Poincaré gauge theory of (2+1)-dimensional gravity

Toshiharu Kawai*

Department of Physics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558, Japan

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A Poincaré gauge theory of (2+1)-dimensional gravity is developed. Fundamental gravitational field variables are dreibein fields and Lorentz gauge potentials, and the theory is underlain with the Riemann-Cartan space-time. The most general gravitational Lagrangian density, which is at most quadratic in curvature and torsion tensors and invariant under local Lorentz transformations and under general coordinate transformations, is given. Gravitational field equations are studied in detail, and solutions of the equations for weak gravitational fields are examined for the case with a static, "spin"less point like source. We find, among other things, the following. (1) Solutions of the vacuum Einstein equation satisfy gravitational field equations in the vacuum in this theory. (2) For a class of the parameters in the gravitational Lagrangian density, the torsion is "frozen" at the place where "spin" density of the source field is not vanishing. In this case, the field equation actually agrees with the Einstein equation, when the source field is "spin"less. (3) A teleparallel theory developed in a previous paper is "included as a solution" in a limiting case. (4) A Newtonian limit is obtainable if the parameters in the Lagrangian density satisfy certain conditions.

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I. INTRODUCTION

Recently, lower dimensional gravity has been attracting considerable attention. The (2+1)-dimensional Einstein theory has no Newtonian limit and no dynamical degrees of freedom, but it has nontrivial global structures. This theory has been studied mainly because of the local triviality and of the global nontriviality [1–3]. For the (3+1)-dimensional gravity, there have been proposed various theories alternative to the Einstein theory, among which we have a teleparallel theory [4] and Poincaré gauge theories [5–9]. A Poincaré gauge theory has been examined [10] also for the (1+1)-dimensional case.

It would be significant to develop various theories of gravity also for (2+1)-dimensional case, which will bring us toy models useful to examine basic concepts in theories of gravity. In a previous paper [11], the present author has proposed a teleparallel theory of (2+1)-dimensional gravity having a Newtonian limit and black hole solutions.

The purpose of this paper is to develop a Poincaré gauge theory of (2+1)-dimensional gravity, in a limiting case of which a teleparallel theory given in Ref. [11] is "included as a solution."

II. DREIBEINS, THREE-DIMENSIONAL LORENTZ GAUGE POTENTIAL, AND RIEMANN-CARTAN SPACE-TIME

The three-dimensional space-time M is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}dx^\mu \otimes dx^\nu$ $(\mu,\nu = 0,1,2)$ related to the fields $\mathbf{e}^k = e^k{}_\mu dx^\mu$ $(k = 0,1,2)$ through the relation $g_{\mu\nu} = e^k{}_\mu \eta_{kl} e^l{}_\nu$ with $(\eta_{kl}) \stackrel{\text{def}}{=} \text{diag}(-1,1,1)$. Here, $\{x^\mu;\ \mu = 0,1,2\}$ is a local coordinate of the space-time. The fields $\mathbf{e}_k = e^\mu{}_k \partial/\partial x^\mu$, which are dual to $\mathbf{e}^k$, are the dreibein fields. The Lorentz gauge potentials $A^{kl}{}_\mu$ $(= -A^{lk}{}_\mu)$ transform accordingly as

$$A'^{kl}{}_\mu(x) = A^{kl}{}_\mu(x) + \omega^k{}_m(x)A^{ml}{}_\mu(x) + \omega^l{}_m(x)A^{km}{}_\mu(x) - \partial_\mu\omega^{kl}(x)\ , \tag{2.1}$$

under the infinitesimal Lorentz gauge transformation of $e^k{}_\mu(x)$:

$$e'^k{}_\mu(x) = e^k{}_\mu(x) + \omega^k{}_l(x)e^l{}_\mu(x)\ , \tag{2.2}$$

where $\omega^{kl} \stackrel{\text{def}}{=} \eta^{lm}\omega^k{}_m$ is an infinitesimal real valued function of x and is antisymmetric with respect to k and l. Here, (η^{kl}) is the inverse matrix of (η_{kl}), and in what follows raising and lowering the indices k, l, m,... are accomplished with the aid of (η^{kl}) and (η_{kl}). The covariant derivative $D_k\varphi$ of the field φ belonging to a representation σ of the three-dimensional Lorentz group is given by

$$D_k\varphi = e^\mu{}_k\left(\partial_\mu\varphi + \frac{i}{2}A^{lm}{}_\mu M_{lm}\varphi\right)\ , \tag{2.3}$$

where $M_{kl} \stackrel{\text{def}}{=} -i\sigma_*(\overline{M}_{kl})$. Here, $\{\overline{M}_{kl}, k,l = 0,1,2\}$ is a basis of the Lie algebra of the three-dimensional Lorentz group satisfying the relation

$$[\overline{M}_{kl}, \overline{M}_{mn}] = -\eta_{km}\overline{M}_{ln} - \eta_{ln}\overline{M}_{km} + \eta_{kn}\overline{M}_{lm} + \eta_{lm}\overline{M}_{kn}\ , \tag{2.4}$$

$$\overline{M}_{kl} = -\overline{M}_{lk}\ , \tag{2.5}$$

*Electronic address: h1758@ocugw.cc.osaka-cu.ac.jp

and σ_* stands for the differential of σ. The field strengths of $e^k{}_\mu$ and of $A^{kl}{}_\mu$ are given by

$$T^k{}_{lm} \stackrel{\text{def}}{=} e^\mu{}_l e^\nu{}_m(\partial_\mu e^k{}_\nu - \partial_\nu e^k{}_\mu) + e^\mu{}_l A^k{}_{m\mu} - e^\mu{}_m A^k{}_{l\mu} , \quad (2.6)$$

$$R^{kl}{}_{mn} \stackrel{\text{def}}{=} e^\mu{}_m e^\nu{}_n(\partial_\mu A^{kl}{}_\nu - \partial_\nu A^{kl}{}_\mu - A^k{}_{r\mu}A^{lr}{}_\nu + A^k{}_{r\nu}A^{lr}{}_\mu) , \quad (2.7)$$

respectively. We have the relation

$$R_{klmn} = \eta_{km}R_{ln} - \eta_{kn}R_{lm} - \eta_{lm}R_{kn} + \eta_{ln}R_{km} - \tfrac{1}{2}(\eta_{km}\eta_{ln} - \eta_{kn}\eta_{lm})R , \quad (2.8)$$

where we have defined $R_{kl} \stackrel{\text{def}}{=} R_k{}^m{}_{lm}$ and $R \stackrel{\text{def}}{=} R^k{}_k$.

For the world vector field $\mathbf{V} = V^\mu\partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\Gamma^\mu_{\lambda\nu}$ is given by

$$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu}V^\lambda . \quad (2.9)$$

We make the requirement

$$D_l V^k = e^\nu{}_l e^k{}_\mu D_\nu V^\mu , \quad (2.10)$$

for the Lorentz vector field V^k, where $V^\mu \stackrel{\text{def}}{=} e^\mu{}_k V^k$. Then the relation

$$A^k{}_{l\mu} \equiv \Gamma^\nu_{\lambda\mu}e^k{}_\nu e^\lambda{}_l + e^k{}_\nu\partial_\mu e^\nu{}_l , \quad (2.11)$$

follows and we have

$$T^k{}_{\mu\nu} \equiv e^k{}_\lambda T^\lambda{}_{\mu\nu} , \quad (2.12)$$

$$R^k{}_{l\mu\nu} \equiv e^k{}_\lambda e^\rho{}_l R^\lambda{}_{\rho\mu\nu} , \quad (2.13)$$

$$D_\lambda g_{\mu\nu} \stackrel{\text{def}}{=} \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\mu\lambda}g_{\rho\nu} - \Gamma^\rho_{\nu\lambda}g_{\mu\rho} \equiv 0 \quad (2.14)$$

with

$$T^\mu{}_{\nu\lambda} \stackrel{\text{def}}{=} \Gamma^\mu_{\lambda\nu} - \Gamma^\mu_{\nu\lambda} , \quad (2.15)$$

$$R^\mu{}_{\nu\lambda\rho} \stackrel{\text{def}}{=} \partial_\lambda\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\tau\lambda}\Gamma^\tau_{\nu\rho} - \Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\lambda} . \quad (2.16)$$

The components $T^\mu{}_{\nu\lambda}$ and $R^\mu{}_{\nu\lambda\rho}$ are those of the torsion tensor and of the curvature tensor, respectively, and they are both nonvanishing in general. Thus, the space-time M is of the Riemann-Cartan type. From (2.14) we obtain

$$\Gamma^\lambda_{\mu\nu} = \left\{{}^{\lambda}_{\mu\,\nu}\right\} + K^\lambda{}_{\mu\nu} , \quad (2.17)$$

where the first term denotes the Christoffel symbol,

$$\left\{{}^{\lambda}_{\mu\,\nu}\right\} \stackrel{\text{def}}{=} \frac{1}{2}g^{\lambda\xi}(\partial_\mu g_{\xi\nu} + \partial_\nu g_{\xi\mu} - \partial_\xi g_{\mu\nu}) , \quad (2.18)$$

and the second stands for the contortion tensor

$$K^\lambda{}_{\mu\nu} \stackrel{\text{def}}{=} -\tfrac{1}{2}\left(T^\lambda{}_{\mu\nu} - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu\right) . \quad (2.19)$$

The field components $e^k{}_\mu$ and $e^\mu{}_k$ will be used, as for the case of V^μ and V^k in the above, to convert Latin and Greek indices.

III. LAGRANGIAN DENSITIES AND GRAVITATIONAL FIELD EQUATIONS

For the matter field φ, $L_M(\varphi, D_k\varphi)$ is a Lagrangian [12] invariant under three-dimensional local Lorentz transformations and under general coordinate transformations, if $L_M(\varphi, \partial_k\varphi)$ is an invariant Lagrangian on the three-dimensional Minkowski space-time.

For the fields $e^k{}_\mu$ and $A^{kl}{}_\mu$, Lagrangians, which are invariant under local Lorentz transformations including also inversions and under general coordinate transformations and at most quadratic in torsion and curvature tensors, are given by

$$L_T = \alpha t^{klm}t_{klm} + \beta v^k v_k + \gamma a^{klm}a_{klm} + \delta , \quad (3.1)$$

$$L_R = a_1 E^{kl}E_{kl} + a_2 I^{kl}I_{kl} + a_3 R^2 + aR . \quad (3.2)$$

Here, t_{klm}, v_k, and a_{klm} are the irreducible components of T_{klm} defined by

$$t_{klm} \stackrel{\text{def}}{=} \tfrac{1}{2}(T_{klm} + T_{lkm}) + \tfrac{1}{4}(\eta_{mk}v_l + \eta_{ml}v_k) - \tfrac{1}{2}\eta_{kl}v_m , \quad (3.3)$$

$$v_k \stackrel{\text{def}}{=} T^l{}_{lk} , \quad (3.4)$$

and

$$a_{klm} \stackrel{\text{def}}{=} \tfrac{1}{3}(T_{klm} + T_{mkl} + T_{lmk}) , \quad (3.5)$$

respectively, and E_{kl} and I_{kl} are the irreducible components of R_{klmn} defined by

$$E_{kl} \stackrel{\text{def}}{=} \tfrac{1}{2}(R_{kl} - R_{lk}) \quad (3.6)$$

and

$$I_{kl} \stackrel{\text{def}}{=} \tfrac{1}{2}(R_{kl} + R_{lk}) - \tfrac{1}{3}\eta_{kl}R , \quad (3.7)$$

respectively. Also, α, β, γ, δ, a_1, a_2, a_3, and a are real constant parameters. Then,

$$\mathbf{I} \stackrel{\text{def}}{=} \frac{1}{c}\int \mathbf{L}d^3x \quad (3.8)$$

is the total action of the system, where c is the light velocity in the vacuum and $\mathbf{L}$ is defined by

$$\mathbf{L} \stackrel{\text{def}}{=} \sqrt{-g}[L_G + L_M(\varphi, D_k\varphi)] \quad (3.9)$$

with $L_G \stackrel{\text{def}}{=} L_T + L_R$ and $g \stackrel{\text{def}}{=} \det(g_{\mu\nu})$. For the case with $a_1 = a_2 = a_3 = 0$, $a \neq 0$, $\alpha = \beta = \gamma = 0$ and with $\delta = 0$, the Lagrangian L_G reduces to the Einstein-Cartan Lagrangian [13,14].

The field equation $\delta\mathbf{L}/\delta e^i{}_\mu = 0$ reads [15]

$$2aR_{ji} + 4J_{[ik][jl]}R^{kl} + 4J^{[kl]}{}_{[jl]}R_{ki} - 2J_{[i}{}^{k]}{}_{[jk]}R - 2D^kF_{ijk} + 2v^kF_{ijk} + 2H_{ij} - \eta_{ij}L_G = T_{ij} , \quad (3.10)$$

where we have defined

$$J_{ijkl} \stackrel{\text{def}}{=} 2a_3 R\eta_{ik}\eta_{jl} + 2\eta_{ik}(a_1 E_{jl} + a_2 I_{jl}) \,, \tag{3.11}$$

$$D^k F_{ijk} \stackrel{\text{def}}{=} e^{\mu k}(\partial_\mu F_{ijk} + A_i{}^m{}_\mu F_{mjk} + A_j{}^m{}_\mu F_{imk} + A_k{}^m{}_\mu F_{ijm}) \,, \tag{3.12}$$

$$F_{ijk} \stackrel{\text{def}}{=} \alpha(t_{ijk} - t_{ikj}) + \beta(\eta_{ij}v_k - \eta_{ik}v_j) + 2\gamma a_{ijk} = -F_{ikj} \,, \tag{3.13}$$

and

$$H_{ij} \stackrel{\text{def}}{=} T_{kli}F^{kl}{}_j - \tfrac{1}{2}T_{jkl}F_i{}^{kl} = H_{ji} \,. \tag{3.14}$$

Also, T_{ij} denotes the energy-momentum density of the field φ defined by

$$\sqrt{-g}T_{ij} \stackrel{\text{def}}{=} e_{j\mu}\frac{\delta \mathbf{L}_M}{\delta e^i{}_\mu} \tag{3.15}$$

with $\mathbf{L}_M \stackrel{\text{def}}{=} \sqrt{-g}L_M$. The field equation $\delta\mathbf{L}/\delta A^{ij}{}_\mu = 0$ reads

$$2D^l J_{[ij][kl]} + \left(\tfrac{4}{3}t_k{}^{[lm]} - \delta_k{}^{[l}v^{m]} + a_k{}^{lm}\right) J_{[ij][lm]} - H_{ijk} = S_{ijk} \tag{3.16}$$

with

$$D^l J_{[ij][kl]} \stackrel{\text{def}}{=} e^{\mu l}(\partial_\mu J_{[ij][kl]} + A_i{}^m{}_\mu J_{[mj][kl]} + A_j{}^m{}_\mu J_{[im][kl]} + A_k{}^m{}_\mu J_{[ij][ml]} + A_l{}^m{}_\mu J_{[ij][km]}) \,, \tag{3.17}$$

$$H_{ijk} \stackrel{\text{def}}{=} -\left(\alpha + \frac{2a}{3}\right)(t_{kij} - t_{kji}) - \left(\beta - \frac{a}{2}\right) \times(\eta_{ki}v_j - \eta_{kj}v_i) + (4\gamma - a)a_{ijk} = -H_{jik} \,. \tag{3.18}$$

Here, S_{ijk} is the "spin" [16] density of φ defined by

$$\sqrt{-g}S_{ijk} \stackrel{\text{def}}{=} -e_{k\mu}\frac{\delta \mathbf{L}_M}{\delta A^{ij}{}_\mu} \,. \tag{3.19}$$

IV. ALTERNATIVE FORMS OF THE GRAVITATIONAL FIELD EQUATIONS

We shall rewrite the gravitational field equations (3.10) and (3.16), by using the expression

$$A_{ij\mu} = \Delta_{ij\mu} + K_{ij\mu} \tag{4.1}$$

with $\Delta_{ij\mu}$ being the Ricci rotation coefficient

$$\Delta_{ij\mu} \stackrel{\text{def}}{=} \tfrac{1}{2}e^k{}_\mu(C_{ijk} - C_{jik} - C_{kij}) \,, \tag{4.2}$$

where

$$C_{ijk} \stackrel{\text{def}}{=} e^\nu{}_j e^\lambda{}_k(\partial_\nu e_{i\lambda} - \partial_\lambda e_{i\nu}) \,. \tag{4.3}$$

There is the relation

$$R_{ij\mu\nu} = R_{ij\mu\nu}(\{\}) + R_{ij\mu\nu}(K) \tag{4.4}$$

with

$$R_{ij\mu\nu}(\{\}) \stackrel{\text{def}}{=} \partial_\mu\Delta_{ij\nu} - \partial_\nu\Delta_{ij\mu} - \Delta_i{}^k{}_\mu\Delta_{jk\nu} + \Delta_i{}^k{}_\nu\Delta_{jk\mu} = e_{i\lambda}e^\rho{}_j R^\lambda{}_{\rho\mu\nu}(\{\}) \,, \tag{4.5}$$

$$R_{ij\mu\nu}(K) \stackrel{\text{def}}{=} \nabla_\mu K_{ij\nu} - \nabla_\nu K_{ij\mu} - K_i{}^k{}_\mu K_{jk\nu} + K_i{}^k{}_\nu K_{jk\mu} \,, \tag{4.6}$$

as is shown by substituting (4.1) into (2.7). Here, $R^\lambda{}_{\rho\mu\nu}(\{\})$ stands for the Riemann-Christoffel curvature tensor

$$R^\lambda{}_{\rho\mu\nu}(\{\}) \stackrel{\text{def}}{=} \partial_\mu\left\{{}^{\,\lambda}_{\rho\,\nu}\right\} - \partial_\nu\left\{{}^{\,\lambda}_{\rho\,\mu}\right\} + \left\{{}^{\,\lambda}_{\tau\,\mu}\right\}\left\{{}^{\,\tau}_{\rho\,\nu}\right\} - \left\{{}^{\,\lambda}_{\tau\,\nu}\right\}\left\{{}^{\,\tau}_{\rho\,\mu}\right\} \,, \tag{4.7}$$

and $\nabla_\mu K_{ij\nu}$ denotes the covariant derivative with respect to the Ricci rotation coefficients when the index is Latin, and with respect to the Levi-Civita connection when the index is Greek. Each irreducible part of R_{ijkl} is split into two parts, as is known by using (4.4) in (3.6) and (3.7), and J_{ijkl} can be expressed as

$$J_{ijkl} = J_{ijkl}(\{\}) + J_{ijkl}(K) \,, \tag{4.8}$$

where $J_{ijkl}(\{\})$ and $J_{ijkl}(K)$ are formed of the irreducible parts of $R_{ijkl}(\{\})$ and of $R_{ijkl}(K)$, respectively. The tensor $J_{ijkl}(\{\})$, in particular, is given by

$$J_{ijkl}(\{\}) = 2a_2\eta_{ik}R_{jl}(\{\}) + 2\left(a_3 - \frac{a_2}{3}\right)\eta_{ik}\eta_{jl}R(\{\}) \,, \tag{4.9}$$

where $R_{ij}(\{\})$ and $R(\{\})$ are the Ricci tensor and the Riemann-Christoffel scalar curvature, respectively,

$$R_{ij}(\{\}) \stackrel{\text{def}}{=} e^\mu{}_i e^\nu{}_j R^\lambda{}_{\mu\lambda\nu}(\{\}) \,, \quad R(\{\}) \stackrel{\text{def}}{=} \eta^{ij}R_{ij}(\{\}) \,. \tag{4.10}$$

The gravitational Lagrangian L_G can be rewritten as

$$L_G = a_2 R^{kl}(\{\})R_{kl}(\{\}) + \left(a_3 - \frac{a_2}{3}\right)[R(\{\})]^2 + aR(\{\}) + L_T' + L_R' - \frac{2a}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu) \,, \tag{4.11}$$

where

$$L'_T \stackrel{\text{def}}{=} \left(\alpha + \frac{2a}{3}\right) t^{klm} t_{klm} + \left(\beta - \frac{a}{2}\right) v^k v_k + \left(\gamma - \frac{a}{4}\right) a^{klm} a_{klm} + \delta \,, \quad (4.12)$$

$$L'_R \stackrel{\text{def}}{=} L_R - a_2 R^{kl}(\{\}) R_{kl}(\{\}) - \left(a_3 - \frac{a_2}{3}\right) [R(\{\})]^2 - aR \,. \quad (4.13)$$

Here, we have used the relation

$$\sqrt{-g}R = \sqrt{-g}R(\{\}) - \sqrt{-g}\left(-\tfrac{2}{3}t^{klm}t_{klm} + \tfrac{1}{2}v^k v_k + \tfrac{1}{4}a^{klm}a_{klm}\right) - 2\partial_\mu(\sqrt{-g}v^\mu) \,. \quad (4.14)$$

Using the above formulas in (3.10) we get the alternative form of the field equation for $e^i{}_\mu$:

$$2aG_{ij}(\{\}) + \frac{1}{3}(5a_2 + 12a_3)R_{ij}(\{\})R(\{\}) - 2a_2 R_i{}^k(\{\})R_{jk}(\{\}) + \eta_{ij}\left\{a_2 R^{kl}(\{\})R_{kl}(\{\}) - \left(\frac{2a_2}{3} + a_3\right)[R(\{\})]^2\right\} + a_2\eta_{ij}\{2R^{kl}(\{\})R_{kl}(K) - R(\{\})R(K)\}$$
$$+ a_2 R_{ij}(\{\})R(K) + \frac{2}{3}(a_2 + 6a_3)R(\{\})R_{ji}(K) - 2a_2 R_i{}^k(\{\})R_{jk}(K) + 4J_{[ik][jl]}(K)R^{kl}$$
$$+ 4J^{[kl]}{}_{[jl]}(K)R_{ki} - 2J_{[i}{}^{k]}{}_{[jk]}(K)R - 2D^k F'_{ijk} + 2v^k F'_{ijk} + 2H'_{ij} - \eta_{ij}(L'_T + L'_R) = T_{ij} \,, \quad (4.15)$$

where $G_{ij}(\{\})$ is the three-dimensional Einstein tensor

$$G_{ij}(\{\}) \stackrel{\text{def}}{=} R_{ij}(\{\}) - \tfrac{1}{2}\eta_{ij}R(\{\}) \,. \quad (4.16)$$

Here, we have defined

$$R_{kl}(K) \stackrel{\text{def}}{=} R^m{}_{kml}(K), \quad R(K) \stackrel{\text{def}}{=} R^k{}_k(K), \quad (4.17)$$

$$F'_{ijk} \stackrel{\text{def}}{=} \left(\alpha + \frac{2a}{3}\right)(t_{ijk} - t_{ikj}) + \left(\beta - \frac{a}{2}\right)(\eta_{ij}v_k - \eta_{ik}v_j) + 2\left(\gamma - \frac{a}{4}\right)a_{ijk} = -F'_{ikj} \,, \quad (4.18)$$

$$H'_{ij} \stackrel{\text{def}}{=} T_{kli}F'^{kl}{}_j - \tfrac{1}{2}T_{jkl}F'_i{}^{kl} = H'_{ji} \,. \quad (4.19)$$

We have the relations

$$F'_{ijk} = \tfrac{1}{2}(H_{ijk} - H_{ikj} - H_{jki}) \quad (4.20)$$

and

$$H_{ijk} = F'_{ijk} - F'_{jik} \,, \quad (4.21)$$

as shown by comparing (3.18) and (4.18). The field equation for $A^{ij}{}_\mu$ is rewritten as

$$-2a_2\nabla_{[i}G_{j]k}(\{\}) - 8\left(a_3 + \frac{a_2}{6}\right)\eta_{k[i}\partial_{j]}G(\{\}) + 2(D^l - \nabla^l)J_{[ij][kl]}(\{\}) + 2D^l J_{[ij][kl]}(K) + \left(\tfrac{4}{3}t_k{}^{[lm]} - \delta_k{}^{[l}v^{m]} + a_k{}^{lm}\right)J_{[ij][lm]} - H_{ijk} = S_{ijk} \,, \quad (4.22)$$

where $G(\{\}) \stackrel{\text{def}}{=} \eta^{ij}G_{ij}(\{\})$. By examining the alternative forms of the gravitational field equations (4.15) and (4.22) we find the following.

(1) For the case with $S_{ijk} \equiv 0$ and with $T_{ij} \equiv 0$, any solution of the equations

$$G_{ij}(\{\}) - \eta_{ij}\Lambda = 0 \,, \quad (4.23)$$

$$T_{ijk} = 0 \,, \quad (4.24)$$

satisfies (4.15) and (4.22) with $\delta = 2\Lambda(a + 6a_3\Lambda)$. We can say shortly, "solutions of the vacuum Einstein equation are solutions of the vacuum gravitational field equations in this theory."

(2) Equation (4.22) does not contain third derivatives of the metric tensor, if and only if

$$a_2 = a_3 = 0 \,. \quad (4.25)$$

(3) When condition (4.25) is satisfied, then (4.15) and (4.22) are considerably simplified. In (4.15), the terms quadratic in the Riemann-Christoffel curvature tensor are all vanishing. In (4.22), the first three terms disappear, and all the remaining terms are linear or quadratic in the torsion tensor. Thus, if the intrinsic "spin" of the source is vanishing, $S_{ijk} \equiv 0$, then (4.22) is satisfied by the vanishing torsion, and when the torsion vanishes, (4.15) reduces to the equation

$$2aG_{ij}(\{\}) - \eta_{ij}\delta = T_{ij} \,. \quad (4.26)$$

For the case with $a = 1/(2\kappa)$ with κ being the "Einstein gravitational constant," (4.26) agrees with the Einstein equation, because $T_{ij}=T_{ji}$ for a vanishing S_{ijk}, as is seen from (5.10) of the next section. The following, however, should be noted: *The torsion tensor does not necessarily vanish, even when condition* (4.25) *is satisfied and the intrinsic "spin" of the source field is vanishing.*

(4) When condition $a_1 = 0$ is satisfied in addition to condition (4.25), then (4.15) and (4.22) reduce to

$$2aG_{ij}(\{\}) - 2D^k F'_{ijk} + 2v^k F'_{ijk} + 2H'_{ij} - \eta_{ij} L'_T = T_{ij} \tag{4.27}$$

and

$$-H_{ijk} = S_{ijk} , \tag{4.28}$$

respectively. The torsion tensor is linearly dependent on S_{ijk}, if

$$(3\alpha + 2a)(2\beta - a)(4\gamma - a) \neq 0 , \tag{4.29}$$

as seen from (3.18) and (4.28). Thus, the torsion is "frozen" at the place where the "spin" density S_{ijk} does not vanish.

If $S_{ijk} \equiv 0$, (4.15) for the present case reduces to (4.26). Also, the field equations for the "spin"less source fields agree with those in the Einstein theory.

V. EQUATION OF MOTION FOR MACROSCOPIC BODIES

We shall derive the equation of motion for macroscopic bodies, which can be done in a way similar to the case of the (3+1)-dimensional theory [7].

From the fact that the gravitational action integral

$$\mathbf{I}_G \stackrel{\text{def}}{=} \frac{1}{c} \int \mathbf{L}_G d^3x \tag{5.1}$$

with

$$\mathbf{L}_G \stackrel{\text{def}}{=} \sqrt{-g} L_G , \tag{5.2}$$

is invariant under general coordinate transformations, the identity

$$\sqrt{-g}(Y^{i\nu}\partial_\mu e_{i\nu} + Z_{ij}{}^{\nu}\partial_\mu A^{ij}{}_\nu) \equiv \partial_\nu\{\sqrt{-g}(Y_\mu{}^\nu + Z_{ij}{}^\nu A^{ij}{}_\mu)\} \tag{5.3}$$

follows, where

$$\sqrt{-g} Y_i{}^\mu \stackrel{\text{def}}{=} -\frac{\delta \mathbf{L}_G}{\delta e^i{}_\mu} \tag{5.4}$$

and

$$\sqrt{-g} Z_{ij}{}^\mu \stackrel{\text{def}}{=} -\frac{\delta \mathbf{L}_G}{\delta A^{ij}{}_\mu} . \tag{5.5}$$

The gravitational field equations $\delta \mathbf{L}/\delta e^i{}_\mu = 0$ and $\delta \mathbf{L}/\delta A^{ij}{}_\mu = 0$ can be expressed as

$$Y_\mu{}^\nu = T_\mu{}^\nu \tag{5.6}$$

and

$$Z_{ij}{}^\nu = -S_{ij}{}^\nu , \tag{5.7}$$

respectively. By using (5.6), (5.7), and the formula

$$\partial_\nu(\sqrt{-g} Y_\mu{}^\nu) - \sqrt{-g} Y^{i\nu}\partial_\mu e_{i\nu} \equiv \sqrt{-g}(\nabla_\nu Y_\mu{}^\nu + \Delta_{\lambda\nu\mu} Y^{\lambda\nu}) , \tag{5.8}$$

in (5.3), we obtain the response equation to gravitation

$$\nabla_\nu T_\mu{}^\nu + \Delta_{\lambda\nu\mu} T^{\lambda\nu} - 2\partial_{[\nu} A^{ij}{}_{\mu]} S_{ij}{}^\nu - \frac{1}{\sqrt{-g}} A^{ij}{}_\mu \partial_\nu(\sqrt{-g} S_{ij}{}^\nu) \equiv 0 . \tag{5.9}$$

Also, the identity

$$\sqrt{-g} T_{[ij]} \equiv \nabla_\mu(\sqrt{-g} S_{ij}{}^\mu) - \frac{i}{2}\frac{\delta \mathbf{L}_M}{\delta \varphi} M_{ij}\varphi \tag{5.10}$$

follows from the invariance of the action

$$\mathbf{I}_M \stackrel{\text{def}}{=} \frac{1}{c} \int \mathbf{L}_M d^3x , \tag{5.11}$$

under local Lorentz transformations. Here, we have defined

$$\nabla_\mu(\sqrt{-g} S_{ij}{}^\mu) \stackrel{\text{def}}{=} \partial_\mu(\sqrt{-g} S_{ij}{}^\mu) + \sqrt{-g} A_i{}^m{}_\mu S_{mj}{}^\mu + \sqrt{-g} A_j{}^m{}_\mu S_{im}{}^\mu . \tag{5.12}$$

We obtain

$$\nabla_\nu T_\mu{}^\nu + \frac{1}{\sqrt{-g}} \Delta_{ij\mu} \nabla_\nu(\sqrt{-g} S^{ij\nu}) - 2\partial_{[\nu} A^{ij}{}_{\mu]} S_{ij}{}^\nu - \frac{1}{\sqrt{-g}} A^{ij}{}_\mu \partial_\nu(\sqrt{-g} S_{ij}{}^\nu) = 0 , \tag{5.13}$$

by use of (5.10) and the field equation $\delta \mathbf{L}/\delta\varphi = \delta \mathbf{L}_M/\delta\varphi = 0$ in (5.9). We apply this equation to the motion of a macroscopic body for which effects due to the "spin" s of the fundamental constituent particles can be ignored, then the energy-momentum tensor of a macroscopic body is symmetric and satisfies the conservation laws

$$\nabla_\nu T_\mu{}^\nu = 0 . \tag{5.14}$$

From (5.14), we can show, in a way quite similar to the case in the four-dimensional Einstein theory, that the world line of a macroscopic body is the geodesic line of the metric $g_{\mu\nu} dx^\mu \otimes dx^\nu$.

VI. TWO LIMITING CASES

A. The case with $a_i \to \infty$ $(i = 1, 2, 3)$

Suppose that the parameters a_i $(i = 1, 2, 3)$ have the expression

$$a_i = \frac{1}{f^2} \bar{a}_i \quad (i = 1, 2, 3) , \tag{6.1}$$

where f is a parameter characterizing the magnitude of a_i's and it can be regarded as standing for the coupling strength between the Lorentz gauge field and the matter field φ. Multiplying both sides of (3.16) by f^2 and taking the limit $f \to 0$, we get

$$2D^l \overline{J}_{[ij][kl]} + \left(\frac{4}{3} t_k{}^{[lm]} - \delta_k{}^{[l} v^{m]} + a_k{}^{lm}\right) \overline{J}_{[ij][lm]} = 0 , \tag{6.2}$$

where

$$\overline{J}_{ijkl} \stackrel{\rm def}{=} 2\overline{a}_3 R\eta_{ik}\eta_{jl} + 2\eta_{ik}(\overline{a}_1 E_{jl} + \overline{a}_2 I_{jl}) \ . \tag{6.3}$$

By substituting $R_{ijkl} = 0$, which is a solution of (6.2), into (3.10), we obtain

$$-2D^k F_{ijk} + 2v^k F_{ijk} + 2H_{ij} - \eta_{ij} L_T = T_{ij} \ . \tag{6.4}$$

Specifically, if we set $A^{ij}{}_\mu = 0$, (6.4) with $\delta = 0$ reduces to (3.7) of Ref. [11], which is the gravitational field equation in a teleparallel theory. Even for this limiting case, the theory given in this paper is not identical to the theory developed in Ref. [11], because (6.2) does not necessarily imply $R_{ijkl} = 0$. To put it briefly, the teleparallel theory developed in Ref. [11] is "included as a solution" in the limiting case with $a_i \to \infty$ $(i = 1,2,3)$ and with $\delta = 0$.

B. The case with $\alpha \to \infty, \beta \to \infty, \gamma \to \infty$

When α, β, and γ become large, the torsion tensor becomes infinitely small with H_{ijk} kept finite, which follows from (3.16). In the limit of $\alpha \to \infty, \beta \to \infty, \gamma \to \infty$, the underlying space-time is of the Riemann type, and H_{ijk} is given by

$$H_{ijk} = -2a_2 \nabla_{[i} G_{j]k}(\{\}) -8\left(a_3 + \frac{a_2}{6}\right)\eta_{k[i}\partial_{j]} G(\{\}) - S_{ijk} \ , \tag{6.5}$$

which is obtained from (4.22). By using (2.8), (4.20), and (6.5) in (4.15) we obtain

$$2aG_{ij}(\{\}) + \tfrac{2}{3}(7a_2 + 6a_3)R_{ij}(\{\})R(\{\}) - 8a_2 R_i{}^k(\{\})R_{jk}(\{\}) -\eta_{ij}\left[-3a_2 R^{kl}(\{\})R_{kl}(\{\}) + \tfrac{1}{3}(5a_2 + 3a_3)(R(\{\}))^2\right] + 2a_2\nabla^k\nabla_k G_{ij}(\{\}) -\tfrac{4}{3}(a_2 + 6a_3)(\eta_{ij}\nabla^k\nabla_k - \nabla_i\nabla_j)G(\{\}) = T_{ij} - \nabla^k(S_{ijk} - S_{ikj} - S_{jki}) \ . \tag{6.6}$$

This equation is obtainable also directly from the Lagrangian $\tilde{L} \stackrel{\rm def}{=} L_{GR} + L_M(\varphi, \nabla_k\varphi)$, where

$$L_{GR} \stackrel{\rm def}{=} a_2 R^{kl}(\{\})R_{kl}(\{\}) + \left(a_3 - \frac{a_2}{3}\right)(R(\{\}))^2 +aR(\{\}) \ , \tag{6.7}$$

$$\nabla_k\varphi \stackrel{\rm def}{=} e^\mu{}_k\left(\partial_\mu\varphi + \frac{i}{2}\Delta^{lm}{}_\mu M_{lm}\varphi\right) \ . \tag{6.8}$$

VII. LINEARIZED GRAVITATIONAL FIELD EQUATIONS AND RELATION TO THE NEWTON THEORY

A. Linearized equations

We now examine the gravitational field equations in the weak field situations in which

$$a^i{}_\mu \stackrel{\rm def}{=} e^i{}_\mu - \delta^i{}_\mu \tag{7.1}$$

and $A^{ij}{}_\mu$ are so small that it is sufficient to keep only terms linear in $a^i{}_\mu$ and in $A^{ij}{}_\mu$. In this approximation, Greek and Latin indices need not be distinguished with each other, and thus we shall use Greek indices throughout the present section with the understanding that they are raised and lowered with $(\eta^{\mu\nu}) \stackrel{\rm def}{=} (\eta_{\mu\nu})^{-1}$ and $(\eta_{\mu\nu}) \stackrel{\rm def}{=} {\rm diag}(-1,1,1)$. For this case, the components $g_{\mu\nu}$ of the metric tensor have the expression

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7.2}$$

with

$$h_{\mu\nu} \stackrel{\rm def}{=} a_{\mu\nu} + a_{\nu\mu} \ . \tag{7.3}$$

Noting (4.1) and (7.1), we employ $a_{\mu\nu}$ and the torsion tensor $T_{\lambda\mu\nu}$ as independent field variables, and express the linearized gravitational field equations in terms of them. In what follows, we consider the case with $\delta = 0$ only, because the "cosmological term" δ in the Lagrangian L_T is not in harmony with our weak field approximation. After some calculations, we find that (4.15) and (4.22) take the forms

$$2aG^{(1)}_{\mu\nu} - 2\partial^\lambda F'^{(1)}_{\mu\nu\lambda} = T_{\mu\nu} \tag{7.4}$$

and

$$Z^{(1)}_{\lambda\mu\nu} = -S_{\lambda\mu\nu} \ , \tag{7.5}$$

respectively. Here, we have defined

$$G^{(1)}_{\mu\nu} \stackrel{\rm def}{=} \partial_\lambda\partial_{(\mu}h^\lambda{}_{\nu)} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\partial_\lambda\partial_\rho h^{\lambda\rho} - \Box h) \ , \tag{7.6}$$

$$F'^{(1)}_{\mu\nu\lambda} \stackrel{\rm def}{=} \left(\alpha + \frac{2a}{3}\right)(t_{\mu\nu\lambda} - t_{\mu\lambda\nu}) + \left(\beta - \frac{a}{2}\right)(\eta_{\mu\nu}v_\lambda - \eta_{\mu\lambda}v_\nu) + 2\left(\gamma - \frac{a}{4}\right)a_{\mu\nu\lambda} = -F'^{(1)}_{\mu\lambda\nu} \ , \tag{7.7}$$

$$Z^{(1)}_{\lambda\mu\nu} \stackrel{\rm def}{=} a_2\left(\partial_\lambda G^{(1)}_{\mu\nu} - \partial_\mu G^{(1)}_{\lambda\nu}\right) + 8\left(a_3 + \frac{a_2}{6}\right)\eta_{\nu[\lambda}\partial_{\mu]}G^{(1)} + (a_1+a_2)\eta_{\nu[\mu}\partial_\rho\partial_\sigma t^{\rho\sigma}{}_{\lambda]}$$
$$+\frac{1}{3}[\partial_\mu\partial_\sigma\{2a_1 t^\sigma{}_{[\lambda\nu]} + 3a_2 t_{\lambda\nu}{}^\sigma\} - \partial_\lambda\partial_\sigma\{2a_1 t^\sigma{}_{[\mu\nu]} + 3a_2 t_{\mu\nu}{}^\sigma\}]$$
$$+\frac{1}{6}(3a_1 + a_2 - 48a_3)\eta_{\nu[\mu}\partial_{\lambda]}\partial_\sigma v^\sigma + \frac{1}{2}(a_1+a_2)\{\partial_\nu\partial_{[\mu}v_{\lambda]} + \eta_{\nu[\lambda}\Box v_{\mu]}\}$$
$$+a_1\partial_\sigma\partial_{[\mu}a_{\lambda]\nu}{}^\sigma + H^{(1)}_{\lambda\mu\nu}, \tag{7.8}$$

with $G^{(1)} \stackrel{\rm def}{=} \eta^{\mu\nu}G^{(1)}_{\mu\nu}$, $h \stackrel{\rm def}{=} \eta^{\mu\nu}h_{\mu\nu}$, $\Box \stackrel{\rm def}{=} \partial^\mu\partial_\mu$, and

$$H^{(1)}_{\lambda\mu\nu} \stackrel{\rm def}{=} -\left(\alpha + \frac{2a}{3}\right)(t_{\nu\lambda\mu} - t_{\nu\mu\lambda}) - \left(\beta - \frac{a}{2}\right)(\eta_{\nu\lambda}v_\mu - \eta_{\nu\mu}v_\lambda) + (4\gamma - a)a_{\lambda\mu\nu} = -H^{(1)}_{\mu\lambda\nu} \; . \tag{7.9}$$

These $G^{(1)}_{\mu\nu}$, $F'^{(1)}_{\mu\nu\lambda}$, $Z^{(1)}_{\lambda\mu\nu}$, $G^{(1)}$, and $H^{(1)}_{\lambda\mu\nu}$ are the linearized expressions for $G_{\mu\nu}(\{\})$, $F'_{\mu\nu\lambda}$, $Z_{\lambda\mu\nu}$, $G(\{\})$, and $H_{\lambda\mu\nu}$, respectively. In the lowest order approximation now considering, we have the differential conservation law

$$\partial_\nu T_\mu{}^\nu = 0 \tag{7.10}$$

and the Tetrode formula

$$\partial_\lambda S_{\mu\nu}{}^\lambda = T_{[\mu\nu]} \; . \tag{7.11}$$

The field equations (7.4) and (7.5) are invariant under the infinitesimal gauge transformations

$$h^*_{\mu\nu} = h_{\mu\nu} + \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu \; , \tag{7.12}$$

$$a^*_{[\mu\nu]} = a_{[\mu\nu]} + \omega_{\mu\nu} \; , \; \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{7.13}$$

with Λ_μ and $\omega_{\mu\nu}$ being both arbitrary infinitesimal functions. The transformations (7.12) and (7.13) are the infinitesimal versions of the general coordinate and local gauge transformations, respectively. The invariance under the transformation (7.12) means that the antisymmetric part $a_{[\mu\nu]}$ does not have physical significance. By virtue of the invariance under the transformation (7.12), we can put the harmonic condition

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \; , \tag{7.14}$$

which is assumed from now on. Here, we have defined

$$\bar{h}_{\mu\nu} \stackrel{\rm def}{=} h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \; . \tag{7.15}$$

The linearized Einstein tensor now takes the form

$$G^{(1)}_{\mu\nu} = -\tfrac{1}{2}\Box\bar{h}_{\mu\nu} \; . \tag{7.16}$$

It is convenient to decompose (7.4) into symmetric and antisymmetric parts:

$$2aG^{(1)}_{\mu\nu} - 3\left(\alpha + \frac{2a}{3}\right)\partial^\lambda t_{\mu\nu\lambda}$$
$$-2\left(\beta - \frac{a}{2}\right)(\eta_{\mu\nu}\partial^\lambda v_\lambda - \partial_{(\mu}v_{\nu)}) = T_{(\mu\nu)} \; , \tag{7.17}$$

$$2\left(\alpha + \frac{2a}{3}\right)\partial^\lambda t_{\lambda[\mu\nu]} + 2\left(\beta - \frac{a}{2}\right)\partial_{[\mu}v_{\nu]}$$
$$-4\left(\gamma - \frac{a}{4}\right)\partial^\lambda a_{\mu\nu\lambda} = T_{[\mu\nu]} \; . \tag{7.18}$$

Taking the trace of (7.17), we obtain

$$2aG^{(1)} - 4\left(\beta - \frac{a}{2}\right)\partial^\lambda v_\lambda = T \tag{7.19}$$

with $T \stackrel{\rm def}{=} \eta^{\mu\nu}T_{\mu\nu}$. Both sides of (7.4) are divergenceless because of (7.10), while the divergence of (7.5) with respect to ν gives (7.18) by virtue of (7.11). Thus, the field equations (7.4) and (7.5) give $(9+9)-(3+3)=12$ independent equations for $3+9=12$ independent field variables.

B. h_{00} due to a static, "spin"less pointlike source

Using (7.8), (7.17), and (7.19) in the the symmetric part of the divergence of (7.5) with respect to x^λ,

$$\partial^\lambda Z^{(1)}_{\lambda(\mu\nu)} = -\partial^\lambda S_{\lambda(\mu\nu)} \; , \tag{7.20}$$

we obtain the fourth-order field equation for $\bar{h}_{\mu\nu}$:

$$A\Box\bar{h}_{\mu\nu} + B\Box^2\bar{h}_{\mu\nu} + C(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Box\bar{h} = T^{(\rm eff)}_{\mu\nu} \tag{7.21}$$

with

$$A \stackrel{\rm def}{=} -a \; , \tag{7.22}$$

$$B \stackrel{\rm def}{=} -\frac{3\alpha a_2}{3\alpha + 2a} \; , \tag{7.23}$$

$$C \stackrel{\rm def}{=} \frac{1}{6(2\beta - a)}\{8\beta(a_2 + 6a_3) - 3aa_2\} - \frac{aa_2}{3\alpha + 2a}, \tag{7.24}$$

and

$$T^{(\text{eff})}_{\mu\nu} \stackrel{\text{def}}{=} T_{(\mu\nu)} - 2\partial^\lambda S_{\lambda(\mu\nu)} - \frac{a_2+24a_3}{6(2\beta-a)}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T - \frac{2a_2}{3\alpha+2a}\left(\Box T_{(\mu\nu)} - \frac{1}{2}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T + 2\partial^\lambda\partial^\rho\partial_{(\mu}S_{\nu)\lambda\rho}\right) . \quad (7.25)$$

It is worth mentioning that the parameters a_1 and γ do not appear in (7.21) with (7.22)–(7.25). We consider now the gravitational field produced by a static, "spin"less source located at the origin, for which $S_{\lambda\mu\nu}$ is vanishing and $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \begin{cases} Mc^2\delta^2(\mathbf{r}) , & \mu=\nu=0 , \\ 0 & \text{otherwise} \end{cases} \quad (7.26)$$

with $\mathbf{r} \stackrel{\text{def}}{=} (x^1, x^2)$. For this case, $T^{(\text{eff})}_{\mu\nu}$ has the expression

$$\begin{aligned} T^{(\text{eff})}_{00} &= Mc^2\{\delta^2(\mathbf{r}) + (P+Q)\Delta\delta^2(\mathbf{r})\} , \\ T^{(\text{eff})}_{0\alpha} &= T^{(\text{eff})}_{\alpha 0} = 0 , \\ T^{(\text{eff})}_{\alpha\beta} &= Mc^2(P-Q)(\partial_\alpha\partial_\beta - \delta_{\alpha\beta}\Delta)\delta^2(\mathbf{r}) , \end{aligned} \quad (7.27)$$

where the indices α and β run over 1 and 2, $\Delta \stackrel{\text{def}}{=} (\partial_1)^2 + (\partial_2)^2$, and

$$P \stackrel{\text{def}}{=} -\frac{a_2+24a_3}{6(2\beta-a)} , \quad Q \stackrel{\text{def}}{=} -\frac{a_2}{3\alpha+2a} . \quad (7.28)$$

Taking the trace of (7.21) we obtain

$$A\Box\bar{h} + (B+2C)\Box^2\bar{h} = T^{(\text{eff})} , \quad (7.29)$$

where $\bar{h} \stackrel{\text{def}}{=} \eta^{\mu\nu}\bar{h}_{\mu\nu}$, and

$$T^{(\text{eff})} \stackrel{\text{def}}{=} \eta^{\mu\nu}T^{(\text{eff})}_{\mu\nu} = -Mc^2\delta^2(\mathbf{r}) - 2Mc^2P\Delta\delta^2(\mathbf{r}) . \quad (7.30)$$

In the following, we shall solve (7.21) and (7.29) to give a time-independent circularly symmetric potential h_{00} for each of the four cases.

1. *The case with $a \neq 0$*

For this case, (7.29) is solved, by utilizing the method of Fourier integration, to give

$$\bar{h}(r) = -\frac{Mc^2}{2\pi A}\ln r + \frac{Mc^2}{4\pi^2 A}(2PA - B - 2C)\int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(B+2C)\mathbf{k}^2 - A}d^2\mathbf{k} + C_1\int \delta[(B+2C)\mathbf{k}^2 - A]e^{i\mathbf{k}\cdot\mathbf{r}}d^2\mathbf{k} + C_2 \quad (7.31)$$

with C_1 and C_2 being integration constants, $r \stackrel{\text{def}}{=} |\mathbf{r}|$ and $\mathbf{k} \stackrel{\text{def}}{=} (k_1, k_2)$. Substituting (7.31) into (7.21) with $\mu=\nu=0$, solving the equation thus obtained and using (7.22), (7.23), and (7.24) and the formula

$$J_0(kr) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\cos\theta}d\theta , \quad (7.32)$$

we obtain

$$\begin{aligned} h_{00}(r) = &-\frac{Mc^2a_2(3\alpha+2a)}{4\pi a}\int_0^\infty \frac{kJ_0(kr)}{3\alpha a_2k^2 - a(3\alpha+2a)}dk \\ &+\frac{Mc^2(a_2+24a_3)(2\beta-a)}{4\pi a}\int_0^\infty \frac{kJ_0(kr)}{2\beta(a_2+24a_3)k^2 + 3a(2\beta-a)}dk \\ &+6\pi C_1|2\beta-a|\int_0^\infty \delta[2\beta(a_2+24a_3)k^2 + 3a(2\beta-a)]kJ_0(kr)dk \\ &+C_3|3\alpha+2a|\int_0^\infty \frac{\delta[3\alpha a_2k^2 - a(3\alpha+2a)]J_0(kr)}{k}dk + C_4 \end{aligned} \quad (7.33)$$

with $k \stackrel{\text{def}}{=} |\mathbf{k}|$, and C_3 and C_4 being integration constants. Here, J_0 denotes the Bessel function of the first kind and of index zero.

2. *The case with $a = 0$ and $\alpha\beta a_2(a_2+24a_3) \neq 0$*

In a way similar to case *1*, the solution

$$h_{00}(r) = \frac{Mc^2(3\alpha+4\beta)}{24\pi\alpha\beta}\ln r - \frac{Mc^2(a_2+6a_3)}{4\pi a_2(a_2+24a_3)}r^2\ln r + C_5r^2 + C_6 \quad (7.34)$$

is obtained, where C_5 and C_6 are integration constants.

3. *The case with $a_2 \to \infty$, $a_3 \to \infty$, and $\alpha\beta \neq 0$*

The parameters a_2 and a_3 are assumed to have the expression

$$a_i = \frac{\bar{a}_i}{f^2} \quad (i = 2,3) \quad (7.35)$$

with f being a real constant, and we consider the limiting case with $f \to 0$. Multiplying both sides of (7.21) and (7.29) by f^2 and taking the limit $f \to 0$ and following a similar procedure as in the above, we find

$$h_{00}(r) = \frac{3\alpha+4\beta}{24\pi\alpha\beta}Mc^2\ln r + C_7r^2 + C_8 \quad (7.36)$$

with C_7 and C_8 being integration constants.

4. *The case with $\alpha \to \infty$, $\beta \to \infty$, and $a \neq 0$*

The potential for this case is given by

$$h_{00}(r) = -\frac{Mc^2 a_2}{4\pi a}\int_0^\infty \frac{kJ_0(kr)}{a_2k^2 - a}dk + \frac{Mc^2(a_2+24a_3)}{4\pi a}\int_0^\infty \frac{kJ_0(kr)}{(a_2+24a_3)k^2+3a}dk$$
$$+\frac{2C_9(a_2+6a_3)}{3}\int_0^\infty \frac{\delta[(a_2+24a_3)k^2+3a]kJ_0(kr)}{a_2k^2-a}dk$$
$$+C_9\int_0^\infty \frac{\delta[(a_2+24a_3)k^2+3a]J_0(kr)}{k}dk + C_{10}\int_0^\infty \frac{\delta(a_2k^2-a)J_0(kr)}{k}dk + C_{11} \qquad (7.37)$$

with C_i (i=9, 10, 11) being integration constants.

C. Relation to the Newton theory

We consider a macroscopic body for which effects due to the "spin" s of the fundamental constituent particles can be ignored. The world line of motion of this macroscopic body moving freely in the space-time under consideration is the geodesic line of the metric $g_{\mu\nu}dx^\mu \otimes d^\nu$, as has been known in Sec.V. Thus, when the motion is sufficiently slow and the gravitational field is weak, this body obeys the equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\partial U}{\partial \mathbf{r}} \qquad (7.38)$$

with $U \stackrel{\text{def}}{=} -c^2h_{00}/2$. Neither of the solutions (7.33) and (7.37) can give a Newton potential. The potential U given by the solution (7.34) satisfies the Newton equation for the gravitational potential,

$$\Delta U = 4\pi GM\delta(\mathbf{r}) \ , \qquad (7.39)$$

if the conditions

$$3\alpha + 4\beta = -\frac{96\alpha\beta\pi G}{c^4} \ , \qquad (7.40)$$

$$a_2 + 6a_3 = 0 \ , \qquad (7.41)$$

are both satisfied *and when the integration constants* C_5 *is chosen to be zero*: $C_5 = 0$. Here, G stands for "Newton gravitational constant." Also the solution (7.36) gives the Newton potential, if the condition (7.40) is satisfied *and if* $C_7 = 0$. Thus, each of the case *2* with the conditions (7.40) and (7.41) and of case *3* with condition (7.40) can give a Newtonian limit by a suitable choice of the integration constant. But, it should be noted that *the field equations for* $h_{\mu\nu}$ *are fourth-order differential equations for both cases.*

VIII. SUMMARY AND COMMENTS

We have formulated a Poincaré gauge theory of (2+1)-dimensional gravity and the results can be summarized as follows.

(1) The theory is underlain with the Riemann-Cartan space-time, and the gravity is attributed to the curvature and the torsion. The most general gravitational Lagrangian, which is at most quadratic in curvature and torsion tensors, is given by $L_G \stackrel{\text{def}}{=} L_T + L_R$ with L_T and L_R being given by (3.1) and (3.2), respectively.

(2) The gravitational field equations are given by (3.10) and (3.16), the alternative forms of which are (4.15) and (4.22), respectively.

(3) Solutions of the vacuum Einstein equation with the cosmological constant Λ satisfy the vacuum gravitational field equations with $\delta = 2\Lambda(a + 6a_3\Lambda)$.

(4) Equation (4.22) does not contain third derivatives of the metric tensor, if and only if $a_2 = a_3 = 0$. For the case with $a_2 = a_3 = 0$, the vanishing torsion satisfies (4.22) with $S_{ijk} \equiv 0$, and (4.15) reduces to the Einstein equation (4.26) for the vanishing torsion.

(5) For the case with $a_1 = a_2 = a_3 = 0$ and with $(3\alpha + 2a)(2\beta - a)(4\gamma - a) \neq 0$, the torsion is "frozen" at the place where the "spin" density S_{ijk} does not vanish. If $a = 1/(2\kappa)$ in addition and the source field is "spin"less, field equations for the gravitational and source fields agree with those in the Einstein theory.

(6) The world line of the macroscopic body is the geodesic line of the metric $g_{\mu\nu}dx^\mu \otimes dx^\nu$, if the effects due to the "spin" of the fundamental constituent particles can be ignored.

(7) In the sense mentioned in Sec. VI A, the teleparallel theory developed in Ref. [11] is "included as a solution" in the limiting case with $a_i \to \infty$ ($i = 1,2,3$) and with $\delta = 0$.

(8) For the case with $\alpha \to \infty, \beta \to \infty, \gamma \to \infty$, the underlying space-time is of the Riemann type.

(9) The linearized field equations lead to the fourth-order differential equation (7.21) for weak gravitational potentials. For the gravitational field produced by a static "spin"less pointlike source, (7.21) has been solved to give the potential $U \stackrel{\text{def}}{=} -c^2h_{00}/2$. The solutions are classified into four cases *1*, *2*, *3*, and *4*. Each of the case *2* with the conditions (7.40) and (7.41) and of case *3* with condition (7.40) can give a Newtonian limit by a suitable choice of the integration constant.

The following is worth mentioning.

(a) Even for the case with $a_1 = a_2 = a_3 = 0, a = 1/(2\kappa)$ and $(3\alpha + 2a)(2\beta - a)(4\gamma - a) \neq 0$, field equations are different from those in the Einstein theory, if the source field has nonvanishing "spin." For this case, space-times in the vacuum regions are locally the same as those in the Einstein theory, but the quantized theory and nonlocal properties such as the gravitational Aharonov-Bohm effect [17, 18] due to the "spin"ning source fields are presumably different from those of the latter theory.

(b) Condition (7.40) agrees with condition (5.11) in Ref. [11] in its form. This is quite naturally understood, if we note the discussions in Secs. VI A and VII B *3* and the fact that the parameter a_1 does not appear in the linearized equation (7.21) with (7.22)–(7.25).

(c)As is known from (3), black hole solutions [19,20] of the ordinary three-dimensional vacuum Einstein equation with a negative cosmological constant satisfy the vacuum gravitational field equations in our theory. Also, it is worth adding that these solutions are independent of the black holes in a teleparallel theory discussed in Ref. [11].

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[15] We denote symmetrization and antisymmetrization of tensor indices by parentheses ( ) and square brackets [ ], respectively. For example, $J_{[ik][jl]} \stackrel{\text{def}}{=} (1/4)(J_{ikjl} - J_{kijl} - J_{iklj} + J_{kilj})$.

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