M'\bar{\epsilon},\ \tilde{J}'=U\tilde{J}-M'^T\bar{\epsilon},\ \tilde{\bar{J}}'=U\tilde{\bar{J}}-M'\epsilon\ ,$$
$$C'\equiv C(g')=UC(g)U^{-1},\ \ M'=UMU^{-1},\ \ V'=UVU^{-1},\ \ \tilde{V}'=U\tilde{V}U^{-1}\ , \tag{2.19}$$
$$U_{ab}\equiv C_{ab}(u^{-1})=C^T_{ab}(u),\ \ U^T=U^{-1},\ \ u^{-1}T_au=C_{ab}(u)T^b,\ \ \epsilon_a\equiv\mathrm{Tr}(T_au^{-1}\partial u),\ \ \bar{\epsilon}_a\equiv\mathrm{Tr}(T_au^{-1}\bar{\partial}u)\ .$$

Using (2.19) one can show that the fields in (2.17) transform as

$$A'_*=UA_*-\epsilon,\quad \bar{A}'_*=U\bar{A}_*-\bar{\epsilon}\ ,\qquad \tilde{A}'_*=U\tilde{A}_*-\bar{\epsilon},\quad \tilde{\bar{A}}'_*=U\tilde{\bar{A}}_*-\epsilon\ , \tag{2.20}$$

i.e., as the gauge fields in (2.2) and (2.11). Note also that

$$(\tilde{A}_*-\bar{A}_*)'=U(\tilde{A}_*-\bar{A}_*)\ ,$$
$$(A_*-\tilde{\bar{A}}_*)'=U(A_*-\tilde{\bar{A}}_*)\ ,$$
$$[(A_*-\tilde{\bar{A}}_*)^a(\tilde{A}_*-\bar{A}_*)_a]'=(A_*-\tilde{\bar{A}}_*)^a(\tilde{A}_*-\bar{A}_*)_a\ .$$

Therefore, substituting (2.17) into (2.14) we can eliminate the gauge fields in a way that preserves gauge invariance.

A natural question is whether the described procedure (which we shall call "corrected" prescription) based on (2.17) is a unique one preserving gauge invariance. In fact, it is not—there exists another similar prescription (which will be described at the end of this section and will be called "semiclassical") in which the local part of the quantum correction (2.14′) does not contain a parity-odd part and thus the expression for $B_{\mu\nu}$ is not modified. To see that the prescription using (2.17) is not unambiguous note that we have dropped nonlocal terms in the process of solving (2.15) [the last two relations in (2.17) imply also that we assume a specific prescription of separating out local terms in (2.14′)]. Eliminating $A,\bar{A}$ from the actions (2.5) and (2.14) one has to be careful about doing integrations by parts: total derivative terms of nonlocal structure may, in fact, contribute to the local part of the result (disregarding total derivative terms in the action does not commute with dropping out nonlocal terms after the insertion of the "truncated" classical solution into the action). For example, let us consider the effect of adding a total derivative term

$$-\tfrac{1}{2}\mu b(A\bar{A}-\tilde{A}\tilde{\bar{A}})$$

to the action (2.15) (μ is an arbitrary parameter). Such a term will not change the equations of motion (2.15) but will produce a nontrivial contribution to the antisymmetric tensor coupling after the solution (2.17) is substituted back into the action (the expression for the metric will not change). It should be possible to modify the above term by other g-dependent total derivative terms [which should compensate for the use of the Polyakov-Wiegmann relation in the process of representing the manifestly gauge invariant action (2.5) in the standard "AJ" form (2.14)] in order to preserve the gauge invariance of the result. It turns out to be necessary to account for this ambiguity in the case of a nonsimple group G (see Appendix B). For simple G adding such terms seems unnatural since they change the coefficient of the $A\bar{A}$ term in the local part of (2.14).

Starting with (2.14) we find the following local action for g

$$\Gamma_{\mathrm{tr}}(g,A)=(k+g_G)[I(g)+\Delta I(g,A_*)+\cdots]$$
$$=\Gamma_{\mathrm{loc}}(g)+\cdots\ , \tag{2.21}$$

where[4]

[4]In the treatment of the general model in [16,5] gauge invariance was maintained only in the zero mode sector. As a result, the expression for the antisymmetric tensor we find below (2.25) is different from the one given in [16,5]. The form of the truncated effective action used in [16,5] was different from (2.8) by the total derivative term (with $\mu=1$) and as a result a gauge-variant expression for the B_{MN} term was obtained after separation of the local part of the action.

$$\Gamma_{\rm loc}(g)=(k+g_G)\left[I(g)+\frac{1}{2\pi}\int d^2z[-A_*\bar{J}-J\bar{A}_*+b(A_*\bar{A}_*-\tilde{\bar{A}}_*\tilde{A}_*)]\right]$$
$$=(k+g_G)\left[I(g)+\frac{1}{2\pi}\int d^2z\{J[-2\tilde{V}^{-1}N^T+b\tilde{V}^{-1}(N^TN^T-b^2I)V^{-1}]\bar{J}\right.$$
$$-b\tilde{\bar{J}}V^{-1}(NN-b^2I)\tilde{V}^{-1}\tilde{J}+bJ[\tilde{V}^{-1}+b\tilde{V}^{-1}(N-N^T)\tilde{V}^{-1}]\tilde{J}$$
$$\left.+b\tilde{\bar{J}}[V^{-1}+bV^{-1}(N-N^T)V^{-1}]\bar{J}\}\right] . \tag{2.22}$$

As follows from the above discussion, this action is invariant under the transformation $g\rightarrow u^{-1}gu$. Parametrizing g in terms of group coordinates x^M [see (2.12)] we can represent (2.22) as the following σ model with the group space G as a configuration space:[5]

$$S(x)=\Gamma_{\rm loc}(g)=-\frac{1}{\pi\alpha'}\int d^2z\, \mathcal{G}_{MN}(x)\partial x^M\bar{\partial}x^N, \quad \alpha'=\frac{2}{k+g_G} , \tag{2.23}$$

where the metric is

$$G_{MN}\equiv\mathcal{G}_{(MN)}=G_{0MN}+2(\tilde{V}^{-1}N^T)_{ab}E^a_{(M}\tilde{E}^b_{N)}-b(\tilde{V}^{-1})_{ab}E^a_ME^b_N-b(V^{-1})_{ab}\tilde{E}^a_M\tilde{E}^b_N , \tag{2.24}$$

and the antisymmetric tensor is

$$B_{MN}\equiv\mathcal{G}_{[MN]}=B_{0MN}+2[\tilde{V}^{-1}N^T-b\tilde{V}^{-1}(N^TN^T-b^2I)V^{-1}]_{ab}E^a_{[M}\tilde{E}^b_{N]}$$
$$-b^2[\tilde{V}^{-1}(M-M^T)\tilde{V}^{-1}]_{ab}E^a_{[M}E^b_{N]}-b^2[V^{-1}(M-M^T)V^{-1}]_{ab}\tilde{E}^a_{[M}\tilde{E}^b_{N]} . \tag{2.25}$$

G_{0MN} and B_{0MN} are the original WZNW (group space) couplings,

$$G_{0MN}=\eta_{AB}E^A_ME^B_N, \quad 3\partial_{[K}B_{0MN]}=E^A_KE^B_ME^C_Nf_{ABC} . \tag{2.26}$$

The expression for the dilaton coupling follows from the (local part of) determinant of the $(A,\bar{A})$ quadratic form in (2.14)

$$\phi=\phi_0-\tfrac{1}{4}\ln\det V . \tag{2.27}$$

It is useful to rewrite expressions (2.24) and (2.25) separating quantum corrections from the semiclassical expressions

$$G^{(s)}_{MN}=G_{0MN}+2M^{-1}_{ab}E^a_{(M}\tilde{E}^b_{N)}, \quad B^{(s)}_{MN}=B_{0MN}+2M^{-1}_{ab}E^a_{[M}\tilde{E}^b_{N]} . \tag{2.28}$$

Namely,

$$G_{MN}=G^{(s)}_{MN}+2b(\tilde{V}^{-1}M^TM^{-1})_{ab}E^a_{(M}\tilde{E}^b_{N)}-b(\tilde{V}^{-1})_{ab}E^a_ME^b_N-b(V^{-1})_{ab}\tilde{E}^a_M\tilde{E}^b_N \tag{2.29}$$

and

$$B_{MN}=B^{(s)}_{MN}+2b^2[\tilde{V}^{-1}M^TM^{-1}(M-M^T)V^{-1}]_{ab}E^a_{[M}\tilde{E}^b_{N]}$$
$$-b^2[\tilde{V}^{-1}(M-M^T)\tilde{V}^{-1}]_{ab}E^a_{[M}E^b_{N]}-b^2[V^{-1}(M-M^T)V^{-1}]_{ab}\tilde{E}^a_{[M}\tilde{E}^b_{N]} . \tag{2.30}$$

Thus the first quantum correction to the metric G_{MN} is one-loop $[O(\alpha')]$ and to B_{MN}—a two-loop $[(O(\alpha'^2)]$ one. Let us note that the above expressions were derived in the case of the vector gauging of a simple group G.

The σ-model action (2.23) is gauge invariant as a consequence of the invariance of (2.22) [note that the semiclassical and quantum correction terms in (2.22) and (2.23) are separately gauge invariant]. The general σ model (2.23) is invariant under the *local* transformation $\delta x^M=Z^M_a(x)\epsilon^a(z,\bar{z})$ (e.g., induced by the gauge transformation $g'=u^{-1}gu$ with generators $Z^M_a=E^M_a+\tilde{E}^M_a$) if the metric and the antisymmetric tensor satisfy the constraints[6]

$$Z^K_a\partial_KG_{MN}+G_{MK}\partial_NZ^K_a+G_{KN}\partial_MZ^K_a=0 ,$$
$$G_{MN}Z^M_a=0, \quad H_{MNK}Z^K_a=0 , \tag{2.31}$$

[5]We assume that the coordinates x^M are rescaled by the overall "radius" of the space so that α' is dimensionless.

[6]Since we are deriving the constraints for the couplings G, B (which transform only as functions of x) the variation of the action should vanish for arbitrary x and ϵ. In the metric part of the variation one should equate to zero separately the $O(\epsilon)$ and $O(\partial\epsilon)$ terms [integration by parts in $O(\partial\epsilon)$ terms induces $\partial\bar{\partial}x$ terms].

so that Z_a^M should be the Killing vectors as well as the null vectors of the metric and the antisymmetric tensor field strength should have a zero projection on Z (cf. [29]). These conditions are thus satisfied for the metric (2.24) (see [5]) and the antisymmetric tensor (2.25). The dilaton term is also invariant [$\phi(g')=\phi(g)$ or $Z_a^M\partial_M\phi=0$] as it is clear from (2.27) and (2.19).

One can now fix a gauge, e.g., by restricting the coordinates x^M on G to coordinates x^μ on G/H. This can be done by solving a gauge condition $R^s(x^M)=0$ [$s=1,\dots,\dim H$, $\det(Z_b^M\partial_M R^s)\neq 0$], i.e., $x^M=x^M(x^\mu)$ [$\mu=1,\dots,\dim(G/H)$] and $\partial_M R^s\partial_\mu x^M=0$. In particular, it is possible to choose such x^μ that are invariant under the transformations of x^M, i.e., $Z_a^M\partial_M x^\mu=0$.[7] If H_i^M is a basis in the tangent space of G/H orthogonal (with respect to G_{0MN}) to the gauge symmetry generators Z_a^M then, in particular, we can choose $\partial_\mu x^M\equiv H_\mu^M=L_\mu^i H_i^M$ so that the couplings of the resulting σ model on G/H are given by

$$G_{\mu\nu}=G_{MN}H_\mu^M H_\nu^N=\bar g_{ij}\bar H_\mu^i \bar H_\nu^j\ ,\qquad \bar g_{ij}=\eta_{ij}-bV_{ab}^{-1}C^a{}_iC^b{}_j\ , \tag{2.32}$$

$$B_{\mu\nu}=B_{MN}H_\mu^M H_\nu^N=\bar b_{ij}\bar H_\mu^i\bar H_\nu^j\ ,\qquad \bar b_{ij}=b_{0ij}-b^2\{V^{-1}(M-M^T)V^{-1}\}_{ab}C^a{}_iC^b{}_j\ , \tag{2.33}$$

$$H_{\mu\nu\lambda}=\bar h_{ijk}\bar H_\mu^i\bar H_\nu^j\bar H_\lambda^k\ .$$

Here $\bar H_\mu^i=\bar H_M^i\partial_\mu x^M$, with $\bar H_M^i$ being a particular basis orthogonal to Z_a^M [5]

$$\bar H_M^i=E_M^i-M_{ab}^{-1}C^{bi}E_M^a\ , \tag{2.34}$$

and b_{0ij} is the projection of B_{0MN}. Using that

$$C_{ai}(u)=0\ ,\qquad uT^iu^{-1}=T^jC_j{}^i(u)\ ,\qquad C_i^k(u)C_{jk}(u)=\eta_{ij}\ , \tag{2.35}$$

one can show that under $g\to u^{-1}gu$ the one-form $\bar H^i=\bar H_M^i dx^M$ and the tensors $\bar g_{ij},\bar h_{ijk}$ transform as

$$\bar H^{i\prime}=\bar H^jC_j{}^i(u),\quad \bar g_{ij}'=C^k{}_i(u)\bar g_{kl}C^l{}_j(u)\ ,\qquad \bar h_{ijk}'=\bar h_{qpn}C^q{}_i(u)C^p{}_j(u)C^n{}_k(u)\ . \tag{2.36}$$

As it is clear from (2.29) and (2.30), in the case when C_{ab} (and hence M_{ab} and N_{ab}) is symmetric the metric still receives $1/k$ corrections while the antisymmetric tensor remains semiclassical:

$$G_{MN}=G_{MN}^{(s)}-b[M(M-2b)]_{ab}^{-1}(E_M^a-\tilde E_M^a)(E_N^b-\tilde E_N^b)\ ,\qquad B_{MN}=B_{MN}^{(s)}\ . \tag{2.37}$$

It is possible to prove that the matrix C_{ab} is symmetric only when the subgroup H is Abelian with $\dim H=1$. Expanding the group element $g=\exp(X^AT_A)$ around the identity one can compute the expansion of C_{ab} in (2.12) to X^3 order. Demanding that the $O(X)$ term is symmetric we obtain the condition $f_{abc}X^c=0$. For this to be true for all X^c the structure constants $f_{ab}{}^c$ of the subgroup H should vanish so that H should be Abelian. Assuming that H is Abelian we find an additional condition from the $O(X^3)$ term: $X^if_{ai}{}^j\mathrm{Tr}[T_jT_b(X_CT^C)^2]-(a\leftrightarrow b)=0$. This condition can be satisfied for all X's only if $\dim H=1$. One can also see this explicitly using the general form of the expansion of C_{AB} in normal coordinates X^M near the unit element of the group (see, e.g. [31])

$$E_M^A=\left[\frac{e^f-1}{f}\right]_M^A=(I+\tfrac12 f+\tfrac16 f^2+\cdots)_M^A\ ,\qquad f_{AB}\equiv f_{ABC}X^C,\quad E_M^AX^M=X^A\ , \tag{2.38}$$

$$C^A{}_B=(e^{-f})^A{}_B=(I-f+\tfrac12 f^2-\tfrac16 f^3+\cdots)^A{}_B\ . \tag{2.39}$$

Imposing the symmetry condition on the $O(X)$ and $O(X^3)$ terms in C_{ab} one concludes that $\dim H=1$.

In the case of a simple group G considered in this section the condition that C_{ab} should be symmetric or that $\dim H=1$ is a sufficient one in order for the quantum corrections in (2.30) to be absent. The matrix C_{ab} is not symmetric in a generic case of Abelian or non-Abelian H.

The situation is different when G is not simple. The general expressions for the effective action (2.5) and (2.22) and hence for the σ-model couplings in (2.23) become more complicated but can be worked out by taking into account that the renormalizations of the levels k may be different for different simple factors in G. It turns out also to be necessary to fix the "total derivative" ambiguity in the derivation of the expression for the antisymmetric tensor in a particular way, consistent with conformal invariance of the resulting σ model. One of the consequences is that the antisymmetric tensor [computed according to our first prescription based on (2.17)] may contain quantum corrections even when $\dim H=1$. An example of such model will be discussed in Sec. IV and Appendix B. Another example is provided by the SO(2,2)/SO(2,1) model (with nonsymmetric C_{ab} [16] and vanishing semiclassical $B_{\mu\nu}$ [11,12,14]) for which we have numerically checked (in a certain gauge) that the quantum correction to the antisymmetric tensor is also vanishing [the $O(E\tilde E)$ term in (2.30)] vanishes by itself while the contributions of the $O(EE)$ and $O(\tilde E\tilde E)$ terms cancel each other). The matrix C_{ab} is nonsymmetric also in the case of the $D=4$ model [$SL(2,\mathbb{R})\times SU(2)/\mathbb{R}\times U(1)$] considered in [7,10,4] but here it appears that the antisymmetric tensor receives nontrivial corrections to all orders in $1/k$.

Let us now describe another natural prescription of separating a local gauge-invariant part of the quantum

[7]Since the action (2.23) has gauge invariance it really depends on the dim(G/H) invariants x^μ (global coordinates) that one can build out of dim$G$ group parameters x^M [30]. Such choice has certain advantages over a generic procedure of gauge fixing. Fixing a gauge one usually restricts consideration to one patch of the entire space only, whereas choosing the G/H coordinates x^μ as group invariants one may determine their range of values by using group theoretic methods [30].

term in the effective action (2.14′) under which $B_{\mu\nu}$ does not receive α' corrections. Suppose that one has managed to truncate (2.14′) in such a way that the resulting local part (quantum contributions to it) is $2d$ parity-even (invariant under $z\to\bar z, \bar z\to z$), i.e., does not contain any α' corrections to the antisymmetric tensor part. That would mean that the resulting metric is still given by (2.24) but $B_{\mu\nu}$ remains semiclassical as in (2.28) [however, with overall coefficient $(k+g_G)$]. The corresponding σ model is obviously gauge invariant since the semiclassical part of (2.14) is gauge invariant by itself and the parity-even and parity-odd parts of (2.14) should be gauge invariant separately [the gauge transformations (2.19), (2.20) do not mix parity-odd and parity-even sectors].

An indication that such a result may be considered as a natural one comes from the structure of the quantum term in the truncated effective action (2.8). This term is manifestly parity-even, suggesting that there should be no correction to the parity-odd part of the σ model that is obtained after solving for $A, \bar A$. The fact that the quantum term [truncated to $O(A^2)$ part] in the effective action (2.8) is parity-even may be considered as a basic "principle" behind this prescription. This may be in turn related to the fact that from the point of view of the operator conformal field theory approach it is the Hamiltonian but not the current algebra (which is sensitive to a definition of currents and thus to the form of $B_{\mu\nu}$) that receives $1/k$ corrections.

It is easy indeed to give a simple prescription for solving for $A, \bar A$ (up to nonlocal terms) in (2.8) that does not generate parity-odd quantum terms in the σ-model action and thus corroborates this suggestion. It is useful to separate the classical term A_{cm} in the solution for A_m from the quantum one $A_{qm}=O(b)$ [see (2.14)]

$$A=A_c+A_q,\quad \bar A=\bar A_c+\bar A_q\ ,\qquad A_c=M^{-1}J,\quad \bar A_c=M^{-1}\bar J\ . \tag{2.40}$$

Then (2.14) takes the following form [cf. (2.22)]

$$\Gamma_{\rm tr}(g,A_q)=(k+g_G)\left[I_c(g)+\frac{1}{\pi}\int d^2z\{A_qM\bar A_q-\tfrac12 b[A_q-\tilde{\bar A}_q+M^{-1}(J-\tilde{\bar J})][\bar A_q-\tilde A_q+M^{-1}(\bar J-\tilde J)]\}\right]\ , \tag{2.41}$$

where I_c is the semiclassical term

$$I_c(g)=I(g)-\frac{1}{\pi}\int d^2z\, JM^{-1}\bar J\ . \tag{2.42}$$

In (2.41) we have already dropped some nonlocal terms by using the rule [similar to that used in (2.17)]: in substituting the classical fields $A_c, \bar A_c$ into the quantum term in (2.14) we replaced $\tilde A_c=(\bar\partial/\partial)(M^{-1}J)$ by $M^{-1}\tilde J$ and $\tilde{\bar A}_c=(\partial/\bar\partial)(M^{-1}\bar J)$ by $M^{-1}\tilde{\bar J}$ [see (2.10), (2.12)]. The second part of our prescription is to maintain the parity-even structure of the quantum part of (2.41) by symmetrizing the $A_qM\bar A_q$ term,

$$A_qM\bar A_q\to\tfrac12(A_qM\bar A_q+\tilde A_qM\tilde{\bar A}_q)\ . \tag{2.43}$$

This is equivalent to dropping the term $\frac12(A_qM\bar A_q-\tilde A_qM\tilde{\bar A}_q)$ which is a total derivative up to nonlocal terms depending on derivatives of M [cf. a discussion above Eq. (2.21)]. The resulting expression for the quantum term $\Gamma_{\rm tr}(g,A_q)-(k+g_G)I_c(g)$ is parity-even (and gauge-invariant). After one solves for $A_q, \bar A_q$ one finds, therefore, a σ model with no quantum correction to the antisymmetric tensor term, i.e., with the metric and dilaton given by (2.24) and (2.27) and the semiclassical antisymmetric tensor (2.28). It should be kept in mind of course that it is still the shifted level $k+g_G$ that appears in front of the semiclassical $B_{\mu\nu}$ in the σ-model action.

As we shall check below on the example of a $D=3$ gauged WZNW model (see Secs. IV, V, and Appendix B) both backgrounds derived according to the two ("corrected" and "semiclassical") prescriptions for $B_{\mu\nu}$ described in this section are solutions of the σ-model conformal invariance conditions in the two-loop approximation.

III. EXACT σ MODELS CORRESPONDING TO BOSONIC AND HETEROTIC CHIRAL GAUGED WZNW MODELS

Using the equivalence of the (1,1) supersymmetric gauged WZNW models with the $N=1$ superconformal coset models it was shown in [2,21] that the expressions for the bosonic background fields remain semiclassical. The (local part of) the effective action of the (1,1) supersymmetric gauged WZNW model is also given just by the classical gauged WZNW action itself [5]. The (1,0) supersymmetric gauged WZNW model [obtained by truncation of the (1,1) supersymmetric one] is anomalous [32], i.e., is not well defined at the quantum level.

In addition to the bosonic gauged WZNW model there exist also two other similar nontrivial models with corresponding σ models containing nonzero quantum correction terms (but with configuration space G instead of G/H one). They are based on the "chiral gauged" WZNW model [19], i.e., are bosonic [25,17] and heterotic [(1,0) supersymmetric] [18] chiral gauged WZNW theories. The classical action of the chiral gauged WZNW model is obtained by dropping out the $A\bar A$ term in the action of the gauged WZNW model (2.1). It reads [9,19]

$$I_{ch}=I(g)+\frac{1}{\pi}\int d^2z\,{\rm Tr}(A\bar\partial gg^{-1}-\bar Ag^{-1}\partial g+g^{-1}Ag\bar A)\ . \tag{3.1}$$

This action is no longer invariant under the true gauge transformations (2.2) but is still invariant under the "chiral" gauge transformations $g\to u^{-1}(z)g\bar u(\bar z)$ with holomorphic and antiholomorphic parameters. Since the "chiral" gauge transformations do not actually eliminate dynamical degrees of freedom (it is more appropriate to consider them as an infinite set of global transformations as in the standard WZNW model case [33]) the configuration space of the σ model one obtains upon elimination of the vector gauge field from the action is the group space G and not G/H. Though the gauged and chiral gauged WZNW models are closely related before one integrates out the vector fields, the associated σ models are, in general, different [17] (with the exception of the case when H is Abelian, when the chiral gauged WZNW model is actually equivalent to a specific class of axially gauged $G\times H/H$ WZNW models [18]).

Like the classical action, the effective action of the chiral gauged WZNW model can be obtained from the effective action of the gauged WZNW model (2.5) by omitting the local counterterm $A\bar{A}$ [17]. This is easy to understand by noting that in the parametrization (2.3) the action (3.1) reduces to a combination of WZNW actions [cf. (2.4)]

$$I_{ch}(g,A)=I(h^{-1}g\bar{h})-I(h^{-1})-I(\bar{h})\ . \tag{3.2}$$

The corresponding quantum effective action is then [17] [cf. (2.5)]

$$\begin{aligned}\Gamma_{ch}(g,A)&=(k+g_G)I(h^{-1}g\bar{h})\\&\quad-(k+g_H)[I(h^{-1})+I(\bar{h})]\\&=(k+g_G)(I_{ch}(g,A)\\&\quad-b[\omega(A)+\bar{\omega}(\bar{A})])\ ,\end{aligned} \tag{3.3}$$

where $b\equiv-(g_G-g_H)/(k+g_G)$ and $\omega(A)$ and $\bar{\omega}(\bar{A})$ are the same nonlocal functionals of $A,\bar{A}$ defined in (2.7).

The classical action of the (1,0) supersymmetric chiral gauged model is obtained by replacing the fields in (3.1) or (3.2) by (1,0) superfields [18]. The bosonic part of the action is still given by (3.1) but the fermionic contribution changes the structure of the effective action: as in the (1,1) supersymmetric case [5] there is no quantum shift of the level k and also there is no $\bar{A}$-dependent term in the quantum part of the effective action [18]. The bosonic part of the resulting effective action is thus given by [18] [cf. (3.3)]

$$\Gamma_{\rm ch}^{(1,0)}(g,A)=k[I_{\rm ch}(g,A)-b\omega(A)]\ , \tag{3.4}$$

where $b\equiv-(1/k)(g_G-g_H)$. To find the corresponding σ models one needs to eliminate $A,\bar{A}$ from (3.3) and (3.4). It is instructive to do this by starting with the most general ansatz for the effective action which formally includes all the cases of the gauged and chiral gauged WZNW models mentioned above [cf. (2.5), (3.3), (3.4)]:

$$\hat{\Gamma}(g,A)=\kappa\left[I(g)+\frac{1}{\pi}\int d^2z\,{\rm Tr}[A\bar{\partial}gg^{-1}-\bar{A}g^{-1}\partial g+g^{-1}Ag\bar{A}+(a-1)A\bar{A}]-b\omega(A)-\bar{b}\bar{\omega}(\bar{A})\right]\ , \tag{3.5}$$

where the values of the constants $\kappa,a,b,\bar{b}$ are

$$\begin{aligned}&\text{Gauged WZNW:}\ \ \kappa=k+g_G,\ \ a=-b,\ \ b=\bar{b}=-\frac{1}{\kappa}(g_G-g_H)\ ,\\&(1,1)\text{ supersymmetric (SUSY) gauged WZNW:}\ \ \kappa=k,\ \ a=b=\bar{b}=0\ ,\\&\text{Chiral gauged WZNW:}\ \ \kappa=k+g_G,\ \ a=1,\ \ b=\bar{b}=-\frac{1}{\kappa}(g_G-g_H)\ ,\\&(1,1)\text{ SUSY chiral gauged WZNW:}\ \ \kappa=k,\ \ a=1,\ \ b=\bar{b}=0\ ,\\&(1,0)\text{ SUSY chiral gauged WZNW:}\ \ \kappa=k,\ \ a=1,\ \ b=-\frac{1}{\kappa}(g_G-g_H),\ \bar{b}=0\ .\end{aligned} \tag{3.6}$$

In this section we shall use the same notation as in (2.12), (2.13), and (2.18) with the exception that the matrices N, V, and $\tilde{V}$ are now defined as

$$N_{ab}\equiv M_{ab}+a\eta_{ab}=C_{ab}+(a-1)\eta_{ab},\ \ V\equiv NN^T-b\bar{b}I,\ \ \tilde{V}\equiv N^TN-b\bar{b}I\ . \tag{3.7}$$

Truncating the functionals $\omega(A),\bar{\omega}(\bar{A})$ in (3.5) to quadratic terms as in (2.8) we obtain [cf. (2.14)]

$$\hat{\Gamma}_{\rm tr}(g,A)=\kappa[I(g)+\Delta I(g,A)],\ \ \Delta I(g,A)=\frac{1}{\pi}\int d^2z(-A\bar{J}-\bar{A}J+AN\bar{A}+\tfrac{1}{2}bA\tilde{A}+\tfrac{1}{2}\bar{b}\bar{A}\tilde{\bar{A}})\ , \tag{3.8}$$

where $\tilde{A}$ and $\tilde{\bar{A}}$ were defined in (2.10). Eliminating $A,\bar{A}$ we get again the nonlocal expression similar to (2.14′),

$$\Delta I(g)=-\frac{1}{2\pi}\int d^2z[J(N^TQ^{-1}N-b\bar{b}Q^{-1})^{-1}(N^TQ^{-1}\bar{J}-bJ)+\bar{J}(NQN^T-b\bar{b}Q)^{-1}(NQJ-\bar{b}\bar{J})]\ . \tag{3.8′}$$

Note that in the case of the gauged WZNW model with the values of the parameters given by the first line in (3.6) the action (3.8) differs from (2.14) by a total derivative term. Since in the case of an arbitrary a we do not have, in general, an extra symmetry analogous to the gauge invariance, we are lacking the principle we used in the previous section in the process of extracting the local part of the effective action. One possibility is that the resulting ambiguity in the expression for the B_{MN} term should be fixed in each particular case. For example, one may expect that there should be no extra total derivative term in (3.8) in the chiral gauged WZNW model case where the form of the effective action (3.3) is fixed by the condition of conformal invariance (the action is expressed in terms of WZNW actions).

On the other hand one can try again to interpret the parity-even nature of the quantum term in (3.8) as implying that the antisymmetric tensor should retain its semiclassical form, i.e., to adopt the "semiclassical" prescription described at the end of the previous section. In what follows we shall first consider the "corrected" prescription.

The equations for the gauge fields $A^a, \bar{A}^a$ that follow from (3.8)

$$-\bar{J}+N\bar{A}+b\tilde{A}=0, \quad -J+N^T A+\bar{b}\tilde{\bar{A}}=0 , \qquad (3.9)$$

imply also

$$\begin{aligned} &-\bar{\tilde{\bar{J}}}+N\bar{\tilde{\bar{A}}}+bA+\cdots=0 , \\ &-\tilde{J}+N^T\tilde{A}+\bar{b}\bar{A}+\cdots=0 . \end{aligned} \qquad (3.10)$$

The local parts of the corresponding solutions are

$$\begin{aligned} &A_*=V^{-1}(NJ-\bar{b}\,\bar{\tilde{\bar{J}}}), \quad \bar{A}_*=\tilde{V}^{-1}(N^T\bar{J}-b\tilde{J}) , \\ &\tilde{A}_*=V^{-1}(N\tilde{J}-\bar{b}\,\bar{J}), \quad \bar{\tilde{\bar{A}}}_*=\tilde{V}^{-1}(N^T\bar{\tilde{\bar{J}}}-bJ) . \end{aligned} \qquad (3.11)$$

If we substitute these solutions into the action (3.8) and again ignore the nonlocal terms we obtain the following expression for the local part of the effective action:

$$\begin{aligned} \hat{\Gamma}_{\mathrm{loc}}(g)&=\kappa\left[I(g)+\frac{1}{2\pi}\int d^2z(-A_*\bar{J}-\bar{A}_*J)\right] \\ &=\kappa\left[I(g)+\frac{1}{2\pi}\int d^2z(-2J\tilde{V}^{-1}N^T\bar{J} \right. \\ &\qquad +\bar{b}\,\bar{\tilde{\bar{J}}}V^{-1}\bar{J} \\ &\qquad \left. +bJ\tilde{V}^{-1}\tilde{J})\right] . \end{aligned} \qquad (3.12)$$

This is the expression one finds by dropping out the nonlocal terms in the expression (3.8′) in the most straightforward way.

The resulting σ model

$$\begin{aligned} \hat{S}(x)&=\hat{\Gamma}_{\mathrm{loc}}(g) \\ &=-\frac{1}{\pi\alpha'}\int d^2z\, \mathcal{G}_{MN}(x)\partial x^M\bar{\partial}x^N , \\ &\qquad \alpha'=\frac{2}{\kappa} , \end{aligned} \qquad (3.13)$$

has a metric which is very similar to the one in (2.24),

$$\begin{aligned} G_{MN}&\equiv\mathcal{G}_{(MN)} \\ &=G_{0MN}+2(\tilde{V}^{-1}N^T)_{ab}E^a_{(M}\tilde{E}^b_{N)}-b(\tilde{V}^{-1})_{ab}E^a_M E^b_N \\ &\quad -\bar{b}(V^{-1})_{ab}\tilde{E}^a_M\tilde{E}^b_N , \end{aligned} \qquad (3.14)$$

while the antisymmetric tensor is given by [cf. (2.25), (2.30)]

$$B_{MN}\equiv\mathcal{G}_{[MN]}=B_{0MN}+2(\tilde{V}^{-1}N^T)_{ab}E^a_{[M}\tilde{E}^b_{N]} . \qquad (3.15)$$

The dilaton is given by (2.27), but now with the definition (3.7) for the matrix V. In general, there is no residual gauge invariance so that this model has the group space G as a configuration space. In the case of the gauged WZNW model the above expressions were given in [16,5]. Notice that the corresponding expression for the antisymmetric tensor (3.15) is different from the result (2.25), (2.30) we have found in the previous section [in particular, the quantum term in (3.15) is $O(1/k)$, not $O(1/k^2)$ as in (2.30)]. In Sec. II we have extracted the local part of the effective action in a way that preserved gauge invariance [which is present in the original nonlocal effective action (3.5) in the special case of the gauged WZNW model]. This gauge invariance is missing in the local part of the effective action (3.12) and (3.13) as we obtained it above, but *is* present in (2.22) and (2.23).

The expressions for the σ-model couplings in the case of the bosonic chiral gauged WZNW model [third line in (3.6)] were already found in [17] [they are given by (3.14), (3.15) and (2.27) with $N=C$ and $\bar{b}=b$, see (3.6)]. In Sec. IV we shall check (in the two-loop approximation) the conformal invariance of the σ model corresponding to the chiral gauged SL(2,$\mathbb{R}$)/$\mathbb{R}$ WZNW model.

The heterotic case [fifth line in (3.6)] was treated in [18] with the result

$$\begin{aligned} &G_{MN}=G^{(s)}_{MN}-b[(C^TC)^{-1}]_{ab}E^a_M E^b_N , \\ &G^{(s)}_{MN}=G_{0MN}+2C^{-1}_{ab}E^a_{(M}\tilde{E}^b_{N)} , \\ &B_{MN}=B^{(s)}_{MN}=B_{0MN}+2C^{-1}_{ab}E^a_{[M}\tilde{E}^b_{N]} , \\ &\phi=\phi^{(s)}=\phi_0-\tfrac{1}{2}\ln\det C . \end{aligned} \qquad (3.16)$$

It is quite remarkable that in this case the antisymmetric tensor and the dilaton retain their semiclassical values while the metric receives just one $1/k$ correction.

For all the models defined in (3.6) one can prove the following theorem (see Appendix A)[8]

$$\begin{aligned} e^{-2\phi}\sqrt{\det G_{MN}}&=K\sqrt{\det G_{0MN}} \\ &=K\ (\text{Haar measure factor for } G) , \end{aligned} \qquad (3.17)$$

where K is a constant k-dependent factor given in (A4) [in the degenerate case of the gauged WZNW model one should fix a gauge and include the Faddeev-Popov ghost determinant making the right-hand side of (3.17) equal to the invariant measure factor on G/H]. In view of the relation of the expression for the dilaton (2.27) to the determinant of the matrix in the $O(A^2)$ term in the effective action one can also interpret (3.17) in the following way:

[8]This relation was conjectured in [34] for the Abelian SL(2,$\mathbb{R}$)/$\mathbb{R}$ coset case and in [12] where it was formulated in a general form for any gauged WZW model. Subsequently, its validity was explicitly checked for many Abelian and non-Abelian cases [13,2,3,4] and proved in general for any gauged WZNW model [5] and for any chiral gauged WZNW model [17].

the quadratically divergent part of the determinant [35,36] resulting from integration over the gauge fields combines with the Haar measure for the group G to give the correct measure $\sqrt{\det G_{MN}}$ for the σ model (3.13) [15,5] (the finite part of this determinant produces the dilaton term [27]).

In the case of the Abelian subgroup H we have shown [18] that the G/H chiral gauged WZNW model is equivalent to the axially gauged $(G\times H)/H$ WZNW model (with a special embedding of the subgroup H into the group $G\times H$). Given that chiral gauged WZNW model must obviously be conformally invariant, one should fix the ambiguity in the procedure of extracting of a local part of the effective action of the corresponding axially gauged $(G\times H)/H$ WZNW model in such a way that to make it identical to that in the case of the chiral gauged model. This will be discussed in the example of the $SL(2,\mathbb{R})\times\mathbb{R}/\mathbb{R}$ model in Appendix B. The conformal invariance of the resulting σ model will be demonstrated in Sec. IV.

If one uses the "semiclassical" prescription for extracting the σ-model coupling from (3.5) one finds the same exact metric (3.14) but the semiclassical antisymmetric tensor. For example, for the chiral gauged WZNW model the corresponding semiclassical B_{MN} is given by (3.16). This result is in agreement with the above-mentioned relation between chiral gauged and gauged WZNW models (assuming that the same "semiclassical" prescription is used for both classes of models).

IV. $D=3$ σ MODEL CORRESPONDING TO $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ GAUGED WZNW MODEL: CONFORMAL INVARIANCE AT TWO LOOPS

The aim of this section is to provide a nontrivial check that the exact σ models of the type discussed in the previous sections are actually conformally invariant beyond the semiclassical (one-loop) approximation, i.e., that the exact expressions for the background fields $G_{\mu\nu},B_{\mu\nu},\phi$ solve the string effective equations beyond the leading α' approximation. Such a check was already done [20,21] for the simplest exact $D=2$ σ model corresponding to the gauged $SL(2,\mathbb{R})/\mathbb{R}$ WZNW model [1]. However, this $D=2$ model has trivial antisymmetric tensor background. In view of the subtleties associated with the derivation of the exact expression for $B_{\mu\nu}$ (see Appendix B) it is important to confirm that the exact backgrounds with $B_{\mu\nu}\neq 0$ do actually solve the σ-model conformal invariance conditions.

A. Description of the model

The simplest model with a nontrivial antisymmetric tensor coupling is the $D=3$ "charged black string" σ-model associated with the $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW model [6]. The exact expression for its metric, dilaton [3] and the nonvanishing component of the antisymmetric tensor (B17) can be represented in the form

$$ds^2=-\frac{z-q-1}{z+b}dt^2+\frac{z-q}{z}dx^2+\frac{dz^2}{4(z-q-1)(z-q)}\ , \tag{4.1}$$

$$\phi=\phi_0-\tfrac{1}{4}\ln[z(z+b)]\ , \tag{4.2}$$

$$B_{tx}=-\frac{[q(q+1+b)]^{1/2}}{1+b}\left[\frac{q+1}{z}-\frac{q+b}{z+b}\right]\ , \tag{4.3}$$

where

$$q\equiv q_0(1+b),\quad \alpha'=\frac{1}{\kappa},\quad \kappa=k-2\ ,$$
$$b=+\frac{1}{\kappa}(g_G-g_H)=\frac{2}{\kappa}=2\alpha'\ , \tag{4.4}$$

and q_0 is a free parameter related to the coefficient which determines the embedding of the subgroup $\mathbb{R}$ into $SL(2,\mathbb{R})\times\mathbb{R}$. We discuss the derivation of (4.1)–(4.3) in Appendix B. Equation (4.3) is found if one uses the "corrected" prescription of Sec. II and fixes an ambiguity in the expression for B_{tx} in a particular way. In case one adopts the "semiclassical" prescription [see (2.41),(2.43)] instead of (4.3) one finds the following expression [see (B17′)]:

$$B_{tx}=-\frac{[q(q+1+b)]^{1/2}}{(1+b)[(1+b)z-bq]}\ . \tag{4.3′}$$

In (B15) we explain the relation of the coordinates z,x,t to the "classical" group space coordinates.[9] Because of the coordinates (and the definition of q) used the "semiclassical" expression (4.3′) still contains $O(b)$ corrections. It is easy to see that the two alternative expressions (4.3) and (4.3′) coincide to the order $O(b^2)$ and thus cannot be distinguished at the two-loop order, i.e., if the background (4.1)–(4.3) is conformally invariant, the background (4.1),(4.2),(4.3′) is conformally invariant as well.

As was shown in [18], the axial gauged $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ WZNW model with the embedding parameter $q_0=-\frac{1}{2}$ is equivalent to the $SL(2,\mathbb{R})/\mathbb{R}$ chiral gauged WZNW model. According to (3.14) and (2.27) the σ model corresponding to the latter theory has the same metric and dilaton as in (4.1) and (4.2) [17] with q now fixed to be

$$q_0=q_0^{(\mathrm{ch})}=-\tfrac{1}{2}\ ,$$
$$q=q^{(\mathrm{ch})}=-\tfrac{1}{2}(1+b)=-\frac{1}{2}\frac{k}{k-2}\ , \tag{4.5}$$

i.e.,

[9]In the notation of this section the metric (and the antisymmetric tensor) or α' are rescaled by a factor of 2 as compared to the previous Secs. II and III [cf. (2.23)]. Note also that the relation of our present notation to the notation used in [3,16,17] is $b=\lambda=2/k'$, $z=(\lambda+1)r-\lambda, q_0=\rho^2$, $k'=\kappa$.

$$(ds^2)^{(\mathrm{ch})} = -\frac{z-(1-b)/2}{z+b}dt^2 + \frac{z+(1+b)/2}{z}dx^2 + \frac{dz^2}{4[z-(1-b)/2][z+(1+b)/2]} . \tag{4.6}$$

The antisymmetric tensor [given by (3.15)] [17][10]

$$B_{tx}^{(\mathrm{ch})} = \frac{i(1-b)}{4}\left[\frac{1}{z}+\frac{1}{z+b}\right] , \tag{4.7}$$

is equal to (4.3) with $q=q^{(\mathrm{ch})}$,

$$B_{tx}(q=q^{(\mathrm{ch})}) = B_{tx}^{(\mathrm{ch})} . \tag{4.8}$$

In addition to the special case of $q=q^{(\mathrm{ch})}$, the $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ model has two other obvious limits: $SL(2,\mathbb{R})$ group space and the direct product $[SL(2,\mathbb{R})/\mathbb{R}]\times\mathbb{R}$ (or "neutral black string"). The two limiting σ models are already known to be conformally invariant so it is useful to keep them in mind while analyzing conformal invariance of the general background (4.1)–(4.3). The first limit corresponds to $q_0=q=\infty$ and the second – to $q_0=q=0$ (see also Appendix B). Introducing the new coordinates t',x',z' according to

$$t=\sqrt{q}\,t', \quad x=\sqrt{q}\,x', \quad z=z'+q+1 , \tag{4.9}$$

and taking the limit $q\to\infty$ $(z/q\to 1)$ one finds

$$ds^2 = -z'dt'^2 + (z'+1)dx'^2 + \frac{dz'^2}{4z'(z'+1)} , \tag{4.10}$$

$$B_{t'x'} = z', \quad H_{t'x'z'} = 1, \quad \phi = \mathrm{const} , \tag{4.11}$$

where in $B_{\mu\nu}$ we have dropped out an infinite constant [this would be unnecessary if we have kept a constant term in the derivation of (B9) from (B6)]. This background corresponds to the $SL(2,\mathbb{R})$ WZNW model. An equivalent model is found using the following generalization of the transformation (4.9) (the case of $n=\frac{1}{2}$ was discussed in [3]): $t=q^{(1-n)/2}t'$, $x=q^{(1-n)/2}x'$, $z=q^n z'+q+1$. Then assuming that $0<n<1$ and taking the limit $q\to\infty$ we get instead of (4.10) $ds^2=-z'(dt'^2-dx'^2)+dz'^2/4z'^2$. The $q\to 0$ $(q_0\to 0)$ limit of (4.1)–(4.3) has the form

$$ds^2 = -\frac{z-1}{z+b}dt^2 + dx^2 + \frac{dz^2}{4z(z-1)} ,$$
$$\phi = \phi_0 - \tfrac{1}{4}\ln[z(z+b)], \quad B_{\mu\nu}=0 , \tag{4.12}$$

that represents the direct product of the exact "black hole" background [1] and the x line. Let us note also that the large z asymptotics of the background (4.1)–(4.3) [or of (4.12)] is described by the flat space and linear dilaton

$$ds^2 = -dt^2 + dx^2 + dy^2 ,$$
$$\phi = \phi_0 - y, \quad B_{\mu\nu}=0, \quad z=e^{2y} . \tag{4.13}$$

B. Two-loop string effective action and field redefinitions

The general strategy of a proof that a background $\varphi^i=(G_{\mu\nu},B_{\mu\nu},\phi)$ solves (in the two-loop or α'^2 approximation) the σ-model conformal invariance conditions $\bar\beta^i=0$ is the following. One should expand the background fields in α',

$$\varphi^i = \varphi_1^i + \alpha'\varphi_2^i + \cdots , \tag{4.14}$$

and first check that the equations

$$\bar\beta^i = \bar\beta_1^i(\varphi) + \alpha'\bar\beta_2^i(\varphi) + \cdots = 0$$

are satisfied to the leading order in α', i.e., in the one-loop approximation, $\bar\beta_1^i(\varphi_1)=0$. Then one needs to find the most general form of the two-loop $\bar\beta^i$, e.g., by starting with the $\bar\beta^i$ functions computed in a particular scheme and making the most general local redefinition of couplings. The resulting $\bar\beta^i$ will be parametrized by a number of free parameters d_r. The problem is then to show that there exists such a "scheme," i.e., a choice of the parameters d_r, that the equations $\bar\beta^i=0$ are satisfied at the next order, i.e.,

$$\left[\frac{\partial\bar\beta_1^i}{\partial\varphi^j}(\varphi_1)\right]\varphi_2^j + \bar\beta_2^i(\varphi_1) = 0 . \tag{4.15}$$

If $\Phi^{ir}(\varphi)$ is a basis in the set of all possible covariant terms with the tensor structure of φ^i constructed out of two derivatives of φ^i then the general coupling redefinition

$$\varphi^{i'} \to \varphi^i + \alpha' d_r \Phi^{ir} + \cdots$$

induces the following change in the two-loop term in $\bar\beta^i$:

$$\bar\beta_2^i \to \bar\beta_2^i + \left[\frac{\partial\bar\beta_1^i}{\partial\varphi^j}\right] d_r \Phi^{jr} ,$$

i.e., Eq. (4.15) changes to

[10]The imaginary unit i in (4.7) can be absorbed into a rescaling of x or t which will change the corresponding sign in the metric, i.e., will change the signature from $(-++)$ to $(--+)$ or $(+++)$.

$$\left[\frac{\partial\bar\beta_1^i}{\partial\varphi^j}(\varphi_1)\right][\varphi_2^j+d_r\Phi^{jr}(\varphi_1)]+\bar\beta_2^i(\varphi_1)=0 \ . \qquad (4.16)$$

Therefore, we can ignore such field redefinitions that vanish on the solution of the leading order equations.

Equivalent approach is to start with the effective action that generates $\bar\beta^i$ functions, make the general field redefinition there and then take the variational derivatives.[11] There exists a simple scheme in which the order α'^2 effective action has the form [24][12]

$$\begin{aligned}S&=\int d^Dx\sqrt{G}\, e^{-2\phi}\bar\beta^\phi\\&=\int d^Dx\sqrt{G}\, e^{-2\phi}\{\tfrac{1}{6}(D-C)-\tfrac{1}{4}\alpha'[R+4D^2\phi-4(\partial_\mu\phi)^2-\tfrac{1}{12}H^2_{\lambda\mu\nu}]\\&\qquad-\tfrac{1}{16}\alpha'^2[R^2_{\mu\nu\lambda\kappa}-\tfrac{1}{2}R^{\mu\nu\kappa\lambda}H^\rho{}_{\mu\nu}H_{\kappa\lambda\rho}+\tfrac{1}{24}H_{\mu\nu\lambda}H^\nu{}_{\rho\alpha}H^{\rho\sigma\lambda}H_\sigma{}^{\mu\alpha}-\tfrac{1}{8}(H_{\mu\alpha\beta}H_\nu^{\alpha\beta})^2]+O(\alpha'^3)\}\ , \qquad (4.17)\end{aligned}$$

where $H_{\mu\nu\lambda}=3\partial_{[\mu}B_{\nu\lambda]}$ and C is the total central charge (we shall ignore the trivial part $C_0=26$ of C). The corresponding leading-order equations are linear combinations of

$$\bar\beta^G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{4}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}+2D_\mu D_\nu\phi=0\ , \qquad (4.18)$$

$$\bar\beta^B_{\mu\nu}=-\tfrac{1}{2}D_\lambda H^\lambda{}_{\mu\nu}+H^\lambda{}_{\mu\nu}\partial_\lambda\phi=0\ , \qquad (4.19)$$

$$\begin{aligned}\bar\beta^\phi&=\tfrac{1}{6}(D-C)+\alpha'[-\tfrac{1}{2}D^2\phi+(\partial\phi)^2-\tfrac{1}{24}H^2_{\lambda\mu\nu}]\\&=0\ . \qquad (4.20)\end{aligned}$$

They are corrected by α' terms which contain a number of free parameters corresponding to the most general local field redefinition [24][13]

$$G'_{\mu\nu}=G_{\mu\nu}+\alpha' S_{\mu\nu},\quad S_{\mu\nu}\equiv T_{\mu\nu}+G_{\mu\nu}X\ ; \qquad (4.21)$$

$$\begin{aligned}&T_{\mu\nu}=d_1R_{\mu\nu}+d_2\partial_\mu\phi\partial_\nu\phi+d_3H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}\ ,\\&X=d_4R+d_5H^2_{\lambda\mu\nu}+d_6D^2\phi+d_7(\partial\phi)^2\ ;\end{aligned} \qquad (4.22)$$

$$\begin{aligned}&B'_{\mu\nu}=B_{\mu\nu}+\alpha' K_{\mu\nu}\ ,\\&K_{\mu\nu}=d_8D_\lambda H^\lambda{}_{\mu\nu}+d_9\partial_\lambda\phi H^\lambda{}_{\mu\nu}\ ;\end{aligned} \qquad (4.23)$$

$$\begin{aligned}&\phi'=\phi+\alpha' Y\ ,\\&Y=d_{10}R+d_{11}H^2_{\lambda\mu\nu}+d_{12}D^2\phi+d_{13}(\partial\phi)^2\ .\end{aligned} \qquad (4.24)$$

In view of the observation that we can use the leading-order equations (4.18)–(4.20) to simplify the field redefinition terms, the number of free parameters reduces to ten. Moreover, in the case of our interest the target space dimension is $D=3$ so that

$$\begin{aligned}&H_{\mu\nu\lambda}=H\epsilon_{\mu\nu\lambda}\ ,\\&H^2_{\mu\nu}\equiv H_{\mu\alpha\beta}H^{\alpha\beta}{}_\nu=2H^2G_{\mu\nu}\ ,\\&H^2_{\lambda\mu\nu}=6H^2\ ,\end{aligned} \qquad (4.25)$$

where $\epsilon_{\mu\nu\lambda}$ is the totally antisymmetric tensor and the metric is assumed to have Euclidean signature. We have also

$$R^2_{\mu\nu\lambda\kappa}=4R^2_{\mu\nu}-R^2,\quad R^{\mu\nu\kappa\lambda}H_{\mu\nu}{}^\rho H_{\kappa\lambda\rho}=2RH^2\ ,$$

$$H_{\mu\nu\lambda}H^\nu_{\rho\alpha}H^{\rho\sigma\lambda}H^{\mu\alpha}_\sigma=6H^4,\quad (H_{\mu\alpha\beta}H^{\alpha\beta}_\nu)^2=12H^4\ .$$

As a result, the relevant field redefinitions are given by (4.21)–(4.23) with

$$\begin{aligned}&T_{\mu\nu}=c_1R_{\mu\nu}+c_2\partial_\mu\phi\partial_\nu\phi\ ,\\&X=c_3H^2+c_4R+c_5\ ,\end{aligned} \qquad (4.26)$$

$$K_{\mu\nu}=c_6\epsilon_{\mu\nu\lambda}D^\lambda H,\quad Y=c_7H^2+c_8R+c_9\ . \qquad (4.27)$$

The constant terms in X and Y appear as a consequence of the elimination of the $(\partial\phi)^2$ terms using (4.20). We can set $c_1=0$ [in view of (4.18)–(4.20) and the previous footnote c_1 can be absorbed into c_3,c_6,c_7,c_8,c_9] and $c_9=0$ (our equations will depend on ϕ only through its derivatives). We are thus left with seven free parameters $c_2,\ldots,c_8$.

To simplify the analysis we shall consider only the following three scalar equations (this turns out to be sufficient in order to prove that our background is a solution): the trace of the metric $\bar\beta$ equation, the equation for the antisymmetric tensor (in $D=3$ it is equivalent to a scalar equation) and the equation for the dilaton. These equations have the following form in the scheme in which the effective action is given by (4.17) [24] [we make use of $D=3$, (4.25) and (4.18)–(4.20)]

[11] If the effective action is known (e.g., is determined from the string S matrix) then one can by-pass the problem of computing independently the "diffeomorphism" vector W_μ that appears in the Weyl anomaly coefficients [37]. Let us use this occasion to correct a confusion in [24] concerning the two-loop value of W_μ. Since the standard β function is computed only modulo the diffeomorphism terms, Eq. (5.35) of [24] should depend on the scheme parameters p_1,p_2,f_1 only through the two combinations: $p_1'=p_1-\frac{1}{4}f_1,p_2$ [the derivative p_1 term in (5.35) should have coefficient $p_1-\frac{1}{4}f_1$]. Then the correct expression for W_μ in the scheme $p_1'=-\frac{1}{4}$, $p_2=0$ (see the discussion after Eq. (6.7) in [24]) should be $W_\mu=\frac{1}{4}D^\lambda(H^2_{\lambda\mu})-\frac{5}{24}D_\mu H^2$.

[12] More general actions related to (4.17) by field redefinitions are discussed in [38–40].

[13] We ignore the term $D_\mu D_\nu\phi$ since it can be eliminated by a coordinate transformation, $\delta x^\mu=d_0\alpha' D^\mu\phi$, $\delta G_{\mu\nu}=2d_0\alpha' D_\mu D_\nu\phi$ (the equations we shall study are covariant so that one combination of the parameters will not be present in the variation).

$$\bar{\beta}^G = R - \tfrac{3}{2}H^2 + 2D^2\phi + \tfrac{1}{2}\alpha'[R^2_{\mu\nu\lambda\kappa} - 6RH^2 + \tfrac{9}{4}H^4 + 5(D_\mu H)^2 - \tfrac{1}{2}D^2H^2] + \alpha' Q = 0 , \tag{4.28}$$

$$\bar{\beta}^B_{\nu\lambda}\epsilon_\mu{}^{\nu\lambda} = -e^{2\phi}\partial_\mu\{e^{-2\phi}[H + \tfrac{1}{4}\alpha'(2RH + 5H^3) + \alpha' F]\} = 0 , \tag{4.29}$$

$$\tilde{\beta}^\phi = \bar{\beta}^\phi - \tfrac{1}{4}\bar{\beta}^G = \tfrac{1}{6}(D - C) - \tfrac{1}{4}\alpha'[R + 4D^2\phi - 4(\partial\phi)^2 - \tfrac{1}{2}H^2 + \tfrac{1}{4}\alpha'(R^2_{\mu\nu\lambda\kappa} - RH^2 - \tfrac{5}{4}H^4) + \alpha' P] = 0 . \tag{4.30}$$

The terms Q, F, and P [that vanish in the scheme of (4.17)] are included to indicate corrections which will appear once a general field redefinition (4.21)–(4.24),(4.26),(4.27) is performed in the leading-order terms [we use again (4.18)–(4.20)]

$$Q = D_\mu D_\nu S^{\mu\nu} - D^2 S + H^2 S - 2D_\mu S^{\mu\nu}\partial_\nu\phi + \partial^\mu S\partial_\mu\phi + 2D^2 Y - 3c_6 HD^2H , \tag{4.31}$$

$$F = -\tfrac{1}{2}SH - 2YH + c_6 D^2 H, \quad S \equiv S^\mu_\mu = T + 3X , \tag{4.32}$$

$$P = D_\mu D_\nu S^{\mu\nu} + S^{\mu\nu}(-2D_\mu D_\nu\phi + 4\partial_\mu\phi\partial_\nu\phi) - 4D_\mu S^{\mu\nu}\partial_\nu\phi + 2\partial^\mu S\partial_\mu\phi + 4D^2Y - 8\partial^\mu Y\partial_\mu\phi - c_6 HD^2H . \tag{4.33}$$

C. Solution of two-loop conformal invariance conditions

Let us now compute various terms in (4.28)–(4.33) in the special case of (4.1)–(4.3), expanding the leading $O(\alpha')$ terms to the first power in α' or b. The necessary geometrical quantities and their expansions in powers of b are given in Appendix C. Since q in (4.1)–(4.4) is a free parameter it is not necessary to expand it in b (i.e., we can treat it as being b independent). For generality we have done the computation of (4.28)–(4.30) for arbitrary values of the constants s_1, s_2 which are used to parametrize B_{tx} in (C9), i.e.,

$$B_{tx} = -\frac{[q(q+1)]^{1/2}}{z}\left[1 + bv_1 + \frac{bv_2}{z} + O(b^2)\right], \quad v_1 = -\tfrac{1}{2}s_1(1+q)^{-1}, \quad v_2 = \tfrac{1}{4}(s_2 - 1) . \tag{4.34}$$

The antisymmetric tensors (4.3) and (4.3′) coincide to this order and both have

$$s_1 = 3 + 4q, \quad s_2 = 1 + 4q . \tag{4.34′}$$

Introducing the notation

$$I_1 = (2q+1)z^{-1}, \quad I_1^2 = (2q+1)^2 z^{-2}, \quad I_2 = q(q+1)z^{-2} ,$$

$$I_3 = I_1 I_2 = q(q+1)(2q+1)z^{-3}, \quad I_4 = I_2^2 = q^2(q+1)^2 z^{-4} ,$$

and using (C9)–(C15) we get ($b = 2\alpha'$)

$$R - \tfrac{3}{2}H^2 + 2D^2\phi = b[-(4 + 6q + 6qs_1)z^{-2} + (6 + 6s_2)q(q+1)z^{-3}] + O(b^2) , \tag{4.35}$$

$$\tfrac{1}{2}\alpha'[R^2_{\mu\nu\lambda\kappa} - 6RH^2 + \tfrac{9}{4}H^4 + 5(D_\mu H)^2 - \tfrac{1}{2}D^2H^2] = \tfrac{1}{4}b[16z^{-2} - 16I_2 + 80I_3 - 240I_4] + O(b^2) , \tag{4.36}$$

$$e^{-2\phi}H = 2i[1 - \tfrac{1}{2}bs_1(1+q)^{-1} + \tfrac{1}{2}b(1+s_2)z^{-1}] + O(b^2) , \tag{4.37}$$

$$\tfrac{1}{4}\alpha' e^{-2\phi}(2RH + 5H^3) = 2ib(6I_2 - I_1) + O(b^2) , \tag{4.38}$$

$$R + 4D^2\phi - 4(\partial\phi)^2 - \tfrac{1}{2}H^2 = -4 + 2b[-(q + 1 + qs_1)z^{-1} + (1 + s_2)I_2]z^{-1} + O(b^2) , \tag{4.39}$$

$$\tfrac{1}{4}\alpha'(R^2_{\mu\nu\lambda\kappa} - RH^2 - \tfrac{5}{4}H^4) = 2b[z^{-2} + 2I_2 - 2I_3] + O(b^2) . \tag{4.40}$$

We have fixed ϕ_0 so that $e^{-2\phi_0}[q(q+1)]^{1/2} = 1$. Calculating (using the software package Mathematica) similar expansions in powers of $1/z$ for P, F, Q in (4.31)–(4.33) and combining them with (4.35)–(4.40) we have found that the leading-order terms in the conformal invariance equations (4.28), (4.29), and (4.30) take the form

$$\bar{\beta}^G = 2\alpha'[e_1(1+2q)z^{-1} + 2e_2 z^{-2} + e_3 q(q+1)z^{-3} + 2e_4 q^2(q+1)^2 z^{-4}] + O(\alpha'^2) , \tag{4.41}$$

$$\bar{\beta}^B_{tx} = 2\alpha' G^{zz}\sqrt{G}\, e^{2\phi}\partial_z[t_1 z^{-1} + t_2 q(q+1)z^{-2}] + O(\alpha'^2) , \tag{4.42}$$

$$\tilde{\beta}^\phi = \tilde{\beta}^\phi_0 - \tfrac{1}{2}\alpha'^2[h_2 z^{-2} + 2h_3 q(q+1)z^{-3} + 12h_4 q^2(q+1)^2 z^{-4}] + O(\alpha'^3) , \tag{4.43}$$

where

$$\begin{aligned}
e_1 &= c_2 - 12c_4 + 16c_8 \,, \\
e_2 &= 14c_4 - 16c_8 + (-5 - c_2 + 28c_3 + 154c_4 - 3c_5 + 12c_6 - 32c_7 - 176c_8 - 3s_1)q \\
&\quad + (-2 - c_2 + 28c_3 + 154c_4 - 3c_5 + 12c_6 - 32c_7 - 176c_8)q^2 \,, \\
e_3 &= 26 - c_2 - 88c_3 - 376c_4 - 48c_6 + 96c_7 + 384c_8 + 6s_2 \\
&\quad + (40 - 2c_2 - 176c_3 - 752c_4 - 96c_6 + 192c_7 + 768c_8)q \,, \\
e_4 &= -30 + c_2 + 72c_3 + 252c_4 + 36c_6 - 64c_7 - 224c_8 \,, \\
t_1 &= 6 + c_2 - 12c_4 - 16c_6 - 16c_8 + 2s_2 + (8 + 2c_2 - 24c_4 - 32c_6 - 32c_8)q \,, \\
t_2 &= -24 - c_2 + 12c_3 + 42c_4 + 24c_6 + 16c_7 + 56c_8 \,, \\
h_2 &= -c_2 + 24c_4 - 32c_8 + (2 - 6c_2 + 40c_3 + 236c_4 - 2c_5 + 8c_6 - 64c_7 - 352c_8 - 2s_1)q \\
&\quad + (4 - 6c_2 + 40c_3 + 236c_4 - 2c_5 + 8c_6 - 64c_7 - 352c_8)q^2 \,, \\
h_3 &= -1 + c_2 - 40c_3 - 164c_4 - 8c_6 + 64c_7 + 256c_8 + s_2 + (-4 + 2c_2 - 80c_3 - 328c_4 - 16c_6 \\
&\quad + 128c_7 + 512c_8)q \,, \\
h_4 &= 10c_3 + 35c_4 + 2c_6 - 16c_7 - 56c_8, \quad \tilde{\beta}_0^{\phi} = \tfrac{1}{6}(D - C) + \alpha' + \alpha'^2 h_0 + O(\alpha'^3), \quad h_0 = -c_2 - c_5 \,,
\end{aligned} \tag{4.44}$$

where c_i are the scheme dependence parameters and s_1, s_2 appear in H in (C9).

The conformal invariance conditions

$$e_1 = e_2 = e_3 = e_4 = t_1 = t_2 = h_2 = h_3 = h_4 = 0$$

(the equation $h_0 = 0$ that determines the value of central charge will be discussed below) is a system of nine equations, while the scheme ambiguity is represented only by seven parameters. That is why the existence of a solution is nontrivial. For generic $q \neq 0, -1$ a solution exists only if $s_2 = 1 + 4q$ which is the value corresponding to (4.3) [i.e., for the special value (B5) of the parameter μ in (B9), (B15), (C9), (C10)] and is given by

$$\begin{aligned}
&c_2 = c_4 = c_8 = 0, \quad c_3 = c_6 = \tfrac{1}{2}, \quad c_7 = \tfrac{3}{8} \,, \\
&c_5 = \frac{1 + 2q - s_1}{1 + q} = -2 \,,
\end{aligned} \tag{4.45}$$

where we have finally set s_1 to its value $s_1 = 4q + 3$ in (4.3). We conclude that there exists a "conformal" scheme in which the background (4.1)–(4.3) is a solution of the conformal invariance equations (4.28) and (4.29) in the two-loop approximation. Equivalently, a background which is related to (4.1)–(4.3) by the field redefinitions (4.21)–(4.24), (4.26), (4.27) with parameters given by (4.45) is a solution of the string equations in the scheme where the effective action has the form (4.17). The transformation that defines the "conformal" scheme in terms of that of (4.17) is thus simply ($c_5 = -2$)

$$\begin{aligned}
G'_{\mu\nu} &= G_{\mu\nu} + \tfrac{1}{4}\alpha' H^2_{\mu\nu} + c_5 \alpha' G_{\mu\nu} + O(\alpha'^2) \\
&= G_{\mu\nu} + \tfrac{1}{2}\alpha' H^2_{\mu\nu} + \alpha' D^2\phi G_{\mu\nu} \\
&\quad - 2\alpha'(\partial\phi)^2 G_{\mu\nu} + O(\alpha'^2) \,, \\
B'_{\mu\nu} &= B_{\mu\nu} + \tfrac{1}{2}\alpha' D^\lambda H_{\lambda\mu\nu} + O(\alpha'^2) \,, \\
\phi' &= \phi + \tfrac{1}{12}\alpha' H^2_{\mu\nu\lambda} + \tfrac{3}{8}\alpha' R + O(\alpha'^2) \,.
\end{aligned} \tag{4.46}$$

In (4.46) we have made use of the leading-order equation (4.20) and replaced $H^2 G_{\mu\nu}$ by $\frac{1}{2}H^2_{\mu\nu}$. For $\phi = 0$ Eq. (4.46) is equivalent to the transformation in [24] between the scheme ("$f_1 = 1$") corresponding to (4.17) and the "conformal" scheme ("$f_1 = -1$") in which the parallelizable space (e.g., a group space) is automatically a solution of the conformal invariance equations. Note that the solutions of the equations in the different schemes are related by the inverse transformations, i.e., if $G_{\mu\nu}$ is the solution in the scheme (4.45) then $G'_{\mu\nu}$ in (4.46) is the solution in the $c_i = 0$ scheme of (4.17).

In particular, the background corresponding to the chiral gauged SL(2,R)/R WZNW model [which is the special case of (4.1)–(4.3) for $q = -\frac{1}{2}(1 + b)$] is also conformally invariant.[14] In the two other special cases: $q = 0$ [direct product of SL(2,R)/R gauged WZNW model and R] and $q \to \infty, z \to \infty$ [SL(2,R) WZNW model] we find the solutions for the following values of nonvanishing c_i

$$q = 0{:}\quad c_2 = c_4 = c_8 = 0, \quad h_0 = 0 \,, \tag{4.47}$$

$$q = \infty{:}\quad 4c_3 + 6c_4 - c_5 = 4, \quad h_0 = 2 \,. \tag{4.48}$$

Equation (4.47) thus reproduces the known result about the two-loop conformal invariance of the exact black hole background (which is true in the "standard" or "conformal" scheme) [20]. Equation (4.48) specifies the relation

[14] A careful analysis of the system of the conformal invariance equations (4.43)–(4.45) in the limit $q = -\frac{1}{2} + O(b)$ gives the following solution [which again exist only if $s_2 = 1 + 4q = -1 + O(b)$ but is more general than (4.45) with $q = -\frac{1}{2}$]: $c_2 = 0$, $c_3 = \frac{1}{2} - \frac{7}{4}c_4$, $c_6 = \frac{1}{2}$, $c_7 = \frac{3}{8} - \frac{7}{2}c_8$, $c_5 = -2s_1$.

between the scheme of (4.17) and the scheme in which the SL(2,$\mathbb{R}$) group space is a solution of $\bar{\beta}^i=0$. Using that for this group space [cf. (C11),(C13)] $R=-6$, $H^2=-4$ we can put the corresponding redefinition of the metric into the form [cf. (4.46)]

$$G'_{\mu\nu}=G_{\mu\nu}+\alpha'(c_3H^2+c_4R+c_5)G_{\mu\nu}+O(\alpha'^2)$$
$$=G_{\mu\nu}+\tfrac{1}{2}\alpha' H^2_{\mu\nu}+O(\alpha'^2)\ . \quad (4.46')$$

D. Scheme dependence and the value of central charge

The conditions of conformal invariance of the σ model $\bar{\beta}^G_{\mu\nu}=0$, $\bar{\beta}^B_{\mu\nu}=0$ imply [41,42,40] that $\tilde{\beta}^\phi=\bar{\beta}^\phi=\bar{\beta}^\phi_0$ is a constant which is proportional to the central charge. It is natural to expect that the latter should be consistent with the conformal field theory expression. It is necessary, however, to note that the correspondence between the σ-model and conformal field theory results may hold only in a specific "conformal" scheme.

As follows from (4.39) and (4.44) the leading order form of (4.30) is satisfied if the value of the central charge is

$$C=D+6\alpha'+O(\alpha'^2)=3[1+2\alpha'+O(\alpha'^2)]\ .$$

This is consistent with the value of the central charge of the [SL(2,$\mathbb{R}$)$\times\mathbb{R}$]/$\mathbb{R}$ conformal field theory

$$C=\frac{3k}{k-2}=3+\frac{6}{k}+\frac{12}{k^2}+O\left[\frac{1}{k^3}\right]=3+\frac{6}{k-2}$$
$$=3+6\alpha'\ , \quad (4.49)$$

where we have used the relation $\alpha'=1/(k-2)$ in (4.4). This suggests that if $\alpha'=1/(k-2)$ all higher order $O(\alpha'^n)$ contributions to C should vanish in the "conformal" scheme in which the whole expression for C should be given just by the "one-loop" $O(\alpha')$ contribution (4.49).

According to (4.44) and (4.45) the $O(\alpha'^2)$ term in (4.48) has the form

$$C=3+6\alpha'-6\alpha'^2\frac{1+2q-s_1}{1+q}+O(\alpha'^3)$$
$$=3+6\alpha'+12\alpha'^2+O(\alpha'^3)\ , \quad (4.50)$$

where we have used that the value of s_1 that corresponds to our background (4.3),(C10) is $s_1=3+4q$. This expression would be in perfect agreement with (4.49) if α' were equal to $1/k$ as in the semiclassical approximation. The conformal invariance of the σ model holds of course for an arbitrary choice of the overall coefficient α'. However, since the α' dependence of the background (4.1)–(4.3) was established by starting from the gauged WZNW effective action which led also to the "shifted" value $\alpha'=1/(k-2)$ in (4.4) one would expect to reproduce the conformal field theory value of C (4.49) with such α' (as indeed was in the $D=2$ black hole model [1,20]).

This, in fact, is possible by noting that there is an additional freedom of rescaling $B_{\mu\nu}$ (or its field strength) by a constant factor $[1+O(\alpha')]$ that can be included in the field redefinition ambiguity [this is the analog of the transformation $G'_{\mu\nu}=G_{\mu\nu}+\alpha' c_5 G_{\mu\nu}+O(\alpha'^2)$ in (4.21) and (4.26)]

$$B'_{\mu\nu}=B_{\mu\nu}+\alpha' c_0 B_{\mu\nu}+O(\alpha'^2)\ . \quad (4.51)$$

By the combined rescalings of $G_{\mu\nu}$ and $B_{\mu\nu}$ one can effectively rescale α'. As it is clear from (C9) and (C10) the transformation (4.51) shifts the value of s_1 to $s'_1=s_1-c_0(1+q)$ or c_5 in (4.45) to $c'_5=c_5+c_0$ and thus h_0 in (4.44) to $-c_2-c_5-c_0$. To have the zero "two-loop" term in C we thus need $c_0=-c_5$ or (since $s_1=4q+3$) $c_0=2$. This is the expected result since, in general, it is clear that to change $\alpha'=1/(k-2)$ into $\alpha'=1/k$ one is to rescale $B_{\mu\nu}$ by the factor $k/(k-2)=1+b=1+2\alpha'$. The rescaled $B_{\mu\nu}$ in (4.3) takes the simpler form

$$B'_{tx}=-[q(q+1+b)]^{1/2}\left[\frac{q+1}{z}-\frac{q+b}{z+b}\right]\ ,$$
$$s'_1=1+2q,\quad c'_5=0\ . \quad (4.52)$$

In what follows we shall use the present $D=3$ example to clarify further the crucial role of coupling redefinitions or scheme dependence in understanding a relation between conformal field theory and σ-model results. If the background (4.1)–(4.3) solves the conformal invariance conditions, the corresponding central charge or $\bar{\beta}^\phi=$const may be computed at any point, e.g., at $\ln z=\infty$. In this limit our background becomes a flat space with linear dilaton (4.13). Assuming that higher order α'^n contributions to $\bar{\beta}^\phi$ are constructed in terms of the curvature R and H which vanish in this limit, one concludes that the exact value of C should be determined just by the "one-loop" dilaton term,

$$C=6\bar{\beta}^\phi=3+6\alpha'(\partial\phi)^2=3+6\alpha'\ . \quad (4.53)$$

This, in fact, gives the conformal field theory value for C (4.49) under the identification $\alpha'=1/(k-2)$ in (4.4). This argument is true only in a special scheme in which there are no higher loop contributions to $\bar{\beta}^\phi$ on the flat linear dilaton background. *A priori* such scheme need not necessarily be the one in which (4.1)–(4.3) is conformally invariant. The general redefinition of the metric (4.21) will induce, e.g., higher order $O((\partial\phi)^n)$ terms in $\bar{\beta}^\phi$ which will *not* vanish on the linear dilaton background and will thus shift the value of C. For example, the two-loop shift h_0 in C (4.44) is expressed in terms of the coefficients c_2,c_5 of the dilatonic $(\partial\phi)^2$ terms in $G'_{\mu\nu}$. In fact, consider the asymptotic large distance form of the redefinitions (4.21),(4.24)

$$G'_{\mu\nu}=G_{\mu\nu}+\alpha'(c_2\delta_{\mu 2}\delta_{\nu 2}+c_5G_{\mu\nu})\ ,$$
$$B'_{\mu\nu}=B_{\mu\nu},\quad \phi'=\phi+\alpha' d_{13}\ , \quad (4.54)$$

where we have made use of (4.13). As a result of (4.54) we get a shift in the overall scale of the metric, $G'_{22}=1+\alpha'(c_2+c_5)$ so that the one-loop expression

(4.53) now produces a two-loop correction to C

$$C=6\tilde{\beta}^{\phi}=3+6\alpha' G'^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$=3+6\alpha'-6\alpha'^2(c_2+c_5)+O(\alpha'^3) \ . \qquad (4.55)$$

The above redefinition amounts to a constant rescaling of (the relevant component of) the metric, or to an effective redefinition of α'.

We conclude that the fact that the total contribution to C comes just from the one-loop correction is scheme dependent, i.e., is true only in a specific scheme. Another illustration of this point can be given on the example of the WZNW theory or the group space σ model. This case is "complementary" to the gauged WZNW or "coset" σ model one where the derivatives of $G_{\mu\nu}$ and $B_{\mu\nu}$ are decreasing with distance while that of the dilaton is approaching a constant: for the group space the derivative of the dilaton is zero but $R=$const, $H=$const so that C is expected to receive contributions to all orders in α'. In fact, identifying α' with $1/k$ and computing $\tilde{\beta}^{\phi}$ in the same scheme in which the two-loop $\tilde{\beta}^{G}$ and $\tilde{\beta}^{B}$ functions naturally vanish, one correctly reproduces [24] the $(1/k)^2$ term in the standard conformal field theory expression [43,44] for C,

$$C=D-\alpha' R+\alpha'^2R^2D^{-1}+O(\alpha'^3)=\frac{kD}{k+g_G} \ ,$$
$$R=Dg_G, \quad \alpha'=1/k \ . \qquad (4.56)$$

This can be also seen directly from (4.28) (with $P=0$) in the SL(2,$\mathbb{R}$) group manifold case ($D=3$, $R_{\mu\nu}=\frac{1}{3}RG_{\mu\nu}=\frac{1}{4}H^2_{\mu\nu}=\frac{1}{2}H^2G_{\mu\nu}$). At the same time, it is possible in principle to find such a "nonstandard" scheme in which the full contribution to C comes just from the one-loop $O(\alpha' H^2)$ term in (4.20),

$$C=D+6\alpha'[-\tfrac{1}{2}D^2\phi+(\partial\phi)^2-\tfrac{1}{24}H^2_{\mu\nu\lambda}]$$
$$=D-\alpha' R=D-\frac{Dg_G}{k+g_G} \ . \qquad (4.57)$$

with all higher-order contributions being now "hidden" in the H^2 term. This happens if one rescales both the metric *and* the antisymmetric tensor $G'_{\mu\nu}=(\kappa/k)G_{\mu\nu}$, $B'_{\mu\nu}=(\kappa/k)B_{\mu\nu}$, i.e., effectively replaces $\alpha'=1/k$ by $1/\kappa$. If one starts with the effective action of the WZNW model [15] where one has κ as an overall coefficient, one is to use such a scheme in order to reproduce the standard expression for C.[15] Since the group space is a particular limit of our $D=3$ background (4.1)–(4.3) this explains also why we need to rescale $B_{\mu\nu}$ (4.51) in order to reproduce the correct expression for C under the identification of α' with $1/\kappa$.[16]

V. SEMICLASSICAL BACKGROUND AS AN EXACT SOLUTION OF CONFORMAL INVARIANCE EQUATIONS

As is well known, there exists a "standard" scheme [46,24] in which the "semiclassical" (α'-independent) group space background of the WZNW model remains the solution of the conformal invariance equations at each order in α' expansion (in such a scheme $\alpha'=1/k$ and the central charge C receives corrections of all orders in α'). Similar statement is true for the SL(2,$\mathbb{R}$)/$\mathbb{R}$ black hole model [47]: there exists a "nonstandard" scheme in which the semiclassical background [27] is an exact solution. This follows from the fact that the exact "black hole" background [1] (that solves the conformal invariance equations in a "standard" scheme [20,21]) is related to the leading-order one [27] by a local, covariant and background-independent field redefinition. Given that the SL(2,$\mathbb{R}$) group space and the "neutral black string" are the two particular limits ($q=\infty$ and $q=0$) of our general background it is natural to ask if there exists such a scheme in which the semiclassical limit of the background (4.1)–(4.3)

$$ds^2=-\frac{z-q_0-1}{z}dt^2+\frac{z-q_0}{z}dx^2$$
$$+\frac{dz^2}{4(z-q_0-1)(z-q_0)} \ , \qquad (5.1)$$
$$p=\phi_0-\tfrac{1}{2}\ln z \ ,$$
$$B_{tx}=-\frac{[q_0(q_0+1)]^{1/2}}{z} \ , \qquad (5.2)$$

is an exact solution of the conformal invariance equations for arbitrary q_0. As we shall show below, this in indeed the case in the two-loop approximation.[17]

[15]Such a rescaling of $B_{\mu\nu}$ would be unnecessary if the semiclassical expression for $B_{\mu\nu}$ (B9') or (4.3'),(B17') were "truly semiclassical" being multiplied by k and not by $k+g_G$ in the σ-model action, i.e., contained an extra factor of $1+b$.

[16]A scheme in which C receives only the one-loop contribution may look natural from the conformal field theory point of view where the renormalization of k originates from normal ordering and so is effectively "one-loop." The above observations seem to resolve the puzzle discussed in [45]. On one hand, the exact current algebra result for C in the (1,1) supersymmetric WZNW model contains just one $1/k$ correction, $C=D(k-g_G)/(\kappa-g_G)=D(1-g_G/k)$. On the other hand, there is no reason to expect that four and higher loop contributions to the σ-model $\bar{\beta}$ functions and thus to $\tilde{\beta}^{\phi}$ should vanish in general in this model (see [45] and references there). The paradox disappears once one notes that the correspondence between the conformal field theory and σ-model results should hold only in a particular scheme; there exists such a scheme in which the whole contribution to $C=6\tilde{\beta}^{\phi}$ comes just from the one-loop term.

[17]A different argument suggesting that there should exist a scheme in which the semiclassical background in a related $D=3$ model is exact to all orders was recently given in [48].

To establish the form of the corresponding two-loop conformal invariance equations (4.28)–(4.30) one is to omit the contributions in (4.35), (4.37), (4.39) which came from the α' dependence of the background (4.1)–(4.3). As a result, the constants in (4.41)–(4.43) change by some c_i-independent numbers. Setting (combinations of) these constants to zero gives the following system of nine equations (now of course $s_1=s_2=0$)

$$c_2-12c_4+16c_8=0\ ,$$

$$2+14c_4-16c_8+(-2-c_2+28c_3+154c_4-3c_5+12c_6-32c_7-176c_8)q_0 +(-2-c_2+28c_3+154c_4-3c_5+12c_6-32c_7-176c_8)q_0^2=0\ ,$$

$$20-c_2-88c_3-376c_4-48c_6+96c_7+384c_8=0\ ,$$

$$-30+c_2+72c_3+252c_4+36c_6-64c_7-224c_8=0\ ,$$

$$4+c_2-12c_4-16c_6-16c_8=0,\quad -24-c_2+12c_3+42c_4+24c_6+16c_7+56c_8=0\ ,$$

$$2-c_2+24c_4-32c_8+(4-6c_2+40c_3+236c_4-2c_5+8c_6-64c_7-352c_8)q_0 +(4-6c_2+40c_3+236c_4-2c_5+8c_6-64c_7-352c_8)q_0^2=0\ ,$$

$$-2+c_2-40c_3-164c_4-8c_6+64c_7+256c_8=0,\quad 10c_3+35c_4+2c_6-16c_7-56c_8=0\ ,$$

with the general solution [cf. (4.45)]

$$c_2=-2,\quad c_3=1,\quad c_4=c_5=c_6=0\ ,\qquad c_7=\tfrac{3}{16},\quad c_8=\tfrac{1}{8}\ . \tag{5.3}$$

We also get the correct two-loop contribution to the central charge [$h_0=-2$ in (4.44), cf. (4.49)] with the choice of $\alpha'=1/k$ appropriate for a semiclassical background. The redefinition [cf. (4.46)]

$$G'_{\mu\nu}=G_{\mu\nu}-2\alpha'\partial_\mu\phi\partial_\nu\phi+\tfrac{1}{2}\alpha' H^2_{\mu\nu}+O(\alpha'^2)\ , \tag{5.4}$$

$$B'_{\mu\nu}=B_{\mu\nu}+O(\alpha'^2)\ ,\qquad \phi'=\phi+\tfrac{1}{32}\alpha' H^2_{\mu\nu\lambda}+\tfrac{1}{8}\alpha' R+O(\alpha'^2)\ , \tag{5.5}$$

that relates the "nonstandard" scheme (5.3) to the scheme of (4.17) is similar to the one found in the $D=2$ black hole case in [47]. In particular, (5.4) contains the term $-2\alpha'\partial_\mu\phi\partial_\nu\phi$ that can be related [47] to a nontrivial derivative term in the determinant [36] resulting from the integration over the gauge fields in the gauged WZNW model.

The existence of the schemes (4.45) and (5.3) in which both the exact (4.1)–(4.3) and semiclassical (5.1),(5.2) backgrounds are solutions of the conformal invariance equations implies that these two backgrounds are related by a local field redefinition [which is found explicitly by combining (4.46) with (5.4) and (5.5) and also taking into account a coordinate transformation involved].[18] It is thus very likely that the full α' dependence of the exact background (4.1)–(4.3) can be generated by a local field redefinition of the semiclassical background (5.1),(5.2).

This does not, however, imply, that it is the semiclassical and not the exact background that has a direct physical interpretation. The reason is that the "conformal" or "standard" scheme associated with the exact background is more directly related to the corresponding conformal field theory. In particular, the equations for the perturbations of the background (e.g., the tachyonic equation) have simple forms only in the "conformal" scheme and become complicated in a "nonstandard" one, i.e., after the field redefinition that transforms the exact background into the semiclassical one [47]. The propagation of a first-quantized string in a given background is described by the Klein-Gordon-type equations for the string modes, i.e., by the equations for the states

$$(L_0+\bar{L}_0)\psi_n=N_n\psi_n\ , \tag{5.6}$$

of the corresponding conformal field theory. The stress tensor or the operator L_0 of the conformal theory can be considered as a functional of the background fields $\varphi^i=(G_{\mu\nu},B_{\mu\nu},\phi,\dots)$. Its structure is fixed in a given theory and does not depend on particular identification of the background fields. Namely, Eqs. (5.6) (and hence their solutions ψ_n) do not change under redefinitions of φ^i. However, the *form* of their representation in terms of the background fields φ^i does depend on a particular choice of φ^i. That is why one should be careful to take into account the scheme dependence in comparing the conformal field theory and σ-model results. For example, (5.6) should correspond to the linearized terms in the corresponding $\bar\beta$-function equations. The latter are scheme dependent and so the agreement is possible only in a particular scheme.

Let us illustrate the above remarks on the example of the tachyon $\bar\beta$-function equation [49–51]

$$\bar\beta^T=-\gamma T+W^\mu\partial_\mu T-2T\ , \tag{5.7}$$

[18] Any background related to (4.1)–(4.3) or (5.1),(5.2) by a local field redefinition will also represent a solution of the two-loop conformal invariance equations in a particular scheme.

$$\gamma=\tfrac{1}{2}\alpha'\Omega^{\mu\nu}D_\mu D_\nu+\cdots\ ,$$
$$W_\mu=\alpha'\partial_\mu\phi+M_\mu(G,H)\ , \tag{5.8}$$
$$\Omega^{\mu\nu}=G^{\mu\nu}+\alpha' l_1 R^{\mu\nu}+\alpha' l_2 H^{2\mu\nu}+O(\alpha'^2)\ . \tag{5.9}$$

Ellipsis in γ stand for higher derivative terms which are absent in the two-loop approximation we shall consider below. M_μ and $\Omega^{\mu\nu}$ do not depend on ϕ in a "standard" class of schemes. In the dimensional regularization and/or minimal subtraction scheme $l_1=0$ [50,51]; $l_2=\frac{1}{4}$ in the scheme of (4.17) (this value can be found from the linear term in the dilaton β function in [24]). In the "standard" scheme for the group space which is related to the scheme of (4.17) by (4.46′) one thus has [40] $l_2=-\frac{1}{4}$, i.e., for the SL(2,$\mathbb{R}$) group space with $\alpha'=1/k$

$$\alpha'\Omega^{\mu\nu}=\alpha'[G^{\mu\nu}-\tfrac{1}{4}\alpha' H^{2\mu\nu}+O(\alpha'^2)]$$
$$=\frac{1}{k}\left[1+\frac{2}{k}+O\left[\frac{1}{k^2}\right]\right]G^{\mu\nu}=\frac{1}{k-2}G^{\mu\nu}\ . \tag{5.10}$$

This relation is true (in the "standard" scheme) to all orders for a general WZNW model [52]. In the scheme (4.45),(4.46) that corresponds to the exact background (4.1)–(4.3) we find

$$\Omega^{\mu\nu}=G^{\mu\nu}-c_5\alpha' G^{\mu\nu}+O(\alpha'^2)\ . \tag{5.11}$$

With $\alpha'=1/k$ and $c_5=-2$ we reproduce the expression (5.10). If $\alpha'=1/\kappa$ and one rescales $B_{\mu\nu}$ as in (4.52) (i.e., $c_5'=0$) then one finds (as in the case of the central charge) that the exact result $\gamma=(1/2\kappa)D^2$ is obtained already in the one-loop approximation. We conclude that the exact background corresponds to the scheme in which the tachyon equation has the same simple form as in the coset conformal field theory.

At the same time, making the redefinition (5.4) which relates the scheme of (4.17) to the "nonstandard" scheme where the semiclassical background is the solution, we find that in the latter scheme

$$\Omega^{\mu\nu}=G^{\mu\nu}+2\alpha' D^\mu\phi D^\nu\phi-\tfrac{1}{4}\alpha' H^{2\mu\nu}+O(\alpha'^2)\ . \tag{5.12}$$

Being computed on the relevant background (5.1),(5.2) the resulting tachyon equation (5.7) is of course the same as the one in the "standard" scheme or (5.6); however, its form in terms of the semiclassical background fields is nonstandard.

The lesson that can be drawn from the above discussion (see also [47]) is that in string theory it is not sufficient just to find the expression for a few nonvanishing background fields that solve the conformal invariance equations in a particular scheme. One is also to specify how the equations for the propagation of a first-quantized string (i.e., the equations for the marginal perturbations of the background) look like in that scheme. The answer to the latter question is simplified if it is known which conformal field theory corresponds to a given solution of the string effective equations, i.e., to a given conformal σ model. In that case a preferred or a "standard" scheme is the one in which the equations for perturbations take a simple canonical form when expressed in terms of background fields.

As demonstrated in [20,21] and in this paper, the background fields which correspond to the coset conformal models in the $\alpha'\to 0$ limit and which solve the all-order conformal invariance equations in the "standard" scheme are nontrivial functions of α'. It still remains to be explained on general grounds why there should exist a scheme in which the semiclassical background is also a solution, i.e., why the α' dependence of the exact background can be generated from the semiclassical background by a local covariant field redefinition. An obvious indication that this may be the case is the quadratic dependence of the classical gauged WZNW action (2.1) on the gauge field suggesting that there may exist a scheme in which the naive Gaussian integral over $A,\bar{A}$ gives the exact answer.[19] Such an argument does not, however, explain why the tachyon and similar higher-mode equations should have a noncanonical form in this "semiclassical" scheme.

ACKNOWLEDGMENTS

We are grateful to Yu.N. Obukhov for his crucial help with the GRG computer algebra system. We would like to thank I. Bars for comments. K.S. is indebted to the University of Southern California for its hospitality during part of this work. A.A.T. acknowledges the support of SERC.

APPENDIX A: THEOREM ON THE MEASURE FACTOR (3.17)

Below we shall prove the validity of the relation (3.17) for arbitrary values of the parameters a, b, and $\bar{b}$ in (3.5). We shall essentially follow the procedure used in [17] in the particular case of the chiral gauged WZNW model. Let us rewrite the exact metric (2.24) in the following way

$$G_{MN}=G_{0MK}\tilde{G}^K{}_N\ , \tag{A1}$$

where G_{0MK} was defined in (2.26) and

$$\tilde{G}^K{}_N=\delta^K{}_N+(\tilde{V}^{-1}N^T)_{ab}(E^{aK}\tilde{E}^b_N+E^a_N\tilde{E}^{bK})$$
$$-b(\tilde{V}^{-1})_{ab}E^{aK}E^b_N-\bar{b}(V^{-1})_{ab}\tilde{E}^{aK}\tilde{E}^b_N$$
$$=\delta^K{}_N+\tilde{C}_{a'b'}S^{a'K}S^{b'}_N\ , \tag{A2}$$

where $S^{a'}_M=(E^a_M,\tilde{E}^a_M)$ and the $2\,\mathrm{dim}H\times 2\,\mathrm{dim}H$ dimensional symmetric matrix $(\tilde{C}_{a'b'})$ is defined as

$$(\tilde{C}_{a'b'})=\begin{bmatrix}-b\tilde{V}^{-1} & \tilde{V}^{-1}N^T\\ V^{-1}N & -\bar{b}V^{-1}\end{bmatrix}\ . \tag{A3}$$

In order to compute $\det G_{MN}$ we need $\det\tilde{G}^K{}_N$. We have

[19] The quantum gauged WZNW theory (2.4) on the full configuration space $g,h,\bar{h}$ is obviously conformally invariant being a combination of the ungauged WZNW models. The corresponding "standard" scheme is the same as in the WZNW model case. When one integrates out the gauge fields (or $h,\bar{h}$) remaining in this "standard" scheme then to maintain the conformal invariance the σ-model couplings become α' dependent (and the dilaton coupling is induced). Preserving the "semiclassical" form of the σ-model couplings corresponds to switching to a nontrivial "semiclassical" scheme.

$$\det\tilde{G}^K{}_N=\det(\delta^K{}_N+\tilde{C}_{a'b'}S^{a'K}S^{b'}_N)=\det(\eta_{a'b'}+\tilde{C}_{a'c'}S^{c'M}S_{b'M})$$

$$=\det\begin{bmatrix} I-\tilde{V}^{-1}(N^TC+bI) & \tilde{V}^{-1}(N^T+bC^T) \\ V^{-1}(N+\bar{b}C) & I-V^{-1}(NC^T+\bar{b}I)\end{bmatrix}$$

$$=\det\begin{bmatrix}\tilde{V}^{-1} & 0 \\ 0 & V^{-1}\end{bmatrix}\det\begin{bmatrix}(a-1)N^T-b(\bar{b}+1)I & N^T+bC^T \\ N+\bar{b}C & (a-1)N-\bar{b}(b+1)\end{bmatrix}$$

$$=\Delta^{\dim H}(\det V^{-1})^2\det\begin{bmatrix} bI & N^T \\ -N & -\bar{b}I\end{bmatrix}=\Delta^{\dim H}\det V^{-1}\ , \tag{A4}$$

where $\Delta=(a-1)^2-(b+1)(\bar{b}+1)$. The Haar measure for the group G is given by $\sqrt{\det G_{0MN}}$ where G_{0MN} is the Killing metric in (2.26). Therefore, by using (A1), (A4), and (2.27) one establishes the validity of the theorem (3.17) for arbitrary values of the parameters a, b, and $\bar{b}$ [including the special ones in (3.6)].

The case of the gauged WZNW model in (3.6) is special: the above derivation formally breaks down since the metric G_{MN} (2.24) is degenerate [has $\dim H$ null Killing vectors, see (2.31)] as a consequence of gauge invariance. This is also clear from (A4): in this case $\Delta=0$ and its power $\dim H$ in (A4) is the dimension of the null vector space of G_{MN}. Defining $\det G_{MN}$ by projecting out the zero modes by a gauge condition one obtains (3.17) with the measure on G replaced by the group-invariant measure $\sqrt{\det(E^i_\mu E^j_\nu\eta_{ij})}$ on G/H (see, for instance [5]).

APPENDIX B: EXACT BACKGROUND CORRESPONDING TO THE [SL(2,R)×R]/R GAUGED WZNW MODEL

Below we shall derive the exact σ-model couplings for the "charged black string" model [6,3]. The present case with nonsimple group $G=SL(2,\mathbb{R})\times\mathbb{R}$ needs a special treatment in what concerns the antisymmetric tensor coupling: the general expression (2.25) derived for a simple G does not apply. To get a background that solves the (two-loop) conformal invariance conditions it turns out to be necessary to take into account an ambiguity present in the extraction of the local antisymmetric tensor part of the σ-model action from the effective action of the gauged WZNW model.

We shall use the following parametrization for the $SL(2,\mathbb{R})$ group element:

$$g=\exp\left[\frac{i}{2}\theta_L\sigma_2\right]\exp\left[\frac{1}{2}r\sigma_1\right]\exp\left[\frac{i}{2}\theta_R\sigma_2\right]\ ,$$
$$\theta_L=\theta+\tilde{\theta},\quad \theta_R=\tilde{\theta}-\theta\ . \tag{B1}$$

Taking $A,\bar{A}$ to be in the axial subgroup generated by $\frac{1}{2}\sigma_2$, the classical gauged WZNW action (2.1) can be represented in the form[20]

$$S(g,A)=kI(g,A)$$
$$=\frac{k}{2\pi}\int d^2z\,[L_0(r,\theta_L,\theta_R)-A\bar{J}-\bar{A}J$$
$$-(C+1+2q_0)A\bar{A}]\ , \tag{B2}$$

$$L_0=\tfrac{1}{2}(\partial r\bar{\partial}r-\partial\theta_L\bar{\partial}\theta_L-\partial\theta_R\bar{\partial}\theta_R-2C\bar{\partial}\theta_L\partial\theta_R)\ ,$$

$$C=C(r)\equiv\cosh r,\quad J=\partial\theta_L+C\partial\theta_R\ ,$$

$$\bar{J}=\bar{\partial}\theta_R+C\bar{\partial}\theta_L\ .$$

We have added the term

$$-k\pi\int d^2z\,(\partial y+\rho A)(\bar{\partial}y+\rho\bar{A}),\quad q_0=\rho^2$$

with an extra scalar degree of freedom y coupled to $A,\bar{A}$ and then gauged it away (ρ is the free parameter of embedding of the subgroup). The action in (B2) is thus written already in a particular gauge ($y=0$). The effective action is found in the similar way as in (2.5)–(2.9) and up to a total derivative is given by

[20]The normalization for the trace of the square of the generators we use below differs by a factor of 2 from that used in Secs. II and III. As a result, the relation between α' and $\kappa=k-2$ we get here and use in Sec. IV is $\alpha'=1/\kappa$ and not $\alpha'=2/\kappa$ as in (2.23) or (3.13).

$$\Gamma(g,A)=\frac{\kappa}{2\pi}\int d^2z\left[L_0-A\bar{J}-\bar{A}J-(C+1+2q+b)A\bar{A}+\frac{1}{2}bA\frac{\bar\partial}{\partial}A+\frac{1}{2}b\bar{A}\frac{\partial}{\bar\partial}\bar{A}\right],$$
$$1+2q+b=(1+b)(1+2q_0),\quad q\equiv q_0(1+b),\quad 1+b=\frac{k}{\kappa}=\frac{k}{k-2}\ . \tag{B3}$$

The coefficient k of the extra scalar term is not renormalized since the subgroup is Abelian so that in the combined expression q_0 is replaced by $q=q_0k/\kappa=q_0(1+b)$. Using the notation (2.10) we can rewrite (B3) in the form

$$\Gamma(g,A)=\frac{\kappa}{2\pi}\int d^2z[L_0-A\bar{J}-\bar{A}J-(C+1+2q)A\bar{A}-\tfrac{1}{2}b(A-\tilde{\bar{A}})(\bar{A}-\tilde{A})-\tfrac{1}{2}\mu b(A\bar{A}-\tilde{A}\tilde{\bar{A}})]\ , \tag{B4}$$

where we have put the quantum $O(b)$ term in the manifestly gauge-invariant form [cf. (2.9)] and also included a total derivative term with a constant coefficient μ. The gauge-invariant form of this latter term (before fixing $y=0$ as a gauge) is

$$-\frac{\mu b}{2q_0}[(\partial y+\rho A)(\bar\partial y+\rho\bar{A})-(\partial y+\rho\tilde{\bar{A}})(\bar\partial y+\rho\tilde{A})]\ .$$

Such term does not change anything at the level of the full effective action but will influence the expression for the antisymmetric tensor term in the local part of (B6) derived under a specific prescription (see Sec. II) of how to drop out the nonlocal terms in the process of elimination of $A,\bar{A}$.

One can think that the role of this total derivative term is to account for the fact that part of the $qA\bar{A}$ term is to be taken in the "symmetrized" $\frac{1}{2}(\bar{A}\bar{A}+\tilde{A}\tilde{\bar{A}})$ form since it originates from an "extra" $\mathbb{R}$ part of the full gauged WZNW action. This suggests that the coefficient μ should be proportional to q_0. In fact, a natural choice seems to be

$$\mu=-2q_0=-2q(1+b)^{-1}\ , \tag{B5}$$

since in this case the local part of (B4) takes the same form as the *classical* action (B2) (with k replaced by κ)

$$\Gamma(g,A)=\frac{\kappa}{2\pi}\int d^2z[L_0-A\bar{J}-\bar{A}J-(C+1+2q_0)A\bar{A}-\tfrac{1}{2}(1+2q_0)b(A-\tilde{\bar{A}})(\bar{A}-\tilde{A})-q_0b(A\tilde{A}+\bar{A}\tilde{\bar{A}})]\ . \tag{B6}$$

This was the property of the effective action in the case of the simple group G [see (2.14)] and one may try to preserve it in the semisimple case as well. The choice (B5) is distinguished also by the fact that for $q_0=-\frac{1}{2}$ the effective action (B6) of the $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW model automatically reduces to the effective action (3.3),(3.5) of the $SL(2,\mathbb{R})/\mathbb{R}$ chiral gauged WZNW model in agreement with the general statement [18] about the equivalence of the two models. This equivalence is proved at the level of the full nonlocal effective actions and thus it implies the equivalence of the resulting local σ-model actions provided that consistent prescriptions for their derivation are used. Correspondence with the chiral gauged WZNW model is an important consistency check: first, this model must definitely be conformally invariant since its action can be expressed as a combination of the WZNW actions (3.2),(3.3), and, second, its effective action does not contain the quantum $A\bar{A}$ term and thus the determination of $B_{\mu\nu}$ is unambiguous. As shown in Sec. IV, the σ-model background fields that correspond to (B4) with μ given by (B5) solve the two-loop conformal invariance equations.

Starting with (B6) and using the "corrected" prescription of Sec. II, i.e., solving the equations for $A,\bar{A}$ as in (2.15)–(2.17) (note that the total derivative μ term does not influence the equations of motion) and substituting the solution back into the action (B4) we get the following σ-model action in terms of the coordinates $x^\mu=(r,\theta,\tilde\theta)$ [for generality we keep the value of μ in (B4) arbitrary]:

$$S(r,\theta,\tilde\theta)=\frac{\kappa}{\pi}\int d^2z[G_{\mu\nu}\partial x^\mu\bar\partial x^\nu+B_{\theta\tilde\theta}(\partial\theta\bar\partial\tilde\theta-\partial\tilde\theta\bar\partial\theta)]\ , \tag{B7}$$

$$ds^2=G_{\mu\nu}dx^\mu dx^\nu=\frac{1}{4}dr^2+(1+q+b)\frac{C-1}{C+1+2q+2b}d\theta^2-q\frac{C+1}{C+1+2q}d\tilde\theta^2\ , \tag{B8}$$

$$B_{\theta\tilde\theta}=-(2-\mu)\frac{q(q+1)}{C+1+2q}-\mu\frac{(q+b)(q+1+b)}{C+1+2q+2b}\ , \tag{B9}$$

$$\phi=\phi_0-\tfrac{1}{4}\ln[(C+1+2q)(C+1+2q+2b)]\ . \tag{B10}$$

As expected, the parameter μ appears only in the expression for the antisymmetric tensor (dependence on μ disappears of course in the classical limit $b=0$). Note that in the limit $q=0$ the model (B7)–(B10) reduces to the direct product of the $SL(2,\mathbb{R})/\mathbb{R}$ (in a patch) and $\mathbb{R}$ as it should [$\tilde{\theta}$ decouples in (B8) but this is a gauge artifact that can be avoided by first rescaling $\tilde{\theta}$ and then taking the limit] and in the limit $q=1$ to the direct product of $SL(2,\mathbb{R})/\mathbb{R}$ (in the dual patch) and $\mathbb{R}$ (in this case one should first rescale θ and then take the limit).

Equation (B9) considered formally for arbitrary values of μ has several special cases. When $\mu=0$ in (B4), i.e., when the effective action is taken in the same form as in (2.14) we get the analogue of (2.25)[21]

$$B_{\theta\tilde{\theta}}=-\frac{2q(q+1)}{C+1+2q} , \tag{B11}$$

where we have dropped the constant term q. Another special case is $\mu=1$. In this case (B6) is exactly equal to the "naive" form of the effective action (B3) [as in (3.8)–(3.12)]. The resulting expression for $B_{\mu\nu}$

$$B_{\theta\tilde{\theta}}=-\frac{q(q+1)}{C+1+2q}-\frac{(q+b)(q+1+b)}{C+1+2q+2b} , \tag{B12}$$

is the same one that one finds (after gauge fixing) from the analogue of (3.15) in the axial subgroup case. For $q_0=-\frac{1}{2}, q=-\frac{1}{2}(1+b)$ (B12) gives $B_{\mu\nu}$ in the $SL(2,\mathbb{R})/\mathbb{R}$ chiral gauged WZNW model.

The expression for the exact antisymmetric tensor coupling of the $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW model that corresponds to μ in (B5) is

$$B_{\theta\tilde{\theta}}=-\frac{2q(q+1+b)}{(1+b)}\left[\frac{q+1}{C+1+2q}-\frac{q+b}{C+1+2q+2b}\right] . \tag{B13}$$

When $q_0=-\frac{1}{2}$ (i.e., $\mu=1$) not only the exact metric (B8) and the dilaton (B10) but also the exact antisymmetric tensor (B13) of the $[SL(2,\mathbb{R})\times\mathbb{R}]/\mathbb{R}$ gauged WZNW model coincide with the background fields of the $SL(2,\mathbb{R})/\mathbb{R}$ chiral gauged WZNW model.[22] The expression (B13) vanishes for $q=0$ and in the $SL(2,\mathbb{R})$ group space limit $q_0\to\infty$ reduces to the correct result $B_{\theta\tilde{\theta}}=\frac{1}{2}C$. The background (B8),(B10),(B13) thus consistently includes (as special cases for $q_0=0,-\frac{1}{2},\infty$) other known $D=3$ exact conformally invariant backgrounds.

If one uses the "semiclassical" prescription for computing the antisymmetric tensor one finds that parity-even quantum term in (B6) does not produce a quantum correction to the antisymmetric tensor, i.e., the latter is given by

$$B_{\theta\tilde{\theta}}=-\frac{2q_0(q_0+1)}{C+1+2q_0}=-\frac{2q(q+1+b)}{(1+b)[(1+b)(C+1)+2q]} . \tag{B9$'$}$$

Introducing the new coordinates z,t,x

$$2z=C+1+2q ,$$
$$it=(1+b+q)^{1/2}\theta , \tag{B14}$$
$$ix=-q^{1/2}\tilde{\theta} ,$$

we find that the metric (B8) and the dilaton (B10) take the forms (4.1) and (4.3) while the antisymmetric tensor B_{tx} corresponding to (B9) becomes

$$B_{tx}=\frac{p_1}{z}+\frac{p_2}{z+b} , \tag{B15}$$

$$p_1=-\tfrac{1}{2}(2-\mu)(q+1)q^{1/2}(q+1+b)^{-1/2} ,$$
$$p_2=-\tfrac{1}{2}\mu(q+b)q^{-1/2}(q+1+b)^{1/2} , \tag{B16}$$

or for the value of μ in (B5)

$$B_{tx}=-\frac{[q(q+1+b)]^{1/2}}{1+b}\left[\frac{q+1}{z}-\frac{q+b}{z+b}\right] . \tag{B17}$$

The semiclassical expression (B9′) for the antisymmetric tensor in the coordinates (B14) is given by

$$B_{tx}=-\frac{[q(q+1+b)]^{1/2}}{(1+b)[(1+b)z-bq]} . \tag{B17$'$}$$

APPENDIX C: GEOMETRICAL OBJECTS FOR THE EXACT "BLACK STRING" $D=3$ BACKGROUND

Below we present the results of computation of some geometrical quantities for the background (4.1)–(4.3) and their expansion in b which were found using GRG computer algebra system [53].[23] All indices below are with respect to the local vierbein corresponding to the metric (4.1) [with the flat metric being $(-1,+1,+1)$].

The exact expressions for the nonvanishing components of the Ricci tensor and curvature scalar are

[21]This expression found in [16] using a manifestly gauge-invariant prescription does not represent a conformally invariant background (cf. Sec. IV). Note that since in the present case the group G is not simple, Eq. (B11) *does contain* quantum corrections even though the subgroup here is one dimensional [(B11) is the same $B_{\mu\nu}$ that follows simply from the classical action (B2) *but* with q_0 replaced by q].

[22]That this is not the case for (B11), i.e., for $\mu=0$ was a puzzle in [17].

[23]We are grateful to Yu.N. Obukhov for helping us with this calculation.

$$R_{00}=(1+q+b)(qb+4qz+bz-2z^2)[z(b+z)^2]^{-1}\ , \quad (C1)$$

$$R_{11}=-q(3b+3qb+4z+4qz-bz-2z^2)[z^2(b+z)]^{-1}\ , \quad (C2)$$

$$R_{22}=(-3qb^2-3q^2b^2-6qbz-6q^2bz+2qb^2z-6qz^2-6q^2z^2-bz^2-b^2z^2+2z^3+4qz^3+2bz^3)[z^2(b+z)^2]^{-1}\ , \quad (C3)$$

$$R=2(-3qb^2-3q^2b^2-7qbz-7q^2bz+qb^2z-7qz^2-7q^2z^2-bz^2-qbz^2-b^2z^2+2z^3+4qz^3+2bz^3)[z^2(b+z)^2]^{-1}\ . \quad (C4)$$

The vierbein components of the second covariant derivative of the dilaton are

$$D_0D_0\phi=-\tfrac{1}{2}(1+q+b)(q-z)(b+2z)[z(b+z)^2]^{-1}\ , \quad (C5)$$

$$D_1D_1\phi=\tfrac{1}{2}q(1+q-z)(b+2z)[z^2(b+z)]^{-1}\ , \quad (C6)$$

$$D_2D_2\phi=\tfrac{1}{2}\{(q-z)z(b+z)(b+2z)+(1+q-z)[z(b+z)(b+2z)-4(q-z)z(b+z)+2(q-z)(b+2z)^2]\}[z^2(b+z)^2]^{-1}\ , \quad (C7)$$

so that

$$D^2\phi=-\tfrac{1}{2}(-3qb^2-3q^2b^2-8qbz-8q^2bz+b^2z+2qb^2z-8qz^2-8q^2z^2+2bz^2+4qbz^2+b^2z^2+4z^3+8qz^3+4bz^3)[z^2(b+z)^2]^{-1}\ .$$

Also,

$$(D_\mu\phi)^2=[(-1-q+z)(-q+z)(b+2z)^2][4z^2(b+z)^2]^{-1}\ .$$

Computing the field strength for $B_{\mu\nu}$ in (B15) and (B17) we get for the scalar H (we use the metric with Euclidean signature)

$$H=G^{-1/2}H_{txz}=i\frac{\sqrt{z(z+b)}}{\sqrt{q(1+q+b)}}\left[(2-\mu)\frac{q(q+1)}{z^2}+\mu\frac{(1+q+b)(b+q)}{(z+b)^2}\right]\ . \quad (C8)$$

Its expansion in powers of b can be represented as (since q_0 is a free parameter we can treat q as being b independent)

$$H^2=-4q(q+1)z^{-2}[1-bs_1(1+q)^{-1}+bs_2z^{-1}]+O(b^2)\ , \quad (C9)$$

$$s_1=1-\mu(2q+1)q^{-1}=4q+3+O(b),\quad s_2=1-2\mu=4q+1+O(b)\ . \quad (C10)$$

The form of the expansions to the first order in b is [for $\mu=-2q+O(b)$]

$$R=4(2q+1)z^{-1}-14q(q+1)z^{-2}+b[4z^{-1}+(-10-18q)z^{-2}+14q(q+1)z^{-3}]+O(b^2)\ , \quad (C11)$$

$$D^2\phi=-2(2q+1)z^{-1}+4q(q+1)z^{-2}+b[-2z^{-1}+3(2q+1)z^{-2}-4q(q+1)z^{-3}]+O(b^2)\ , \quad (C12)$$

$$H^2=-4q(q+1)z^{-2}+b[4q(4q+3)z^{-2}-4q(q+1)(1+4q)z^{-3}]+O(b^2)\ , \quad (C13)$$

$$(D\phi)^2=1-(2q+1)z^{-1}+q(q+1)z^{-2}+b[-z^{-1}+(2q+1)z^{-2}-q(q+1)z^{-3}]+O(b^2)\ . \quad (C14)$$

To the leading order $He^{-2\phi}=$const and

$$R^2_{\mu\nu\lambda\kappa}=4R^2_{\mu\nu}-R^2=16z^{-2}+32q(q+1)z^{-2}-48q(q+1)(2q+1)z^{-3}+76q^2(q+1)^2z^{-4}+O(b). \quad (C15)$$

It is possible to show that for any number p [note that $\phi=\phi_0-\tfrac{1}{2}\ln z+O(b)$]

$$D^2z^p=z^p[4p^2-4p(p-1)(2q+1)z^{-1}+4p(p-2)q(q+1)z^{-2}]+O(b)\ , \quad (C16)$$

where $D^2=(1/\sqrt{G})\partial_\mu(G^{\mu\nu}\sqrt{G}\,\partial_\nu)$.

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