Oscillating neutrinos in the early Universe

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We investigate the physics of an ensemble of oscillating neutrinos in the primordial plasma of the early Universe. The result is a coupled set of quantum kinetic equations both for the neutrino density operator and the momentum distributions of other species in the plasma. In the appropriate momentum-averaged limit, a comparable set of quantum rate equations arises. For the case of oscillations within the weakly interacting sector we find surprising new contributions to the forward scattering effective potential, as well as new quantum contributions to the kinetic and/or rate equations. The possibility that neutrino oscillations might affect primordial nucleosynthesis is discussed.

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I. INTRODUCTION

It is well known that neutrino oscillations may provide an elegant solution to the solar neutrino problem [1]. Since neutrinos play a pivotal role in primordial nucleosynthesis [2], the following question naturally arises: "If neutrino oscillations were to occur in the early Universe, what effect would they have on the standard model of big-bang nucleosynthesis; in particular would the predicted light element abundances be altered?" The first step to answering this question is to understand the physics appropriate to the description of the evolution of oscillating neutrinos in the early Universe. The hot primordial plasma of elementary particles is a very different environment to that of the solar interior, or the vacuum of space, in which neutrino oscillations are usually studied. The principal complication that arises in the early Universe is that, prior to decoupling at $T \sim 1$ MeV, the neutrinos are part of an *interacting* multispecies gas of elementary particles. In contrast, solar neutrinos evolve essentially free from scattering (apart from the forward scattering with the background solar matter which induces mean-field energy shifts in the propagator).

Previously, other authors have examined the physics of oscillating neutrinos in the primordial plasma. The usual approach taken is to work in the rate-equation approximation where the variable of interest is the one-body reduced-density operator normalized to the total particle number [3–5]. Such an approach ignores the momentum degree of freedom by explicitly integrating it out. In contrast, some authors [6–9] have considered the construction of the full momentum-dependent equations of motion. (For earlier approaches see Ref. [10].) Since neutrino oscillations explicitly rely upon the evolution of amplitudes, rather than just probabilities, the form taken by the equations of motion is necessarily nonclassical. In order to make this distinction clear we shall refer to the equations for the momentum-dependent one-body reduced density operator as quantum kinetic equations (QKE's), and to their momentum-averaged counterparts as quantum rate equations (QRE's).

The principal aim of this paper is to improve on the existing body of work by providing a careful and detailed derivation of the QKE's and QRE's pertaining to an assembly of oscillating neutrinos. Full attention is given to determining under what circumstances the simpler QRE's approximation is appropriate, and under what circumstances that even the full QKE's may cease to be valid. In addition, we discuss the form taken by the QKE's, and where appropriate the QRE's, assuming standard model interactions in the context of the early Universe.

As we shall show, the QKE's and/or QRE's include quantum effects beyond those usually associated with simple neutrino oscillations. For example, in the context of sterile-active oscillations, the phenomena of quantum damping [5,11,12] is known to occur. Interestingly, still further new effects arise in the active-active oscillation scenario, effects that have not been identified in previous work [6–9] on QKE's. Leaving aside the application of these results to explicit calculations, these curious new quantum contributions to the equations of motion are fascinating in their own right.

The explicit derivation of the general QKE's is undertaken in Sec. II. Care has been taken to keep the development as general as possible, and, in fact, the final result is equally applicable to a general many-body quantum system so long as the system is sufficiently dilute and a single reasonable ansatz is satisfied. This includes systems with an arbitrarily large discrete eigensubspace or even spatial inhomogeneities. (For an example of an application of these QKE's to another system see Ref. [11].)

In general, neutrino oscillations can occur between weakly interacting SU(2)-doublet neutrinos (active-active oscillations) or between a weakly interacting species and an SU(2)-singlet sterile neutrino (sterile-active oscillations). The latter possibility has been examined in the context of the early Universe by approximate methods [4,13,14] and through a detailed numerical integration of the QRE's [3,15,16]. Surprisingly, the former scenario has received only scant attention [7,17]. In Sec. III we employ the results of the preceding section to construct the QKE's, and the corresponding QRE's, which describe the evolution of oscillating neutrinos (including both active-active and sterile-active schemes) in the early

Universe. In doing so, we shall encompass all possible oscillation scenarios within one formalism at both the QKE's and QRE's level. The question of how the new terms which appear in the QRE's and/or QKE's relate to decoherence and quantum measurement is also briefly addressed in Sec. III. It may be that important points concerning the transition from quantum to classical behavior can be inferred from the QRE's/QKE's formalisms.

Section IV describes how the various parameters appearing in the QKE's/QRE's can be calculated assuming standard model interactions. For the QKE's this requires the specification of kernels for various momentum integrals, whereas in the case of the QRE's the problem reduces to the calculation of a number of temperature-dependent parameters. Having evaluated these various quantities, there follows in Sec. V, a discussion of under what circumstances the QRE's, approximation to the full QKE's is valid. This is important because, although in practical terms the QRE's provide a substantial simplification, the results given by the QKE's and QRE's can vary significantly.

Finally in Sec. VI the main points of this work are summarized and final conclusions are drawn. In particular, future applications of the results developed here are outlined.

II. DERIVATION OF THE GENERAL QUANTUM KINETIC EQUATION

In the usual calculation of time-evolving number densities in the early Universe (e.g., the determination of the primordial n/p ratio), quantum mechanics only enters the problem via the calculation of cross sections for the various reaction channels. In fact, the overall approach is essentially classical—a Boltzmann equation describes the evolution of number densities in terms of probabilities. Such an approach is inadequate when neutrino oscillations occur, as the oscillations depend inherently upon the evolution of *amplitudes* rather than probabilities. In order to retain the quantum amplitudes, we shall adopt a density operator description which allows for overlap between the particle species (and even momentum eigenstates), and further, we shall describe the interactions of the system via an S matrix. As will be shown, such an approach automatically yields reaction rates, effective forward-scattering potentials and even vacuum oscillations, with nothing left to be added in by hand. The classical Boltzmann equation (or perhaps more accurately the Boltzmann-Pauli equation, since quantum-transition probabilities are employed) is reproduced in the appropriate limit.

Consider a multispecies plasma of elementary particles contained in some arbitrary volume. (We defer the problem of accounting for the expansion of the Universe until later.) At the temperatures we shall be interested in, the plasma will be sufficiently dilute that it can be described by a one-particle reduced density operator $\hat{\rho}^{(1)}$:

$$\hat{\rho}^{(1)}=\sum_{\mathbf{ij}}\rho_{\mathbf{ij}}|\mathbf{i}\rangle\langle\mathbf{j}| \ , \tag{1}$$

where the boldface index $\mathbf{i}$ is shorthand for the set of eigenvalues corresponding to some complete set of commuting observables that span the Hilbert space. It is convenient to work in the momentum representation so that the pertinent eigenvalues are the momentum p_i and a discrete index i that describes the type of particle, i.e., $i\in\{\nu_e,\bar{\nu}_e,e^+,e^-,\ldots\}$ (wherein a sum over spins is implied but not explicitly displayed). It follows then that

$$\sum_{\mathbf{i}}\equiv\frac{\Omega}{(2\pi)^3}\sum_i\int d^3p_i \quad \text{and} \quad |\mathbf{i}\rangle\equiv|p_i,i\rangle \ , \tag{2}$$

where Ω is the quantization volume.

It is convenient, for the moment, to normalize the one-particle operator $\hat{\rho}^{(1)}$ to unity so that $\mathrm{Tr}\hat{\rho}^{(1)}=\sum_{\mathbf{i}}\rho^{(1)}=1$. The diagonal matrix elements $\rho_{\mathbf{ii}}=n_{\mathbf{i}}$ are related to the total number density of particles in the state $|\mathbf{i}\rangle$; $N_{\mathbf{i}}$, by $N_{\mathbf{i}}=N\Omega^{-1}n_{\mathbf{i}}$, where N is the total number of particles plus antiparticles in the volume Ω. Similarly the off-diagonal matrix elements $\rho_{\mathbf{ij}}$ measure the extent to which the system may be found in a superposition of the states $|\mathbf{i}\rangle$ and $|\mathbf{j}\rangle$. In order to calculate the evolution of the entire interacting multiparticle system we must adopt an N-body approach. In the limit that the one-body reduced density operator is an adequate description of the system, the N-body density operator is simply the tensor product

$$\hat{\rho}=\prod^{N}\otimes\hat{\rho}^{(1)} \ . \tag{3}$$

This restriction to the canonical ensemble ($N=$const) is sufficient for our purposes because we shall only be concerned with number-conserving, weak interaction, two-body processes (see Sec. III). Changes in the total number of particles will only come about via EM processes involving e^-, e^+, and photons, which, due to the relative disparity in interaction strengths, can always be taken to be in thermal equilibrium.

Our derivation follows an approach similar to that used by Harris and Stodolsky [11] in their investigation of quantum damping of neutrino oscillations by elastic processes. (A very recent alternative derivation of QKE's along these same lines can be found in Ref. [8], it has the advantage over the development presented here of working with a second quantized formalism, thus allowing the effect of quantum statistics to be taken account of. On the other hand, our development has the advantage that it investigates the full range of standard model weak-interaction processes, and in doing so, uncovers unexpected new physics. Moreover, the first quantized approach is arguably simpler and therefore more accessible.) The essential point is to employ an S matrix so that quantum amplitudes, rather than just classical probabilities, are evolved forward in time. Through an interaction, the system is evolved forward in time by the S matrix in the usual manner:

$$\hat{\rho}^{(f)}=\hat{S}\hat{\rho}^{(i)}\hat{S}^{\dagger} \ , \tag{4}$$

where $\hat{\rho}^{(i)}$ and $\hat{\rho}^{(f)}$ are the pre- and postcollision N-body density operators and $\hat{S}$ is the N-body scattering matrix. Using $\hat{S}=\hat{I}+i\hat{T}$ we can rewrite (2) in terms of the T matrix:

$$\hat{\rho}^{(f)}-\hat{\rho}^{(i)}=\tfrac{1}{2}[2\hat{T}\hat{\rho}^{(i)}\hat{T}^{\dagger}-\hat{T}\hat{T}^{\dagger}\hat{\rho}^{(i)}-\hat{\rho}^{(i)}\hat{T}\hat{T}^{\dagger} +i(\hat{T}+\hat{T}^{\dagger})\hat{\rho}^{(i)}-i\hat{\rho}^{(i)}(\hat{T}+\hat{T}^{\dagger})] , \quad (5)$$

where we have employed the optical theorem $i(\hat{T}-\hat{T}^{\dagger})=-\hat{T}\hat{T}^{\dagger}$. The first three terms in (5) refer, very roughly speaking, to the scattering in and out of states by collisions, and the last two terms are the mean-field (forward-scattering) contributions to the effective potential.

In the dilute gas limit we can always restrict our attention to two-body collisions with negligible error. This allows us to replace the N-body operators in (5) by an appropriate sum of two-body operators representing the possible pairwise interactions of the N particles in the system. A relationship between the pre- and postcollision one-body reduced density operators can be then be deduced by partial tracing over all but one of the particle degrees of freedom. In the exclusively two-body interaction limit, this simply gives rise to an overall factor of N, since each particle is able to interact with $(N-1)\approx N$ other particles. Furthermore, due to the feebleness of the weak interaction, we can safely truncate the T matrix at first order in the Fermi constant G_F. It turns out that this is sufficient to yield both the collisional-scattering and forward-scattering effects at their lowest order manifestations, (which are of order G_F^2 and G_F, respectively). The more general case, where higher-order terms become relevant, is nevertheless interesting in its own right. For more details, including a diagrammatic representation of the various contributions, see Ref. [6].

We shall employ the notation $T(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')[V(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')]$ for the two-body T-matrix (interaction-matrix) elements, where the first two indices refer to the initial states and the last two indices refer to the final states. The T matrix, to lowest order in the interaction, can then be written

$$T(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')=(-i)^2\int_{-\infty}^{+\infty}d\tau V(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')(\tau) , \quad (6)$$

where $V(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')(\tau)$ is written in the interaction picture $\hat{V}(\tau)=e^{i\hat{H}_0\tau}\hat{V}e^{-i\hat{H}_0\tau}$. Provided we choose to work in the representation that diagonalizes the free Hamiltonian, (6) becomes

$$T(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')=(-i)^2 2\pi V(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')\delta_E(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}') , \quad (7)$$

where we have adopted the convenient shorthand notation

$$\delta_E(\mathbf{i},\mathbf{j}|\mathbf{i}',\mathbf{j}')=\delta(\omega_{\mathbf{i}}+\omega_{\mathbf{j}}-\omega_{\mathbf{i}'}-\omega_{\mathbf{j}'}) , \quad (8)$$

where $\omega_{\mathbf{i}}$ is the energy eigenvalue belonging to the eigenstate $|\mathbf{i}\rangle$. For example, in the case of a relativistic particle of mass m_i; $\omega_{\mathbf{i}}^2=p_i^2+m_i^2$.

Partial tracing over all but one of the particle degrees of freedom in (5), and noting that (7) implies that the T-matrix elements will be real to first order in the interaction, we obtain the following expression involving the one-body reduced density matrix elements:

$$\rho_{\mathbf{ij}}^{(f)}-\rho_{\mathbf{ij}}^{(i)}=N\tfrac{1}{2}[2T(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')T(\mathbf{j},\mathbf{k}|\mathbf{j}',l')\rho_{\mathbf{i}'\mathbf{j}}^{(i)}\rho_{\mathbf{k}'l'}^{(i)}-T(\mathbf{i}',\mathbf{k}'|\mathbf{j}',l')T(\mathbf{i}',\mathbf{k}'|\mathbf{j},l)\rho_{\mathbf{ij}'}^{(i)}\rho_{ll'}^{(i)}-T(\mathbf{j}',l'|\mathbf{i},\mathbf{k})T(\mathbf{j}',l'|\mathbf{i}',\mathbf{k}')\rho_{\mathbf{i}'\mathbf{j}}^{(i)}\rho_{\mathbf{k}'\mathbf{k}}^{(i)}] -iN[T(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\rho_{\mathbf{i}'\mathbf{j}}^{(i)}\rho_{\mathbf{k}'\mathbf{k}}^{(i)}-T(\mathbf{j}',l'|\mathbf{j},\mathbf{k})\rho_{\mathbf{ij}'}^{(i)}\rho_{\mathbf{k}l'}^{(i)}] , \quad (9)$$

where summation over repeated indices is assumed in the sense of expression (2).

Before proceeding any further with the general development of (9) it will be useful to look at a special case. Consider a system in which only diagonal elements arise in the one-body reduced density operator, that is, a system in which no quantal overlaps occur so that $\rho_{\mathbf{ij}}=\rho_{\mathbf{ii}}\delta_{\mathbf{ij}}$. This corresponds physically (in the momentum representation) to a spatially homogeneous system in which no oscillations occur in the internal degree of freedom. For such a system it is easy to see that (9) becomes

$$\rho_{\mathbf{ii}}^{(f)}-\rho_{\mathbf{ii}}^{(i)}=NT^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')(\rho_{\mathbf{i}'\mathbf{i}'}\rho_{\mathbf{k}'\mathbf{k}'}-\rho_{\mathbf{ii}}\rho_{\mathbf{kk}}) . \quad (10)$$

Consider now the square of the T-matrix element appearing above, substituting from (7) reveals that

$$T^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')=(2\pi)^2V^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\delta_E^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}') =2\pi V^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\delta_E(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\Delta\tau , \quad (11)$$

where we have employed the same arguments used to derive Fermi's "golden rule" to extract a time interval from the squared energy delta function (see Ref. [6]). Provided the system changes sufficiently slowly, we can take $\Delta\tau$ to provide a coarse-grained time interval which allows us to identify the left-hand side (LHS) of (10) as the time derivative of $\rho_{\mathbf{ii}}$. Thus, Eq. (10) then becomes

$$\frac{d}{dt}n_{\mathbf{i}}=2\pi NV^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')(n_{\mathbf{i}'}n_{\mathbf{k}'}-n_{\mathbf{i}}n_{\mathbf{k}})\delta_E(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}') \quad (12)$$

which is immediately identifiable as the Pauli-Boltzmann equation. [See the Appendix for an example of how $2\pi NV^2(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\delta_E(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')$ can be calculated using weak-interaction matrix elements, and collision frequencies thereby deduced.] In fact, this type of development, which connects quantum transitions and classical probability flow, appears in the early work of Pauli and Dirac [18].

The alert reader will have noticed that (12) is not quite complete because no Pauli-blocking factors are manifest. This is a direct consequence of having adopted a first quantized approach which necessarily neglects quantum statistics. This could be remedied by inserting factors proportional to $(1-n_{\mathbf{i}})(1-n_{\mathbf{k}})$ in the appropriate parts of the Pauli-Boltzmann equation (12). However, since quantum statistical effects do not play a dominant role in the behavior of particles in the hot early Universe, we shall, for the sake of simplicity, largely ignore their pres-

ence. (Although, for completeness, we shall display the general form of the QKE's including quantum statistical effects at the conclusion of this section.) Nevertheless, it must be remembered that quantum statistical effects will be very important in circumstances where the neutrino density becomes high, such as in a supernova. For this reason, it is valuable that consistent second quantized approaches have been pursued elsewhere [8,9].

It remains for us to derive from (9) a more general expression than (12) which will describe the evolution of arbitrary density operator matrix elements, not just the diagonal members. Naively taking the time evolution from $-\infty$ to $+\infty$ introduces ill-defined unmanageable phases into the problem. In order to avoid this, it is convenient to deal explicitly with the actual time interval over which the evolution occurs. Taking the time interval from $t^{(i)}=-\Delta\tau/2$ to $t^{(f)}=+\Delta\tau/2$, and assuming that the system changes infinitesimally over the period $\Delta\tau$, allows the LHS of (9) to be written in the form

$$\rho_{ij}^{(i)}-\rho_{ij}^{(f)}=\Delta\tau\frac{d}{dt}(\rho_{ij}e^{i(\omega_i-\omega_j)t})\Big|_{-\Delta\tau/2}$$

$$=\Delta\tau\left[\frac{d}{dt}\rho_{ij}+i(\omega_i-\omega_j)\rho_{ij}\right]\times\exp\left[-i(\omega_i-\omega_j)\frac{\Delta\tau}{2}\right], \tag{13}$$

since, by definition, the interaction picture matrix elements $\rho_{ij}^{(i)}$ and the Schrödinger picture matrix elements ρ_{ij} are related by $\rho_{ij}^{(i)}=\rho_{ij}e^{i(\omega_i-\omega_j)t^{(i)}}$. Note that the vacuum oscillation contribution to the evolution of ρ_{ij} is immediately recognizable in (13). Clearly then, the problem now reduces to that of exhibiting a factor $\Delta\tau e^{i(\omega_i-\omega_j)(\Delta\tau/2)}$ on the right-hand side (RHS) of (9).

Consider the following typical factor in (9) involving the product of two Dirac δ functions and the accompanying phase factor from the $\rho_{ij}^{(i)}$, it is easy to show that

$$\delta_E(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\delta_E(\mathbf{j},\mathbf{k}|\mathbf{j}',l')e^{-i(\omega_{i'}-\omega_{j'}+\omega_{k'}-\omega_{l'})\Delta\tau/2}$$

$$=\delta_E^+(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\delta_E^{+*}(\mathbf{j},\mathbf{k}|\mathbf{j}',l')\exp\left[-i(\omega_i-\omega_j)\frac{\Delta\tau}{2}\right], \tag{14}$$

where

$$\delta^+(x)\equiv\frac{1}{2\pi}\int_0^{\Delta\tau}dt\,e^{+ixt}. \tag{15}$$

Somewhat surprisingly, all the products of delta functions can be manipulated in a similar manner into the same form as the RHS of (14). The single powers of delta functions are also amenable to the same sort of reduction, a typical result being of the form

$$\delta_E(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')e^{-i(\omega_{i'}-\omega_j+\omega_{k'}-\omega_k)\Delta\tau/2}$$

$$=\delta_E^+(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\exp\left[-i(\omega_i-\omega_j)\frac{\Delta\tau}{2}\right]. \tag{16}$$

Thus all the terms in (9) can be brought into a form proportional to the phase factor, $e^{(-i(\omega_i-\omega_j)\Delta\tau/2)}$ which can then be completely factored out.

Further, it is easy to show that

$$\delta^+(x)\delta^{+*}(y)=\frac{1}{2}\delta^+\left[\frac{x-y}{2}\right][\delta^+(x)+\delta^{+*}(y)], \tag{17}$$

which allows us to write the RHS of (14) (remembering to factor out the redundant common phase) as

$$\delta^+\left[\frac{(\omega_i+\omega_k-\omega_{i'}-\omega_{k'})-(\omega_j+\omega_k-\omega_{j'}-\omega_{l'})}{2}\right]\times\frac{1}{2}[\delta_E^+(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')+\delta_E^{+*}(\mathbf{j},\mathbf{k}|\mathbf{j}',l')]. \tag{18}$$

Similarly, all of the second-order terms in (9) can be shown to be proportional to factors of this form.

Consider now the ansatz

$$\forall\rho_{ij}\neq0,\quad|\omega_i-\omega_j|\ll\frac{2\pi}{\Delta\tau}, \tag{19}$$

which states that the density operator only includes overlaps between energy eigenstates that have a small separation relative to the scale set by the coarse-graining time interval $\Delta\tau$. It happens that this is a sufficient condition to ensure that the argument of δ^+ in (16), and the first δ^+ factor in (18) [and all other similar terms of both types that arise in (9)], are small enough to allow us to simply replace them by $\Delta\tau/2\pi$ since

$$\lim_{x\Delta\tau\to0}\int_0^{\Delta\tau}dt\,e^{+ixt}\to\Delta\tau. \tag{20}$$

Further, note that the remaining factor in (18) [and all similar terms on the RHS of (9)] can be written as

$$\frac{1}{4\pi}[\delta^+(x)+\delta^{+*}(y)]=\frac{1}{4\pi}\int_0^{\Delta\tau}dt\exp\left[+i\frac{x-y}{2}t\right]\left[\exp\left[+i\frac{x+y}{2}t\right]+\exp\left[-i\frac{x+y}{2}t\right]\right]$$

$$\simeq\frac{1}{4\pi}\left[\delta^+\left[\frac{x+y}{2}\right]+\delta^{+*}\left[\frac{x+y}{2}\right]\right]=\frac{1}{2\pi}\delta(x+y), \tag{21}$$

where we have used the fact that (19) and the structure of (9) guarantee that all the differences $x-y$ that occur satisfy $e^{+ix-y/2t}\simeq1$ ($\forall t\leq\Delta\tau$) in the same sense as (20).

Note that the considerations of the preceding paragraph show that the ansatz (19) implies that energy differences below a certain size are irrelevant insofar as the arguments of the energy-conserving δ functions are concerned. Thus, the coarse graining of the system's

time variable via $\Delta\tau$, in conjunction with the ansatz (19), implies a coarse graining of the energy variable. This should probably not come as a surprise; recall that a similar result arises in time-dependent perturbation theory. In that context it becomes, somewhat misleading, the so-called "time energy uncertainty principle."

It is important to note that (19) in no way restricts the size of the arguments of $\delta_E(x+y)$ which remain in (9) after the substitution (21) is made, nor does it restrict the matrix elements of the interaction which contribute. It is only the off-diagonal terms of the density operator that are directly constrained in the resultant expression. Thus, provided we are willing to accept the restriction imposed by the ansatz (19), we can deduce from (9) the following QKE's for ρ_{ij}:

$$\frac{d}{dt}\rho_{ij}=N\pi[2V(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')V(\mathbf{j},\mathbf{k}|\mathbf{j}',l')\rho_{i'j'}\rho_{k'l'}\delta_E((\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')+(\mathbf{j},\mathbf{k}|\mathbf{j}',l'))$$
$$-V(\mathbf{i}',\mathbf{k}'|\mathbf{j}',l')V(\mathbf{i}',\mathbf{k}'|\mathbf{j},l)\rho_{ij'}\rho_{ll'}\delta_E((\mathbf{i}',\mathbf{k}'|\mathbf{j}',l')+(\mathbf{i}',\mathbf{k}'|\mathbf{j},l))$$
$$-V(\mathbf{j}',l'|\mathbf{i},\mathbf{k})V(\mathbf{j}',l'|\mathbf{i}',\mathbf{k}')\rho_{i'j}\rho_{k'k}\delta_E((\mathbf{j}',l'|\mathbf{i},\mathbf{k})+(\mathbf{j}',l'|\mathbf{i}',\mathbf{k}'))]-i(E^{ik}\rho_{kj}-E^{kj}\rho_{ik}) , \tag{22}$$

where we have combined the mean-field energy shifts from (9) with the vacuum oscillation contribution from (13) by defining

$$E^{ij}=\omega_i\delta_{ij}+NV(\mathbf{i},\mathbf{k}|\mathbf{j},\mathbf{k}')\rho_{k'k} . \tag{23}$$

It is interesting to compare our result (22) with a more general expression which includes quantum statistical effects. Akhiezer and Peletminski [19] employed a second quantized approach (very different to the approach adopted here) to derive the QKE

$$\frac{d}{dt}\rho_{ij}=N\pi V(\mathbf{k}',l'|\mathbf{m}',\mathbf{n}')V(\mathbf{k},l|\mathbf{m},\mathbf{n})\delta_E^-(\mathbf{k}',l'|\mathbf{m}',\mathbf{n}')[\rho_{n'l}\rho_{m'k}(\delta_{mk'}\pm\rho_{mk'})(\delta_{il'}\pm\rho_{il'})-\rho_{mk'}\rho_{il'}(\delta_{km'}\pm\rho_{km'})(\delta_{n'l}\pm\rho_{n'l})]\delta_{nj}$$
$$+\text{complex conjugate } [i\leftrightarrow j]-i(E^{ik}\rho_{kj}-E^{kj}\rho_{ik}) , \tag{24}$$

where the plus sign refers to Bose-Einstein statistics and the minus sign to Fermi-Dirac statistics. This more general QKE reduces to (22) in the limit that quantum statistical effects are ignored. [N.B. It is necessary to remember that the ansatz (19) results in a coarse graining of the energy when doing so.] In addition, the general expression (24) reproduces the usual Pauli-Boltzmann equation, including Pauli blocking effects, in the appropriate limit.

Let us now examine in some detail the ansatz (19). In particular, it is important to understand the quantity $\Delta\tau$. In an ideal scattering experiment the system is assumed to evolve from a "free" (noninteracting) state in the infinite past through to a similarly "free" state in the infinite future. However, in the practical problem of an interacting gas of particles, the maximum time separating the "free states" before and after a single collision cannot be greater than the inverse collision frequency, f^{-1}. Similarly, if the initial and final states are going to be truly free, the evolution of the system must be considered over an interval very much longer than the actual time the quanta spend interacting, $\Delta\tau^{\text{coll}}$. It follows then that

$$\Delta\tau^{\text{coll}}\ll\Delta\tau\ll f^{-1} . \tag{25}$$

Thus it will only be possibly to find an appropriate time interval Δt if the system is sufficiently weakly interacting, and dilute, that $\Delta\tau^{\text{coll}}\ll f^{-1}$. Physically this simply states that if the quanta spend most of their time interacting, rather than freely propagating, then the procedure developed above will fail. This is not surprising since for a system in such a regime, interparticle correlations (explicitly ignored in this development by the use of a one-body reduced operator description) will be important.

In the particular case of a relativistic weakly interacting gas near equilibrium we can estimate $\Delta\tau^{\text{coll}}\sim(M_{Z_0})^{-1}$ and $f\sim G_F^2T^5$. This implies the existence of a $\Delta\tau$ satisfying (25) provided that $T\ll10^2$ GeV. Since this upper bound is 3–5 orders of magnitude above our temperature range of interest (1–100 MeV) we can safely assume that (25) is always satisfied. From (19) it is apparent that the lower bound on $\Delta\tau$ limits the size of the energy overlaps allowed; once again using $\Delta\tau^{\text{coll}}\sim(M_{Z_0})^{-1}$ we find that $|\omega_i-\omega_j|\ll2\pi/\Delta\tau\ll2\pi M_{Z_0}$; thus, since we will at most be interested in energy splits of order keV, there is ample room for a suitable $\Delta\tau$ to exist.

The QKE (22) describes the evolution of ρ_{ij} through interactions involving scatterings involving both a discrete index and the continuous momentum variable. As it stands the result is equally applicable to spatially homogeneous and inhomogeneous systems alike, provided that (19) is satisfied. That is, provided that the energy split between superimposed states is not too large. Although the discrete index was originally envisaged to describe particle flavors, it is clear to see that the derivation applies to any discrete eigenset of any dimension. In the "classical" limit, $\rho_{ij}=\rho_{ii}\delta_{ij}$, it has already been shown that (22) reduces to the Boltzmann-Pauli equation (12). In the next section we shall examine the specific forms taken by (22) assuming active-active and active-sterile neutrino oscillations scenarios in the early Universe. When doing so we shall keep to the standard big bang model and assume that the system is spatially homogeneous so that the density operator will be diagonal in the momentum variable. However, in order to allow for neutrino oscillations, the overlaps between the discrete flavor eigenvalues must be maintained.

III. DESCRIPTION OF NEUTRINO OSCILLATIONS IN THE EARLY UNIVERSE

We need to be able to describe the evolution of an ensemble of oscillating neutrinos, in the hot thermal environment of the early Universe, from a temperature of around 100 MeV down to about 0.1 MeV—at which point the neutrinos cease to be important for nucleosynthesis. (Above 100 MeV the quark-hadron phase transition introduces unmanageable uncertainties into any prospective calculation.) In this temperature range the primordial plasma can be naturally divided into two parts: the electromagnetically interacting sector (e^+,e^-,γ) and the exclusively weakly interacting sector $(\nu_e,\bar{\nu}_e,\nu_\mu,\bar{\nu}_\mu,\nu_\tau,\bar{\nu}_\tau)$. The natural hierarchy of interaction strengths implies that the electromagnetic sector will remain continuously in equilibrium, even if the weakly interacting species deviate from equilibrium due to neutrino oscillations. Thus, we need only consider how the weakly interacting sector evolves under the influence of collisions and neutrino oscillations.

Assuming that the early Universe is spatially homogeneous, as is standard, we can immediately simplify the structure of the one-body reduced density operator (1) by making it diagonal in the momentum variable. Further, since in (22) the off-diagonal matrix elements remain zero if they are initially zero (and provided that no explicit, or induced, mixing exists), we can also assume that $\tilde{\rho}$ is block diagonal in flavor space. It follows that the only nonzero entries will be the diagonal terms $n_i(k)$ and $n_{\bar{i}}(k)$ corresponding to the populations of nonmixed i species, and the $n\times n$ matrices $\rho_\nu(k)$ and $\rho_{\bar{\nu}}(k)$ for the mixed neutrino sector (assuming mixing among n different flavors). Thus the one-body density matrix, in the rest frame of the plasma, for the kth momentum eigensubspace can be written

$$\rho(k)=\sum_i \oplus n_i(k)\oplus n_{\bar{i}}(k)\oplus\rho_\nu(k)\oplus\rho_{\bar{\nu}}(k)\ , \tag{26}$$

where, and henceforth, the index i runs over all the weakly interacting particle species *excluding* the mixed neutrino sector. In contradistinction, the index j will henceforth be reserved for summation over all the weakly interacting particle species *including* the mixed neutrino sector. Also, in contrast with the notation of the previous section, we shall now always write the flavor index i and the momentum variable $\vec{k}$ separately. Note also that, since we are working in the system's rest frame, $\rho(\vec{k})$ only depends upon $k=|\vec{k}|$. Assuming that mixing occurs between only two neutrino species, the α and β flavors say, the mixed neutrino and antineutrino density submatrices can be written as [20]

$$\rho_\nu(k)=\tfrac{1}{2}[P_0(k)+\mathbf{P}(k)\cdot\sigma],\quad \rho_{\bar{\nu}}(k)=\tfrac{1}{2}[\bar{P}_0(k)+\bar{\mathbf{P}}(k)\cdot\sigma]\ , \tag{27}$$

so that $n_{\nu_\alpha}(k)=\frac{1}{2}[P_0(k)+P_z(k)]$, and $n_{\nu_\beta}(k)=\frac{1}{2}[P_0(k)-P_z(k)]$, and similarly for the antiparticles. [The generalization to the case of mixing among N species is readily accomplished by replacing the Pauli matrices σ_i with λ_i, the Gell-Mann matrices for SU(N).]

Substitution of (26) and (27) into the QKE (22) yields a set of coupled nonlinear differential equations for the particle and antiparticle density operators. We shall concentrate on the equations for the particles only, the corresponding expressions for the antiparticles are easily deduced by the transformation $\nu(k)\leftrightarrow\bar{\nu}(k)$. The mixed-neutrino density matrices evolve according to

$$\begin{aligned}\frac{d}{dt}\mathbf{P}(k)=&\ \mathbf{V}(k)\times\mathbf{P}(k)+[R_\alpha(k)-R_\beta(k)]\hat{\mathbf{z}}\\&-D(k)\mathbf{P}_T(k)+\int dk'd(k,k')\mathbf{P}_T(k')\\&-\mathbf{C}(k)P_0(k)+\int dk'\mathbf{c}(k,k')P_0(k')\ ,\end{aligned} \tag{28}$$

and

$$\frac{d}{dt}\mathbf{P}_0(k)=R_\alpha(k)+R_\beta(k)\ , \tag{29}$$

where $\mathbf{P}_T(k)\equiv P_x(k)\hat{\mathbf{x}}+P_y(k)\hat{\mathbf{y}}$ is the transverse part of the polarization vector and the quantities $R_\delta(k)$ are given by

$$R_\delta(k)=\int dk'dp'dp\left[\sum_j F_{\delta j}(kp|k'p')[n_j(k')n_{\bar{j}}(p')-n_{\nu_\delta}(k)n_{\nu_{\bar{\delta}}}(p)]-\frac{1}{2}\sum_i G_i(k'p'|kp)[\mathbf{P}_T(k)\cdot\bar{\mathbf{P}}_T^*(p)]\right]\ , \tag{30}$$

while the evolution of the nonmixed species is given by

$$\frac{d}{dt}n_i(k)=\int dk'dp'dp\left[\sum_j F_{ij}(kp|k'p')[n_j(k')n_{\bar{j}}(p')-n_i(k)n_{\bar{i}}(p)]+G_i(kp|k'p')[\mathbf{P}_T(k')\cdot\bar{\mathbf{P}}_T^*(p')]\right]\ , \tag{31}$$

where $\int dp$ is shorthand for the full momentum integral $(\Omega/(2\pi)^3)\int d^3p$ defined by (2). The quantities $\mathbf{V}(k)$, $D(k)$, $d(k,k')$, $\mathbf{C}(k)$, $\mathbf{c}(k,k')$, $F_{ij}(kp|k'p')$, and $G_i(kp|k'p')$ are various functions of the interaction Hamiltonian (and density operator) matrix elements, their specific definitions are set out below.

The expressions (28)–(31) are our final set of QKE's. They describe the evolution of the amplitudes involved in the neutrino oscillations, as well as the number densities of all the weakly interacting species. The only classically familiar terms are the Boltzmann-Pauli equation parts of (30) and (31), cf. Eq. (12). They describe the pairwise creation and annihilation of particles and antiparticles, which proceeds at the rate governed by the product of

matrix elements

$$F_{jj'}(kp|k'p') = 2\pi NV^2[j(k),\bar{j}(p)|j'(k'),\bar{j}'(p')]\delta_E(kp|k'p') , \quad (32)$$

where (and henceforth) we have ignored the, in practice unimportant ($k,p \gg m$), mass corrections to the energy eigenvalues within the δ function. All the remaining terms are entirely quantum in nature; let us consider them one by one. (We leave the exact form of the interactions unspecified at this stage except to assume microscopic lepton universality; this reduces the number of terms resulting from neutrino-antineutrino interactions.)

First, terms arise proportional to the mutual coherence between the neutrino and antineutrino oscillations: $\mathbf{P}_T\cdot\bar{\mathbf{P}}_T^*$. Clearly [see (30) and (31)] they indicate the flow of quanta from the oscillating neutrino and antineutrino sectors into the i and $\bar{i}$ species. In fact, they describe the annihilation of *mixed* neutrino-antineutrino pairs via the following cross-flavor product of amplitudes:

$$G_i(kp|k'p') = 2\pi NV(\nu_\alpha(k'),\bar{\nu}_\alpha(p')|i(k),\bar{i}(p)) \times V(\nu_\beta(k'),\bar{\nu}_\beta(p')|i(k),\bar{i}(p))\delta_E(kp|k'p') . \quad (33)$$

That is, because both the neutrino flavors are weakly interacting, neutrino states which are superpositions of flavor eigenstates can annihilate with similarly admixed antineutrinos. Note that the effect vanishes if $\mathbf{P}_T(k)$ and $\bar{\mathbf{P}}_T^*(p)$ are orthogonal for all k and p.

The first term in (28), $\mathbf{V}(k)\times\mathbf{P}(k)$, is responsible for the oscillations between the mixed neutrino flavors. Ignoring the other terms, the polarization vector $\mathbf{P}(k)$ will precess about the vector $\mathbf{V}(k)$ with angular velocity $|\mathbf{V}(k)|$. In terms of the (assumed real) energy and interaction matrix elements $\mathbf{V}(k)$ can be written

$$\mathbf{V}(k) = 2E_{\alpha\beta}(k)\hat{\mathbf{x}} + [E_{\alpha\alpha}(k) - E_{\beta\beta}(k)]\hat{\mathbf{z}} , \quad (34)$$

where

$$E_{\alpha\beta}(k) = \omega_{\alpha\beta}(k) + V_{\alpha\beta}(k), \quad V_{\alpha\beta}(k) = N\int dp \sum_{ll'} V(\nu_\alpha(k)l(p)|\nu_\beta(k)l'(p))\rho_{l'l}(p) , \quad (35)$$

and l,l' run over all particle and antiparticle species in the system. Notice that the processes $\nu_\alpha\nu_\beta \leftrightarrow \nu_\beta\nu_\alpha$ and $\nu_\alpha\bar{\nu}_\alpha \leftrightarrow \nu_\beta\bar{\nu}_\beta$ give rise to off-diagonal contributions to $\mathbf{V}$.

Quantum damping [11,12] by elastic and inelastic processes gives rise to the terms proportional to $\mathbf{P}_T$ in (7) where

$$D(k) = \pi N\int dk'dp'dp\,\delta_E(kp|k'p')\sum_j\{[V^2(\nu_\alpha(k),j(p)|\nu_\alpha(k'),j(p')) + V^2(\nu_\beta(k),j(p)|\nu_\beta(k'),j(p'))]n_j(p) + V^2(\nu_\beta(k),\bar{\nu}_\beta(p)|j(k'),\bar{j}(p'))n_{\bar{\nu}_\beta}(p) + V^2(\nu_\alpha(k),\bar{\nu}_\alpha(p)|j(k'),\bar{j}(p'))n_{\bar{\nu}_\alpha}(p)\} , \quad (36)$$

and

$$d(k,k') = 2\pi N\int dp'dp\ \delta_E(kp|k'p')\sum_j V(\nu_\alpha(k),j(p)|\nu_\alpha(k'),j(p'))V(\nu_\beta(k),j(p)|\nu_\beta(k'),j(p'))n_j(p') . \quad (37)$$

These effects arise because of collisional processes involving the overlap between the mixed-neutrino flavor eigenstates. Note that (36) includes contributions from both elastic and inelastic interactions. Since $D(k)\geq 0$ its effect will always be to damp the transverse part of the polarization vector, thus reducing the overlap (coherence) between the flavor components. In a very heuristic sense, it *might* be described as the result of a "measurement" of the flavor eigenvalue of the oscillating neutrino by collisions with other species [11,12]. In contrast, the term involving $d(k,k')$ can be positive or negative depending upon the relative phase inherent in $\mathbf{P}_T(k)\cdot\mathbf{P}_T^*(k')$. If the k and k' eigenstates happen to be oscillating in phase, the contribution involving $d(k,k')$ will reinforce (rather than destroy) the quantum coherence in the kth eigenstate.

Finally, and perhaps most interestingly, in active-active oscillation scenarios there will also arise dampinglike terms proportional to $\bar{\mathbf{P}}_T^*$. The terms involving $\mathbf{C}(k)$ and $\mathbf{c}(k,k')$, which are due to processes involving the off-diagonal part of the antineutrino density operator $\rho_{\bar{\beta}\bar{\alpha}}$, can be written

$$\mathbf{C}(k) = \pi N\int dk'dp'dp\,\delta_E(kp|k'p')\left|\sum_{\delta=\alpha,\beta} V(\nu_\alpha(k),\bar{\nu}_\alpha(p)|\nu_\delta(k'),\bar{\nu}_\delta(p'))V(\nu_\delta(k'),\bar{\nu}_\delta(p')|\nu_\beta(k),\bar{\nu}_\beta(p)) + \sum_i V(\nu_\alpha(k),\bar{\nu}_\alpha(p)|i(k'),\bar{i}(p'))V(i(k'),\bar{i}(p')|\nu_\beta(k),\bar{\nu}_\beta(p))\right|\bar{\mathbf{P}}_T^*(p) , \quad (38)$$

$$\mathbf{c}(k,k') = 2\pi N\int dp'dp\,\delta_E(kp|k'p')V(\nu_\alpha(k'),\bar{\nu}_\alpha(p')|\nu_\beta(k),\bar{\nu}_\beta(p))V(\nu_\alpha(k),\bar{\nu}_\beta(p)|\nu_\alpha(k'),\bar{\nu}_\beta(p'))\bar{\mathbf{P}}_T^*(p') , \quad (39)$$

where $\mathbf{P}_T^*(p)\equiv P_x(p)\hat{\mathbf{x}}-P_y(p)\hat{\mathbf{y}}$ is the reflection of $\mathbf{P}_T(p)$ in the x-y plane. These expressions bear some similarity to (36) and (37) which define $D(k)$ and $d(k,k')$. However, neither of these new contributions is either positive or negative definite. The contributions involving $\mathbf{C}(k)$ and $\mathbf{c}(k,k')$ will either destroy or reinforce quantum coherence depending upon whether $\mathbf{P}_T(k)\cdot\bar{\mathbf{P}}_T^*(p)$ and $\mathbf{P}_T(k')\cdot\bar{\mathbf{P}}_T^*(p')$, respectively, are positive or negative (cf. $D\geq 0$). Clearly then, no heuristic measurement paradigm can be evoked to explain these effects [nor to explain the $d(k,k')$ contribution]. It should be viewed, as should quantum damping for that matter, as simply resulting from the unitary evolution of quantum amplitudes through the S matrix. That is, the initial density operator is mapped forward in time by all the possible interceding operators contained in the Lagrangian [12].

For many purposes, it is only necessary to know the total number of particles within each species. For example, in the standard calculation of the primordial n/p ratio, rate equations are used to determine the evolution of the particle number densities integrated over momentum. In general, however, it is not possible to extract rate equations (or their quantum generalization) by integrating (28)–(31) over $\vec{k}$. This procedure fails because the momentum-dependent density operator matrix elements are necessarily convoluted together with the amplitudes in such a procedure. However, consider the ansatz

$$\rho_{ij}(k)=f(k)\rho_{ij}(\langle k\rangle)\ , \tag{40}$$

where $f(k)$ is the equilibrium-momentum distribution for a single Weyl degree of freedom normalized to unity so that $(\Omega/(2\pi)^3)\int d^3p\, f(p)=1$ which implies that $\sum_i n_i(\langle k\rangle)=1$. [For clarity we shall henceforth drop the $\langle k\rangle$ dependence and write $n_i(\langle k\rangle)\equiv n_i$. It is important to remember that n_i is normalized to unity under simple summation over the various particle species, whereas $n_i(p)\equiv n_i$ is normalized to unity under particle summation and momentum integration as defined by expression (2).] It then follows from (40) that $\rho_\nu(k)=\frac{1}{2}(P_0+\mathbf{P}\cdot\sigma)f(k)$ and $\rho_{ii}(k)=f(k)n_i$, etc., where P_0, $\mathbf{P}$, and n_i depend only upon the average momentum $\langle k\rangle$ which is simply a function of temperature. (This approximation is similar to one often employed in the classical kinetic theory of gases [21] where it is assumed that the velocity distribution can be factored out of a spatially varying distribution function. In that context, the justification given is that the hierarchy of relaxation times implies that local thermodynamic equilibrium is established much more quickly than spatial equilibrium.) In this approximation the total number of particles in each species can deviate from equilibrium via ρ_{ij}, but the momentum distribution is always proportional to the standard Fermi result. Clearly such an ansatz will only be appropriate so long as the evolution of the system can be taken to be largely independent of momentum. For the moment we shall assume that this is the case and return to examine its validity later.

Substituting (40) into (28)–(31) and integrating $[(\Omega/(2\pi)^3)\int d^3k]$ yields

$$\frac{d}{dt}\mathbf{P}=\mathbf{V}\times\mathbf{P}-D\mathbf{P}_T-C\bar{\mathbf{P}}_T^*+(R_\alpha-R_\beta)\hat{\mathbf{z}}\ , \tag{41}$$

$$\frac{d}{dt}\mathbf{P}_0=(R_\alpha+R_\beta)\ . \tag{42}$$

$$R_\delta=\sum_l F_{\delta j}[h_j n_j n_{\bar{j}}-n_{\nu_\delta}n_{\nu_\delta}]-\frac{1}{2}\sum_i G_i[\mathbf{P}_T\cdot\bar{\mathbf{P}}_T^*]\ , \tag{43}$$

$$\frac{d}{dt}n_i=\sum_j F_{ij}[h_j n_j n_{\bar{j}}-h_i n_i n_{\bar{i}}]+G_i[\mathbf{P}_T\cdot\bar{\mathbf{P}}_T^*]\ , \tag{44}$$

where the factor h_j takes account of the fact that electrons have twice as many Weyl degrees of freedom as neutrinos, thus $h_e=\frac{1}{4}$ and $h_\nu=1$. Further, we have defined the parameters C, D, and $\mathbf{V}$ as

$$C=\frac{P_0}{\bar{\mathbf{P}}_T^*}\int dk\left[\mathbf{C}(k)f(k)-\int dk'\mathbf{c}(k,k')f(k')\right]\ , \tag{45}$$

$$D=\int dk\left[D(k)f(k)-\int dk' d(k,k')f(k')\right]\ , \tag{46}$$

$$\mathbf{V}=\int dk\,\mathbf{V}(k)f(k)\ , \tag{47}$$

and (recalling that microscopic reversibility will hold)

$$\begin{aligned} G_i&=\int dk'dk\ dp'dp\ G_i(k'p'|kp)f(k)f(p)\ ,\\ F_{jj'}&=\int dk'dk\ dp'dp\ F_{jj'}(kp|k'p')f(k')f(p')\ . \end{aligned} \tag{48}$$

Note that the part of D which depends upon elastic processes vanishes if both neutrino flavors have the same interaction with the collision partner, i.e., if $V(\nu_\alpha,j|\nu_\alpha,j)=V(\nu_\beta,j|\nu_\beta,j)$. Or to put it another way, $D=0$ if the collision cannot differentiate between the two superimposed states. Also notice that D is never negative so that the overall effect of interactions is always to destroy quantum coherence and thereby increase the associated entropy. For a thorough discussion of this phenomena see Ref. [12]. Additionally G_i, C, and the off-diagonal contributions to the effective potential, all vanish if one of the mixed-neutrino species is sterile. The remaining expressions then reduce to those previously used in explicit calculations of sterile-active oscillations in the early Universe [3,4,15,16].

The rate-type equations (41)–(46) provide a major computational simplification compared with the full system of quantum kinetic equations (28)–(31). However, care must be taken since they will provide a reasonable description of the system only in some circumstances. In the next section we shall determine the momentum and temperature dependence of the parameters appearing in the QKE's, and QRE's; once this is done we can then discuss under what circumstances the momentum dependence can be factored out. First, however, let us conclude this section by addressing two interesting issues that are somewhat outside of the main development. These being the question of how to implement the QKE's and/or QRE's in the expanding early Universe and some points concerning the relation of the QKE's and/or QRE's formalism to the quantum measurement problem.

QKE's and/or QRE's in the expanding early Universe. The QKE's (28)–(31) and the QRE's (41)–(44) are formulated in terms of the canonical ensemble ($N=$const). However, in the context of the early Universe, it is necessary to take account of the changes in particle number density that occur as the Universe expands and cools. To see how this can be accomplished, consider the QRE's. We can always define a quantity $g_{\rm eff}$ to be the effective number of degrees of freedom via

$$g_{\rm eff}=\frac{1}{n_{\rm eq}}\frac{N}{\Omega}\ , \tag{49}$$

where $n_{\rm eq}$ is the number density of particles corresponding to a single fermionic (Weyl) degree of freedom in thermal equilibrium. It is then possible to rescale the relative populations n_i by the factor $g_{\rm eff}$ to form the new variables $\tilde{n}_i$ where $\sum_i \tilde{n}_i = g_{\rm eff}$. Furthermore, recalling that the electron and positron populations will be continuously in equilibrium so that $\tilde{n}_{e^+}=\tilde{n}_{e^-}=2$, it is easy to see that $g_{\rm eff}$ will *not* in general be a conserved quantity (unless $n_\nu=1$ for all neutrino flavors). Thus $g_{\rm eff}$ $(=\sum_i \tilde{n}_i)$ provides a continuous measure of how many effective neutrino degrees of freedom are in thermal equilibrium, this in turn allows the total energy density of the system to be evaluated. Having done so, it is a simple matter to determine the time evolution of the system's temperature on the basis of thermodynamic considerations. A full account of how this can be accomplished, which we shall not reproduce here, appears in Ref. [3].

The best way to implement the QKE's in the early Universe is to follow the approach adopted by Dodelson and Turner [22] in their analysis of incomplete neutrino decoupling (see also Bernstein [23]). The expansion of the Universe can be accommodated by rescaling the momentum variable by the scale factor $R(t)$ and then using the integrated energy densities to determine, via the Einstein equations, the time dependence of $R(t)$.

Quantum measurement and the QKE's and/or QRE's formalisms. It has been pointed out before [11,12] that the process of quantum damping (i.e., loss of coherence between the superimposed states) can be viewed as resulting from the effective "measurement" of the state of the neutrino by interactions with the environment. Conversely, modern ideas concerning the quantum measurement problem [24,25] stress that the transition from microscopic quantum behavior to macroscopic classical physics, results from the decoherence of quantum overlaps through interactions with the environment. Several authors [26–29] have constructed explicit, though idealized, models to demonstrate this phenomena. We are naturally led to ask do the QKE's and/or QRE's in any way provide a model of decoherence that is pertinent to the quantum measurement problem? In order to answer this question, we need to first clarify exactly what the one-body reduced density operator represents in the QKE's and/or QRE's context.

In their present form the QKE's and/or QRE's describe the evolution of the one-body reduced density operator, i.e., the average one-body quantum state of a neutrino in the system. Thus, statistics enters the problem in two ways: first through the hypothetical *ensemble* of neutrinos that are necessary in any definition of a quantum state; second by the *assembly* of Nn_ν neutrinos that go to make up the many-body system. Ostensibly the appearance of statistics on these two levels makes it hard to pin down exactly what the one-body reduced density operator refers to. For example, in the context of decoherence, we might inquire is the decoherence (quantum damping) something that applies to the relative phase of the two superimposed parts of the individual wave packet? Or does it refer only to the average coherence that can be assigned to the assembly of Nn_ν neutrinos in the plasma?

The answer to the above question is that the one-body reduced density operator refers principally to the theoretical construct of an *ensemble* of neutrinos, but that the distinction between *ensembles* and *assemblies* is, by assumption, moot in this circumstance. In the context of the aforementioned example, decoherence applies to the measurement results arising from the *ensemble* of neutrinos states that pertain to a single neutrino. However, in the reduced one-body approximation, the ability to distinguish between the theoretical construct of an *ensemble,* and the actual *assembly* of individual particles, has been lost. In fact, the distinction between the two concepts has been explicitly abandoned in order to simplify the problem.

There is an important exception to the above interpretation: the distinction between *ensemble* and *assembly* becomes manifest when we include the nonlinear terms in the QKE's and QRE's that depend upon the neutrino number densities. Clearly their presence is predicted on the assumption that the neutrinos are present in an actual physical *assembly* of Nn_ν particles. In contrast, an *ensemble* refers to a set of identically prepared systems each of which usually possesses only a single neutrino. In fact, things are not as contradictory as they may first appear. If the nonlinear terms are to be included, we should simply think of the one-body reduced density operator as describing an *ensemble* of *assemblies.*

Having clarified the significance of the one-body reduced density operator, we can now ask what, if any, relevance this work has to the question of quantum measurement. Consider the derivation from Secs. II and III, it is clearly possible to redo the whole development with $Nn_\nu=1$, that is, with a single neutrino. The resulting expression will take the same form as the QKE's (28)–(31), except that the nonlinear terms dependent on the mixed neutrino number densities will not arise. Thus the QRE's constitute a master equation describing the evolution of a single neutrino interacting with a background assembly of $N-1$ other particles. Or, to be more precise, it describes the evolution of the density operator corresponding to an ensemble of neutrinos, each of which interacts with an $N-1$ body system. Consistent with the discussion above, decoherence still occurs, even for a single neutrino. The fact that the QKE's and/or QRE's constitute master equations describing the decoherence of a single state must be of interest from the viewpoint of quantum measurement: The interactions with the environment destroy the quantum coherence and move the system towards a state whose description is indistinguishable

from that of a classical object. In the context of quantum measurement, this is an indication of how interactions stop microscopic quantum behavior from being manifested in the large scale world.

Earlier in this section it was mentioned that the measurement paradigm for quantum damping is misleading. That is, it is incorrect to assert that interactions between particles constitute microscopic measurements which result in decoherence. Recall that the derivation shows that decoherence results naturally from the smooth unitary evolution of an N-body system. In fact, it is much more illuminating to consider these issues in the reverse order: The unitary evolution of an N-body system leads naturally to decoherence; this in turn provides a microscopic basis for understanding why quantum measurements only ever result in single eigenvalues—why there are no macroscopic superpositions. However, it would be overly naive to assert that the transition from coherent, quantum, microscopic behavior to incoherent, classical, macroscopic physics is entirely accomplished by the above considerations. There is a technical problem in that (within the usual statistical interpretation of QM) the decoherence only really manages to map the initial *pure* quantum state into an *improper* mixture, whereas the classical world involves *proper* statistical mixtures of states (see d'Espagnat Ref. [30]).

Nevertheless, if we are willing to adopt the less conventional interpretation of Everett [31] whereby interactions lead to the continued branching of the Universe, the decoherence of the sort described by the QKE's and/or QRE's is sufficient to explain the apparent dichotomy between the microscopic and macroscopic domains (see Zurek [32] and references therein).

The above discussion has focused on the issue of decoherence (i.e., quantum damping). This is reasonable because the quantum damping terms are the only exotic terms that survive in the single neutrino limit, and it is the single neutrino limit that most clearly relates to the measurement problem. Keeping this in mind, let us consider the other interesting terms that arise in the QKE's and/or QRE's. Without recounting the details, the most important single fact to note is that some of the terms nonlinear in ρ_ν and $\bar{\rho}_\nu$ can result in an actual increase in the coherence in the average neutrino wave function. There are two comments to make in this regard.

(1) It therefore appears possible to contrive interactions which would increase, rather than decrease, the coherence of a quantum (sub)system. In the many-neutrino problem, these interactions arise naturally through neutrino-neutrino-antineutrino interactions. Still, one might consider constructing a series of interactions that are designed to enhance the coherence. Since such a procedure would require knowledge of the state of the system, it is tempting to invoke the concept of a *quantum* Maxwell's demon.

(2) It is possible to show [19], from rather general considerations, that any increase in the states coherence, is more than compensated for by an increase in entropy elsewhere in the system. One might, for example, envisage the entropy (coherence) associated with the neutrino's internal degree of freedom, decreasing (increasing) at the expense of a greater increase in the entropy associated with the kinetics of the system. Thus, as in the classical problem, Maxwell's demon is defeated when all the degrees of freedom are included.

It remains to be seen just how the QKE's and/or QRE's formalism can be exploited to further probe quantum decoherence and related issues. We hope that there may be some applications to realistic systems and possible experimental arrangements.

IV. DETERMINATION OF PARAMETERS FOR PARTICULAR OSCILLATION SCHEMES

During the epoch of interest ($1\ \text{MeV} \lesssim T \lesssim 100\ \text{MeV}$) the presence of the electron-positron plasma breaks lepton universality and singles out the electron neutrino species. Therefore, mixing scenarios involving the muon and τ neutrinos can be considered identical, but the schemes involving electron neutrinos must be treated separately. It follows that there are four distinct oscillation schemes that need to be considered: Those involving two weakly interacting neutrinos, $\nu_e \leftrightarrow \nu_\mu, \nu_\tau$ and $\nu_\mu \leftrightarrow \nu_\tau$, and those involving a weakly interacting neutrino and a sterile partner $\nu_s, \nu_e \leftrightarrow \nu_s$ and $\nu_\mu, \nu_\tau \leftrightarrow \nu_s$. In the remainder of this section we shall calculate the parameters $\mathbf{V}$, $F_{j,k}$, $\mathbf{C}$, $\mathbf{c}$, D, d, and G_i for each of these four schemes assuming the usual weak-interaction processes. (Wherever a choice arises between considering oscillations involving ν_μ or ν_τ, we will consider the ν_μ species to be the one participating in the oscillations. The results for oscillations involving ν_τ are readily deduced by the transformation $\nu_\mu \leftrightarrow \nu_\tau$.)

The effective potential can be extracted from the defining expression (35) by simply inserting the appropriate matrix element and performing the implicit sum over the spin degrees of freedom. Alternatively, the same results can be derived through the standard techniques of finite-temperature field theory. Calculations similar to the former approach can be found in Ref. [9] while the later method has been pursued at length elsewhere [33,34]. As an example, the calculation of the effective potential due to $\nu_e, \nu_\mu - e^-$ elastic scattering is contained in the Appendix wherein the usual Mikheyev-Smirnov-Wolfenstein (MSW) result is recovered. For a neutrino with momentum k propagating in an otherwise equilibrium plasma of temperature T, the results can be expressed as ($x = \sin^2\theta_W = 0.226$)

$$V_{ee}(k) = \sqrt{2} G_F \left[N_\gamma \left[L^{(e)} - 4\frac{\omega_{\nu_e}(k)}{M_W^2} \right] - \frac{8\omega_{\nu_e}(k)}{3M_Z^2} [\langle \omega_{\nu_e} \rangle N_{\nu_e} + \langle \omega_{\bar{\nu}_e} \rangle N_{\bar{\nu}_e}] \right], \tag{50}$$

$$V_{\mu\mu}(k) = \sqrt{2} G_F \left[N_\gamma L^{(\mu)} - \frac{8\omega_{\nu_\mu}(k)}{3M_Z^2} \times [\langle \omega_{\nu_\mu} \rangle N_{\nu_\mu} + \langle \omega_{\bar{\nu}_\mu} \rangle N_{\bar{\nu}_\mu}] \right], \tag{51}$$

where

$$L^{(e)}=(\tfrac{1}{2}+2x)L_e+(\tfrac{1}{2}-2x)L_p-\tfrac{1}{2}L_n+2L_{\nu_e}+L_{\nu_\mu}+L_{\nu_\tau}\,,$$
$$L^{(\mu)}=L^{(e)}-L_e-L_{\nu_e}+L_{\nu_\mu},\quad L_\delta=\frac{N_\delta-N_{\bar\delta}}{N_\gamma}\,, \tag{52}$$

while

$$N_\delta=N\frac{g_\delta}{(2\pi)^3}\int d^3p\, n_{\delta\delta}(p)\,, \tag{53}$$

is the number density of the δ-species particles and

$$\langle\omega_\delta\rangle=\frac{\Omega}{(2\pi)^3}\int d^3p\,\omega(p)n_{\delta\delta}(p)\,. \tag{54}$$

The QRE's limit is recovered by focusing on the evolution of the average momentum eigenstate, and by assuming that the momentum distribution of the neutrinos is proportional to its equilibrium form. Thus if we adopt the equilibrium values $\omega_\nu(k)=\langle\omega_\nu\rangle=(7\xi(4)/2\xi(3))T$ and $N_\nu=\frac{3}{8}N_\gamma n_\nu$, expressions (50) and (51) become

$$V_{ee}=\sqrt{2}G_FN_\gamma\left[L^{(e)}-A\frac{T}{M_W^2}\left[1+\frac{1-x^2}{4}|n_{\nu_e}+n_{\bar\nu_e}]\right]\right]\,, \tag{55}$$

$$V_{\mu\mu}=\sqrt{2}G_FN_\gamma\left[L^{(\mu)}-A\frac{T}{4M_Z^2}[n_{\nu_\mu}+n_{\bar\nu_\mu}]\right]\,, \tag{56}$$

where

$$A=\frac{14\zeta(4)}{\zeta(3)}\sim 12.61\quad\text{and}\quad N_\gamma=2\frac{\xi(3)T^3}{\pi^2}\,. \tag{57}$$

The effective potentials for antineutrinos $\bar V_{\delta\delta}$ follows from the expressions above via the transformation $L^{(\delta)}\to -L^{(\delta)}$.

However it has recently been pointed out that neutrino-neutrino [35] and neutrino-antineutrino [36] forward scattering amplitudes induce off-diagonal contributions to the effective potential. Indeed, it is easy to see from (35) that

$$V_{\alpha\beta}(k)=\sqrt{2}G_F\frac{N}{(2\pi)^3}\int dp\left[[\rho_{\alpha\beta}(p)-\rho_{\bar\alpha\bar\beta}(p)]-\frac{8\omega_\nu(k)}{3M_Z^2}\langle\omega_\nu\rangle\rho_{\alpha\beta}(p)\right]\,, \tag{58}$$

which reduces to the following expression in the QRE's approximation:

$$V_{\alpha\beta}=\frac{3\sqrt{2}G_FN_\gamma}{8}\left[(p_{\alpha\beta}-\rho_{\bar\alpha\bar\beta})-A\frac{2T}{3M_Z^2}\rho_{\alpha\beta}\right]\,. \tag{59}$$

This contribution only arises when both the mixed-neutrino species are weakly interacting. Note also that the higher-order correction due to the mass of the intermediate boson only occurs for neutrino-neutrino scattering. This is easy to understand because the forward-scattering process $\nu_\alpha(k)\nu_\beta(p)\to\nu_\alpha(p)\nu_\beta(k)$ requires the Z_0 to carry momentum so that a correction of order $(E/M_Z)^2$ must arise. In contrast, the corresponding antineutrino contribution comes about via the interaction $\nu_\alpha(k)\bar\nu_\alpha(p)\to\nu_\beta(k)\bar\nu_\beta(p)$ where no momenta is carried by the Z_0.

The order G_F^2 terms in the QKE's (28)–(31) can be written out explicitly by simply substituting in the appropriate products of weak-interaction matrix elements. It would be impractical, and quite unrevealing, to reproduce all the resulting expressions in full. Instead, we have chosen to include a detailed exposition of the contributions arising from a single specific interaction. The process chosen is neutrino-electron elastic scattering, and the explicit details are contained in the Appendix. It is expected that the interested reader will be easily able to generalize the example to include all other weak-interaction processes. Alternative sources that may also be helpful in this regard are Refs. [9,22,37,38] and most especially Ref. [39].

Although the full momentum integrals of the QKE's are cumbersome to deal with, the momentum-averaged results of the QRE's are quite compact. Therefore we shall take the opportunity to present them in full. The Appendix shows how the QRE's parameters can be consistently derived from the QKE's in the case of neutrino-electron elastic scattering. The contributions from other interactions can be similarly derived. (Alternatively, the QRE's parameters can be evaluated via standard techniques by directly averaging the total cross section over the equilibrium distributions of collision partners and the appropriate flux factor [3,37,38].)

To begin, let us look at the collision frequencies F_{ij}. The results for ν_e and ν_μ (ν_τ) collisions are equal to twice the values displayed in the third and fourth columns of Table I [comparison of (32) and (36) reveals that the damping factor is just half the collision frequency in the case of sterile-active oscillations] where the temperature dependent parameter F_0 is

$$F_0=\frac{G_F^2}{6\pi}\left[\frac{7\zeta(4)}{2\zeta(3)}\right]^2T^2N_\gamma(T)=\frac{49G_F^2}{12\pi^3}\frac{\zeta^2(4)}{\zeta(3)}T^5\,. \tag{60}$$

The remaining quantities D, C, and G_i can all be evaluated in much the same way. Except that, instead of squared amplitudes, products of amplitudes for two different processes are in general involved. The dynamics of the interactions are the same throughout so that structure of the calculation remains unchanged. All that needs to be done is to identify the constant factors which premultiply the various amplitudes. (In general the amplitudes will be proportional to functions of $x=\sin^2\theta_W$.) As an example, consider the contributions to D due to elastic scattering with $\bar\nu_e$ in the case of $\nu_e\leftrightarrow\nu_\mu$ oscilla-

TABLE I. The calculated values of D/F_0 for the various weak interaction process. Collision frequencies for the processes are equal to twice the quantities in the third and fourth columns.

	$\nu_e \leftrightarrow \nu_\mu$	$\nu_\mu \leftrightarrow \nu_\tau$	$\nu_e \leftrightarrow \nu_s$	$\nu_\mu \leftrightarrow \nu_s$
$\nu_e\nu \to \nu_e\nu$	$\frac{3}{2}n_{\nu_e}$	0	$6n_{\nu_e}$	$\frac{3}{2}n_{\nu_e}$
$\bar{\nu}_e\nu \to \bar{\nu}_e\nu$	$\frac{1}{2}n_{\bar{\nu}_e}$	0	$2n_{\bar{\nu}_e}$	$\frac{1}{2}n_{\bar{\nu}_e}$
$\nu_\mu\nu \to \nu_\mu\nu$	$\frac{3}{2}n_{\nu_\mu}$	$\frac{3}{2}n_{\nu_\mu}$	$\frac{3}{2}n_{\nu_\mu}$	$6n_{\nu_\mu}$
$\bar{\nu}_\mu\nu \to \bar{\nu}_\mu\nu$	$\frac{1}{2}n_{\bar{\nu}_\mu}$	$\frac{1}{2}n_{\bar{\nu}_\mu}$	$\frac{1}{2}n_{\bar{\nu}_\mu}$	$2n_{\bar{\nu}_\mu}$
$\nu_\tau\nu \to \nu_\tau\nu$	0	$\frac{3}{2}n_{\nu_\tau}$	$\frac{3}{2}n_{\nu_\tau}$	$\frac{3}{2}n_{\nu_\tau}$
$\bar{\nu}_\tau\nu \to \bar{\nu}_\tau\nu$	0	$\frac{1}{2}n_{\bar{\nu}_\tau}$	$\frac{1}{2}n_{\bar{\nu}_\tau}$	$\frac{1}{2}n_{\bar{\nu}_\tau}$
$e^-\nu \to e^-\nu$	$3n_{e^-}$	0	$(4x^2+3x+\frac{3}{4})n_{e^-}$	$(4x^2-3x+\frac{3}{4})n_{e^-}$
$e^+\nu \to e^+\nu$	n_{e^+}	0	$(4x^2+x+\frac{1}{4})n_{e^+}$	$(4x^2-x+\frac{1}{4})n_{e^+}$
$\nu_e\bar{\nu}_e \to e^-e^+$	$(4x^2+2x+\frac{1}{2})n_{\bar{\nu}_e}$	0	$(4x^2+2x+\frac{1}{2})n_{\bar{\nu}_e}$	0
$\nu_\mu\bar{\nu}_\mu \to e^-e^+$	$(4x^2-2x+\frac{1}{2})n_{\bar{\nu}_\mu}$	$(4x^2-2x+\frac{1}{2})n_{\bar{\nu}_\mu}$	0	$(4x^2-2x+\frac{1}{2})n_{\bar{\nu}_\mu}$
$\nu_\tau\bar{\nu}_\tau \to e^-e^+$	0	$(4x^2-2x+\frac{1}{2})n_{\bar{\nu}_\tau}$	0	0
$\nu_e\bar{\nu}_e \leftrightarrow \nu_\mu\bar{\nu}_\mu$	$\frac{1}{2}(n_{\bar{\nu}_e}+n_{\bar{\nu}_\mu})$	$\frac{1}{2}n_{\bar{\nu}_\mu}$	$\frac{1}{2}n_{\bar{\nu}_e}$	$\frac{1}{2}n_{\bar{\nu}_\mu}$
$\nu_e\bar{\nu}_e \leftrightarrow \nu_\tau\bar{\nu}_\tau$	$\frac{1}{2}n_{\bar{\nu}_e}$	$\frac{1}{2}n_{\bar{\nu}_e}$	$\frac{1}{2}n_{\bar{\nu}_e}$	0
$\nu_\mu\bar{\nu}_\mu \leftrightarrow \nu_\tau\bar{\nu}_\tau$	$\frac{1}{2}n_{\bar{\nu}_\mu}$	$\frac{1}{2}(n_{\bar{\nu}_\mu}+n_{\bar{\nu}_\tau})$	0	$\frac{1}{2}n_{\bar{\nu}_\tau}$

tions. Explicit evaluation shows that the process $\bar{\nu}_e\nu_\mu \leftrightarrow \bar{\nu}_e\nu_\mu$ proceeds at the rate $F_{\nu_e\nu_\mu}=F_0$. Since the amplitude for elastic scattering of like flavors is twice that for unlike flavors, $V(\nu_e\bar{\nu}_e|\nu_e\bar{\nu}_e)=2V(\nu_\mu\bar{\nu}_e|\nu_\mu\bar{\nu}_e)$. Substituting into (36), and comparison with (32), yields the following contribution to D: $\frac{1}{2}F_0(2\times2+1-2\times2)n_{\bar{\nu}_e}=\frac{1}{2}F_0n_{\bar{\nu}_e}$. The evaluation of the various contributions to D has been considered at length elsewhere [3,37,39] and the results are summarized in Table I.

The total value of D is calculated by summing all the contributions from the various channels, for example, the total value of D for $\nu_e \leftrightarrow \nu_\mu$ oscillations is, from Table I,

$$D=\tfrac{1}{2}F_0[16+3(n_{\nu_e}+n_{\nu_\mu})+(8x^2+4x+4)n_{\bar{\nu}_e}+(8x^2-4x+4)n_{\bar{\nu}_\mu}], \tag{61}$$

where we have used $n_{e^+}=n_{e^-}=2$. Note that the damping of antineutrino oscillations is given by (26) with the substitution $\nu \leftrightarrow \bar{\nu}$.

The same approach can be used to evaluate the parameters G_i and C since, as was the case above, these are simple products of amplitudes with the same underlying dynamical structure. These quantities have not been calculated before but it is easy to show that, for $\nu_e \leftrightarrow \nu_\mu$ oscillations,

$$G_{\nu_\tau}=4F_0,\quad G_e=(32x^2-4)F_0,\quad C=(16x^2+4)F_0(n_{\nu_e}+n_{\nu_\mu}), \tag{62}$$

while, for $\nu_\mu \leftrightarrow \nu_\tau$ oscillations,

$$G_{\nu_e}=4F_0,\quad G_e=(32x^2-16x+4)F_0,\quad C=(16x^2-8x+8)F_0(n_{\nu_\mu}+n_{\nu_\tau}). \tag{63}$$

In the scenario of $\nu_\mu \leftrightarrow \nu_s$ or $\nu_e \leftrightarrow \nu_s$ oscillations C and G_i both vanish because they depend upon products of amplitudes involving both of the oscillating neutrino flavors and, of course, if one of the neutrino flavors is sterile the product must then vanish.

The parameters G_i and C both enter the equations of motion at order G_F^2 just like the collision frequencies. It follows that they are every bit as important as the other second-order terms in (28)–(31). They can be ignored only after the neutrinos decouple from the rest of the primordial plasma at a temperature of 2–3 MeV.

V. COMPARISON BETWEEN QKE's AND QRE's

Now that we have calculated the particle physics contributions to the QKE's and QRE's, we can now address the question of when the QRE's provide a workable approximation to the full QKE's. Clearly, if the evolution of the system depends strongly on the momentum, then the approximation is poor indeed. Superficially it appears that this will always be so, since the neutrino oscillation frequency depends explicitly on the momentum. Often though, the phase of the oscillations is largely

unimportant, and other aspects of the systems behavior can be largely insensitive to the momentum dependence. To see this, let us examine the momentum dependence of the oscillation frequency, oscillation amplitude, and collision frequency—all as functions of the temperature of the remaining plasma. (Recall that the EM interaction will keep the charged leptons and photons in thermal equilibrium, regardless of the mixed neutrinos, so that a temperature is always definable for the system.)

Assuming that the matter eigenstates $|1\rangle, |2\rangle$ and the interaction eigenstates $|\alpha\rangle, |\beta\rangle$ are mixed according to

$$\begin{aligned} |1\rangle &= \sin\theta_0|\alpha\rangle + \cos\theta_0|\beta\rangle \ , \\ |2\rangle &= \cos\theta_0|\alpha\rangle - \sin\theta_0|\beta\rangle \ , \end{aligned} \tag{64}$$

then diagonalizing the propagation energy $E_{\alpha\beta}(k)$ yields the matter mixing angle $\theta_m(k)$ via

$$\sin^2 2\theta_m(k) = [1 - 2y(k)\cos 2\theta_0 + y(k)^2]^{-1} \sin^2 2\theta_0 \ , \tag{65}$$

where

$$y(k) = \frac{2k[V_{\alpha\alpha}(k) - V_{\beta\beta}(k)]}{\delta m^2}, \quad \delta m^2 = m_1^2 - m_2^2 \ . \tag{66}$$

Similarly, the instantaneous oscillation frequency is given by

$$f_{\rm osc}(k) = |\mathbf{V}| = \frac{|\delta m^2|}{2k}[1 - 2y(k)\cos 2\theta_0 + y^2(k)]^{1/2} \ . \tag{67}$$

In the standard big bang scenario it is usually assumed that the (directly unobserved) neutrino asymmetries L_{ν_α} are comparable to the observed baryon asymmetry $L_b \sim 10^{-10}$. In this case it is possible to show [33,34] that the contributions of $L^{(e)}$ and $L^{(\mu)}$ to (49), (50) are largely unimportant during the epoch of interest. It follows then that $y(k) \propto k^2 T^4$ in the standard scenario. Alternatively, if a significant neutrino asymmetry is postulated, the effective potential will only depend on the number density of scattering particles so that $y(k) \propto kT^3$. These power-law functions for $y(k)$ ignore the feedback contribution of the mixed neutrino species, but this is reasonable since the effective potential due to the remainder of the plasma will dominate.

The temperature and momentum dependence of the collision frequencies have already been examined in the previous section and in the Appendix. From (A16) and (A13) the collision frequency for a neutrino of momentum k will be proportional to kT^4. Therefore the average collision frequency will go like T^5 provided that the neutrinos are in equilibrium, or at least, if the momentum distribution scales like the temperature.

Having determined the momentum and temperature dependence of $y(k)$ [i.e., $\theta_m(k)$ and $f_{\rm osc}(k)$] and $f_{\rm coll}(k)$, we can now ask in what situations one might be satisfied with the QRE's approximation. The important point to note is that the physical quantities of interest are simple (usually monotonic) functions of the momentum. As will be argued below, the QRE's approximation is justified when the results of interest do not depend sensitively on the frequency (and therefore the phase) of oscillation $f_{\rm osc}(k)$.

To begin, let us note that the ansatz (40) is not unreasonable because the mixed-neutrino ensemble will possess a momentum distribution that *at least* roughly resembles the equilibrium distribution at that temperature. By "roughly resembles," we simply mean that the momentum distribution scales like the temperature of the remainder of the plasma. This is a reasonable (and modest) assertion, since at least half of the mixed-neutrino sector will remain in contact with the EM sector until decoupling. Note that this is a far weaker assumption than (40). Indeed, oscillations may still partition this quasiequilibrium distribution in a decidedly momentum-dependent manner between the two flavors—in clear contradiction to the strict QRE's ansatz.

The evolution of a given momentum eigenstate will depend principally on the instantaneous values which $\theta_m(k)$, $f_{\rm osc}(k)$, and $f_{\rm coll}(k)$ acquire. The matter mixing angle θ_m determines the instantaneous amplitude of the neutrino oscillations; therefore, it governs both the rate at which sterile neutrons can be brought into equilibrium, and the degree to which the electron-neutrino population can be depleted. Similarly, the collision frequency $f_{\rm coll}(k)$ is important in determining the rate at which collisions bring additional neutrinos into equilibrium, as well as in defining the point at which the neutrinos decouple. On the other hand, the instantaneous oscillation frequency $f_{\rm osc}(k)$ is not so important, except in so far as it affects the ability of $n_\nu(\langle k \rangle)$ to accurately represent $\langle n_\nu(k) \rangle$ through feedback on the RHS of the QRE's. [The exception to this statement is when the frequency of $\nu_e \leftrightarrow \nu_s$ oscillations is slow, $f_{\rm osc}(k) \lesssim H^{-1}$, because in such a circumstance the phase of the oscillation will be important.] Since temperature provides a convenient reparametrization of the systems evolution, we can reformulate our original question as follows: How do the various momentum eigenstates evolve as a function of temperature, and in particular, is the $\langle k \rangle$ state somehow representative of the evolution of the rest of the momentum distribution?

In terms of the collision frequency $f_{\rm coll}(k)$ and the mixing angle $\theta_m(k)$, the average momentum eigenstate reasonably approximates the evolution of the entire ensemble in the following way. Usually (away from resonance) $f_{\rm coll}(k)$ and $\theta_m(k)$ are monotonic functions of k, so that while the part of the ensemble with $k > \langle k \rangle$ experiences enhanced (quenched) values, the $k < \langle k \rangle$ part experiences quenched (enhanced) values. Thus, the values of $f_{\rm coll}(k)$ and $\theta_m(k)$ attained by the $\langle k \rangle$ state will be midway between the values achieved by the remainder of the distribution. In this sense, the relevant parameters pertaining to the average momentum state are, more or less, the average values of those parameters across the distribution. The one exception to this is around a point of resonant enhancement. In such a circumstance, we must be content with the fact that the various momentum eigenstates will reach resonance in sequence, one after the other. Moreover, the $k = \langle k \rangle$ momentum eigenstate will still represent an average behavior of the entire momentum distribution in the sense that it will attain resonance before the $k < \langle k \rangle$, and after the $k > \langle k \rangle$, parts of the distribution.

The argument above does not apply to the oscillation

frequency $f_{\rm osc}(k)$ which affects the evolution of the system, not only through its instantaneous value, but also through the integrated phase that accrues over time. For this reason, and in contrast with $f_{\rm coll}(k)$ and $\theta_m(k)$, it is irrelevant that $f_{\rm osc}(\langle k \rangle)$ might be considered a reasonable approximation to $\langle f_{\rm osc}(k) \rangle$. The point is that no matter how limited a range the oscillation frequencies occupy, the relative phases will always grow with time. Thus, in order to employ the QRE's approximation, we require that it is possible to ignore the relative phase of the oscillations across the momentum distribution. The phase of the oscillations can be important in two ways: (i) via the feedback terms in the equations of motion, especially for active-active oscillations, and, (ii) through the explicit oscillation-induced time dependence of the neutrino number densities, especially the electron-neutrino number density at the epoch when the n/p ratio is fixed. Thus, although in some respects the momentum dependence can be ignored, we need to keep in mind that circumstances can arise where the accumulated phase, and therefore the momentum dependence, is important.

To briefly conclude, the argument for adopting the QRE's approximation looks reasonable, if the relative oscillation phases can be ignored. Or conversely, we have established that the QRE's approximation will fail when the relative oscillation phase is important in the feedback terms, or if the resulting neutrino populations depend upon the phase of the oscillations. [As they will, for example, in the case where $f_{\rm osc}(k) \lesssim H^{-1}$ for some appreciable range of k values in the distribution.] It remains for us to examine when these criteria are likely to be violated in particular circumstances. This is most easily accomplished by examining the possible neutrino oscillation scenarios in detail. First, however, it will be useful to recall exactly how the various oscillation scenarios might affect nucleosynthesis.

The predicted primordial light element abundances depend upon the neutrino populations in two quite separate ways. First, the total effective number of neutrino degrees of freedom in equilibrium at the time of nucleosynthesis, contributes to the total energy density which in turn, determines the rate of expansion of the Universe. If the Universe expands too quickly, there will be insufficient time for the β decay of the neutrons and the initial neutron-to-proton ratio is driven up. The latest calculation [40] gives the bound $N_\nu \leq 3.3$, where N_ν is the effective total number of neutrino degrees of freedom in equilibrium at about $T \approx$ MeV. Since we know from the recent measurement of the Z_0 width at the CERN e^+e^- collider LEP that there are three light doublet neutrinos, this leaves only 0.3 extra degrees of freedom available. The second way that neutrinos affect primordial nucleosynthesis is through the direct participation of electron neutrinos in nonequilibrium process such as $\nu_e + n \leftrightarrow e^- + p^+$ and $\bar{\nu}_e + n \leftrightarrow e^+ + p^-$. Changes in the relative, or absolute, number densities of ν_e and $\bar{\nu}_e$ will alter the resulting neutron-to-proton ratio. Note that it is only the electron flavor neutrinos and antineutrinos which participate in these lepton-nucleon reactions.

Thus, neutrino oscillations can affect the standard primordial nucleosynthesis scenario in two ways: (a) by altering the total effective number of neutrino species and (b) by altering the electron (anti)neutrino populations. Let us now consider how these mechanisms can arise in the various oscillation schemes, and determine what approach, QKE's or QRE's, is the most appropriate method of calculation.

A. Active-sterile oscillations

This scenario has been considered at length elsewhere; therefore, we shall only need to outline the possibilities that can arise. The interested reader should consult Refs. [4,13,15] for more detail and in particular Ref. [3] for an explicit numerical calculation in the QRE's limit. Briefly though, the two mechanisms outlined above both arise naturally in the case of oscillations between a doublet neutrino and an initially unpopulated sterile state, especially for $\nu_e \leftrightarrow \nu_s$ mixing.

Oscillations between a doublet neutrino and a sterile partner effectively provide an interaction channel for the inert state so that it can be brought into thermal equilibrium. This will occur for both $\nu_e \leftrightarrow \nu_s$ and $\nu_\mu, \nu_\tau \leftrightarrow \nu_s$ oscillations so long as the oscillations arise before the neutrinos decouple, since the depleted doublet neutrino population must be "refilled" by inelastic interactions with the electron-positron plasma if there is to be an increase in the number of neutrino degrees of freedom. Clearly this must be avoided so as not to violate the bound on extra neutrino species.

It turns out that the excitation of additional neutrino degrees of freedom is well described by the QRE's approach. Because the oscillations occur before the neutrinos decouple, the actual oscillations are largely damped out [3,5,34], and the neutrino populations change smoothly with little evidence of any oscillation. (For numerical demonstrations of this phenomena see Refs. [3,34].) Clearly the phase of the oscillations is unimportant is such a circumstance and the QRE's approach can be adopted. However, as the point at which extra neutrinos are brought into thermal equilibrium approaches the point at which the neutrinos decouple, the damping of the oscillations becomes less efficient. Recall however, that in such a scenario the phase of the oscillations only really becomes important through feedback. It follows that, provided we are willing to accept the (presumably small) error introduced by ignoring the oscillations in the feedback, the QRE's approach can still be used. Thus, except perhaps for oscillations that occur just before the neutrinos decouple, the QRE's approach is clearly sufficient for examining the process of extra neutrinos being brought into thermal equilibrium. Moreover, it would be hard to justify the extra numerical work required to implement the QKE's approach in this circumstance, especially given that the appearance of the phase in the feedback terms is only a secondary effect.

Alternatively the electron (anti)neutrino populations can be depleted by oscillations with a sterile partner, but in contrast with the discussion above, the oscillations must arise *after* neutrino decoupling so that inelastic processes do not repopulate the ν_e and $\bar{\nu}_e$ populations. Interestingly, even $\nu_\mu \leftrightarrow \nu_s$ oscillations can alter the ν_e and

$\bar{\nu}_e$ populations provided that the oscillations arise at about the time of decoupling. This is the case because the inelastic channel $\nu_e\bar{\nu}_e \leftrightarrow \nu_\mu\bar{\nu}_\mu$ can partially reduce the ν_e and $\bar{\nu}_e$ populations in an attempt to "refill" the $\nu_\mu\bar{\nu}_\mu$ states which have been depleted by oscillations. This nonequilibrium process can then be "frozen out" if it is incomplete at the time of neutrino decoupling.

In this second scenario, the oscillation are not damped, and the frequencies of oscillation cannot be ignored. However, this is still a circumstance where the QRE's approach is largely justified. Indeed, the phase of the oscillations is not crucial unless the oscillations are slow compared to the expansion of the Universe. Leaving aside slow oscillations, the effect of the oscillations only really impacts through the time average of the electron neutrino population, that is, assuming that many oscillations occur during the epoch when n/p is fixed (see Ref. [3] for more details on this matter). It follows that the QRE's approach is still justified, provided that the oscillation period is small compared to the time scale over which the ratio n/p is fixed. Nevertheless, if slow oscillations occur, one should really go back and integrate the full QKE's. An intermediate approach would be to look at a set of QRE's integrations for different momenta, unfortunately this approach would ignore the feedback contributions. Thus, the fact that the oscillations for various momentum eigenstates are different, can very often be ignored. Similarly, resonant conversion between sterile and electron neutrinos are amenable to the QRE's approach; one just needs to check that most of the distribution has undergone conversion before $T \sim 1$ MeV.

Active-sterile oscillations are usually well-described within the QRE's approach, except for some specific circumstances where the approximation is questionable. However, these only occur for a very restricted range of the possible mixing parameters.

B. Active-active oscillations

It is fair to say that neutrino oscillations are discussed much more often in the context of active-active mixing than in sterile-active mixing. This bias is understandable because no new neutrino species need be added in the former scenario whereas the latter scheme requires new inert flavors to be introduced. However, although active-active neutrino mixing is more naturally realized, it has not attracted much attention in the context of the early Universe. The reason is that the simplest active-active oscillation scenarios have no appreciable effect. More specifically, in a scenario where the doublet (anti)neutrino populations are in equilibrium with zero chemical potential, active-active oscillations cannot excite extra degrees of freedom or change the ν_e or $\bar{\nu}_e$ populations.

Quite simply, active-active oscillations only induce transformations between doublet neutrino species, and if all the doublet neutrino populations are initially equal, the net effect of the oscillations must be no change at all. Still though, one might try to search for some small effect, since lepton universality is destroyed by the e^+e^- plasma. Langacker *et al.* [17] examined this possibility closely, by taking account of the small temperature difference between the ν_e and $\nu_\mu\nu_\tau$ populations. The final conclusion being that any effect must be entirely insignificant. [Although this calculation [17] ignored the off-diagonal contributions to the effective potential (57) and (58) and the new higher order corrections in (27)–(30) and (40)–(43), the conclusion is nevertheless correct. The argument simply rests on the observation that neutrino number is conserved, and if oscillations do occur they can only result in the swapping of quanta between almost equally populated flavors.]

At this point the reader might wonder what the motivation was in presenting the complete equations of motion for active-active oscillations (aside from the fact that they contain curious quantum physical effects that are interesting in their own right). The answer is that the above comments are made with the understanding that there is no neutrino degeneracy. Indeed, the implicit assumption in the above discussion, and in the standard primordial nucleosynthesis calculation, is that the chemical potentials of the neutrino species vanish. In contrast with the baryon number asymmetry which we can readily observe today, the neutral lepton number is hidden from us by the feebleness of the weak interaction. However, the assumption that $L \approx B$ is a prejudice, which is, at best, only based upon a conjectured mechanism for the generation nonzero baryon number. It follows that L_{ν_e}, L_{ν_μ}, and L_{ν_τ} should be treated as free parameters in the initial conditions of big-bang nucleosynthesis calculations. Indeed, this has been recognized for some time, and the possibility of nucleosynthesis with degenerate neutrinos has been repeatedly examined [41] in the last decade. In fact, it is well known that a loophole exists which allows the usual light element abundances to be reproduced despite significant neutrino degeneracies.

If we admit the possibility of neutrino degeneracy, active-active oscillation schemes gain considerable potency. If the various neutrino populations are unequal, oscillations between them can have a profound effect. To calculate the evolution of such a scenario would be a difficult problem. (Although numerical studies have shown that *small* lepton asymmetries are dynamically damped away during oscillations [34].) To begin with, the introduction of nonzero lepton numbers couples the neutrino and antineutrino sectors together in very direct and nonlinear way—no longer do the two sectors decouple. Further, the number of free parameters involves not only the mass-squared difference and the vacuum-mixing angle, but also the three initial lepton asymmetries. Clearly it would not be sensible to scan through the range of possible values for these five parameters to see which combinations do, or do not, leave the light element abundances unaffected. However, if active-active neutrino mixing were to be experimentally seen (we need not remind the reader of the tantalizing hints in this direction from solar neutrino experiments [1]) then the mixing angle and mass splitting would be known, and the problem would then be tractable. In the future, if active-active neutrino oscillations are accepted as the solution to the solar neutrino problem, the motivation for examining scenarios with nonzero L_i would be every bit as compelling as that for examining these scenarios without neutri-

no oscillations today [41].

Clearly, provided that neutrino degeneracy occurs, active-active neutrino oscillations can affect nucleosynthesis. Which approach then, the QKE's or QRE's, is the most appropriate way to model active-active neutrino oscillations? Since there are no extra neutrino degrees of freedom available, the only possible mechanism to consider is the oscillation-induced transformations $\nu_e \leftrightarrow \nu_\mu$ and $\nu_e \leftrightarrow \nu_\tau$, both before and after neutrino decoupling. In modeling such oscillations before decoupling, the new exotic $[O(G_F^2)]$ effects in the QKE's and/or QKE's will come into play. Clearly, they depend explicitly on the relative phase between the various momentum eigenstates, thus implying that the QKE's approach is most appropriate. Nevertheless, one might attempt to implement the QRE's approach in this circumstance by noting that the relative phases of the oscillations will rapidly grow so that a random phase approximation might be constructed. That is, one could explicitly assume that the relative phases between the various momentum eigenstates are distributed randomly, and therefore substitute appropriate average values into the feedback term. (This same approach might also be a useful way of improving calculations of the active-sterile oscillation scenario.) As for oscillations that arise after decoupling, the QRE's approach is probably a reasonable approximation provided that the oscillation frequency is not too slow.

In practice, the most prudent approach would be to begin by examining the dynamics of active-active oscillations using the QRE's approximation (perhaps for a range of momentum eigenvalues) with a random phase assumption. This would provide a base from which to determine when the full QKE's approach needs to be deployed. Of course, such a detailed and time consuming investigation would only be justified if active-active neutrino mixing were to be otherwise established (and perhaps even then only if there were some indication that neutrino degeneracy might arise).

VI. CONCLUSION

We have derived the general QKE's and QRE's describing arbitrary neutrino oscillations in the early Universe. The QKE's and/or QRE's includes curious new contributions to the Boltzmann-Pauli equation that result from the inherent quantum nature of neutrino oscillations. Previously established results involving neutrino oscillations and quantum damping have been recovered in the appropriate limits. However, it should be noted that we have worked from a "first principles" approach throughout, yet have been able to reproduce results that are more usually derived through direct, separate, "by hand" calculations.

By assuming standard weak-interaction processes, the calculation of the collision kernels has been outlined and the explicit results in the QRE's limit have been tabulated for both active-active and active-sterile oscillations.

In addition to outlining the well-known possibility of sterile-active oscillations affecting nucleosynthesis, we have described a scenario whereby active-active neutrino mixing could alter the standard primordial nucleosynthesis model. The QKE's and QRE's both have a role to play in investigating such scenarios. Indeed, the QRE's approach has already been employed elsewhere to study sterile-active oscillations. It is clear that the general investigation of active-active oscillations is at present impractical, however the future results from solar (and terrestrial) neutrino oscillation experiments may not only reduce the number of unknown mixing parameters but may also make such an investigation imperative. In that case, the QKE's and/or QRE's formalisms developed here will need to be used, including especially, the new quantum effects that have been shown to occur with active-active oscillations.

Finally, it remains to be seen exactly how the new quantum effects can be understood. Even the process of quantum damping (which has been known for many years) still lacks a definitive interpretation. It may even be that the QKE's and/or QRE's will provide a useful tool for investigating the quantum measurement problem.

Note added in proof. Since the completion of this work the following papers on related issues have appeared: X. Shi, D. N. Schramm, and B. D. Fields, Phys. Rev. D **48**, 2563 (1993); V. Alan Kostelecký and Stuart Samuel, *ibid.* **49**, 1740 (1994); V. Alan Kostelecký, J. Pantaleone, and Stuart Samuel, Phys. Lett. B **315**, 46 (1993); Stuart Samuel, Phys. Rev. D **48**, 1462 (1993).

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APPENDIX

The purpose of this appendix is to demonstrate how the standard weak-interaction matrix elements can be employed in the QKE's collision integrals (28)–(31), and thereby be used to calculate the corresponding QRE's parameters. It would be impractical to reproduce here every possible weak-interaction contribution to the QKE's. Fortunately, all the important points that arise in practice can be demonstrated with a representative example; elastic scattering between neutrinos and electrons. It is expected that the interested reader will have no trouble in generalizing the analysis below to other interactions. The interaction chosen ($\nu - e^-$ elastic scattering) only contributes to the third and fourth terms in the QKE (28):

$$-D(k)\mathbf{P}_T(k)+\int dk' d(k,k')\mathbf{P}_T(k') \ . \tag{A1}$$

As it happens, the evaluation of products of amplitudes in (A1) is quiet representative of the other second-order $(0G_F^2)$ terms in the QKE's (28)–(31). Let us then proceed to calculate $D(k)$ and $d(k,k')$.

From the defining relations (35) and (36), and for the chosen interaction $\nu(k)$, $e(p) \leftrightarrow \nu(k')$, $e(p')$, $D(k)$, and $d(k,k')$ are given by

$$D(k)=\pi N \int dk' dp' dp \, \delta_E(kp|k'p')[V^2(\nu_\alpha(k),e(p)|\nu_\alpha(k'),e(p'))+V^2(\nu_\beta(k),e(p)|\nu_\beta(k'),e(p'))]n_e(p) \ , \tag{A2}$$

and

$$d(k,k')=2\pi N\int dp'dp\,\delta_E(kp|k'p')V(\nu_\alpha(k),e(p)|\nu_\alpha(k'),e(p'))V(\nu_\beta(k),e(p)|\nu_\beta(k'),e(p'))n_e(p')\ . \tag{A3}$$

Following Rudzsky [9] it is convenient to write the matrix element $V(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')$ in the form

$$V(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')=(\delta_{\mathbf{i}+\mathbf{k},\mathbf{i}'+\mathbf{k}'}/\Omega)\mathcal{N}_{\mathbf{i}}\mathcal{N}_{\mathbf{k}}\mathcal{N}_{\mathbf{i}'}\mathcal{N}_{\mathbf{k}'}\mathcal{V}(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')\ , \tag{A4}$$

where $\mathcal{N}_{\mathbf{i}}=1/\sqrt{2\omega_{\mathbf{i}}}$ for massless fermions and $\mathcal{N}_{\mathbf{i}}=\sqrt{m_i/\omega_i}$ for massive fermions, while $\mathcal{V}(\mathbf{i},\mathbf{k}|\mathbf{i}',\mathbf{k}')$ is the weak-interaction matrix element. For the process in question,

$$\begin{aligned}\mathcal{V}_\alpha&\equiv\mathcal{V}(\nu_\alpha(k),e(p)|\nu_\alpha(k'),e(p'))\\&=\frac{1}{2\sqrt{2}}G_F[\bar\nu_\alpha(k)\gamma_\mu(1-\gamma_5)\nu_\alpha(k')][\bar e(p)\gamma^\mu(C_\alpha(1-\gamma_5)+D_\alpha(1+\gamma_5))e(p')]\ ,\end{aligned} \tag{A5}$$

where we have adopted the convenient notation introduced by Flaig [39], so that $C_e=2X+1$, $D_e=2X$, $C_\mu=2X-1$, $D_\mu=2X$, and $X=\sin^2\theta_W$. In the thermodynamic limit the Kronecker delta $\delta_{k+p,k'+p'}$ in (A4) becomes $[(2\pi)^3/\Omega]\delta^3(k+p,k'+p')$ and the matrix elements can then be substituted into (A2) and (A3) recalling that $\int dp\equiv[\Omega/(2\pi)^3]\int d^3p$ to yield

$$D(k)=\frac{\pi N}{(2\pi)^6}\int d^3k'd^3p'd^3p\,\delta^4(k,p|k'p')\frac{m_e^2}{4}\frac{1}{\omega(k)}\frac{1}{\omega(k')}\frac{1}{\omega(p)}\frac{1}{\omega(p')}[\mathcal{V}_\alpha^2+\mathcal{V}_\beta^2]n_e(p)\ , \tag{A6}$$

and

$$d(k,k')=\frac{2\pi N}{\Omega(2\pi)^3}\int d^3p'd^3p\,\delta^4(k,p|k'p')\frac{m_e^2}{4}\frac{1}{\omega(k)}\frac{1}{\omega(k')}\frac{1}{\omega(p)}\frac{1}{\omega(p')}\mathcal{V}_\alpha\mathcal{V}_\beta n_e(p')\ , \tag{A7}$$

where an average over electron spins is implicit and $\delta^4(x)=\delta_E(x)\delta^3(x)$. It is easy to show [39] that, for the amplitude (A5),

$$\overline{\mathcal{V}}_\alpha\mathcal{V}_\beta=\frac{8G_F^2}{m_e^2}[C_\alpha C_\beta(k\cdot p)(k'\cdot p')+D_\alpha D_\beta(k\cdot p')(k'\cdot p)]\ , \tag{A8}$$

where terms of order (m_e^2/E) have been ignored.

Thus we see that the kernels of the collision integrals follow simply from the weak-interaction matrix element (A5). No extra difficulties arise in performing the same procedure for other amplitudes and their respective contributions to $D(k)$, $d(k,k')$, $C(k)$, $c(k,k')$, etc. A particularly valuable source for problems of this nature is the work of Flaig [39], who considers in detail the calculation of $D(k)$ using standard weak interaction theory. Clearly, it is possible to simplify expressions (A6) and (A7) by explicitly doing the integrations—this is especially easy in the case of (A6). However, let us delay the reduction of the collision integrals until after going to the QRE's limit.

Substituting (A6), (A7), and (A8) into the defining expression for D (46) yields

$$\begin{aligned}D=&\frac{\Omega N}{(2\pi)^8}\int d^3kd^3k'd^3p'd^3p\frac{1}{\omega(k)}\frac{1}{\omega(k')}\frac{1}{\omega(p)}\frac{1}{\omega(p')}\delta_E(k,p|k'p')f(k)f(p)2G_F^2\frac{n_e}{2}\\&\times\left[\left[\frac{1}{2}[C_\alpha^2+C_\beta^2]-C_\alpha C_\beta\right](k\cdot p)(k'\cdot p')+\left[\frac{1}{2}[D_\alpha^2+D_\beta^2]-D_\alpha D_\beta\right](k\cdot p')(k'\cdot p)\right]\ ,\end{aligned} \tag{A9}$$

where we have exploited the time reversal invariance of the products of matrix elements (A8), and used the fact that $n_\nu=f(k)\rho_\nu$, while $n_e(p)=f(p)(n_e/2)$ because the normalization requires that $n_e=2$ at equilibrium. (Although there are two spin degrees of freedom for the electron, the spin sum has already been included in the evaluation of the squared matrix element.) Following the analysis of Flaig [39], the k' and p' integrations can be performed by noting that, in the extreme relativistic limit,

$$\begin{aligned}&\int\frac{d^3k'}{\omega(k')}\frac{d^3p'}{\omega(p')}\delta^4(k,p|k'p')(k\cdot p)(k'\cdot p')=2\pi(k\cdot p)^2\ ,\\&\int\frac{d^3k'}{\omega(k')}\frac{d^3p'}{\omega(p')}\delta^4(k,p|k'p')(k\cdot p')(k'\cdot p)=\frac{2\pi}{3}(k\cdot p)^2\ ,\end{aligned} \tag{A10}$$

so that (A9) becomes

$$D=\frac{\Omega N}{(2\pi)^7}\int d^3k\,d^3p\frac{1}{\omega(k)}\frac{1}{\omega(p)}(k\cdot p)^2f(k)f(p)\frac{n_e}{2}2G_F^2\left[\left[\frac{1}{2}[C_\alpha^2+C_\beta^2]-C_\alpha C_\beta\right]+\frac{1}{3}\left[\frac{1}{2}[D_\alpha^2+D_\beta^2]-D_\alpha D_\beta\right]\right]\ . \tag{A11}$$

Next, the p integration can be performed once it is noticed that

$$\frac{1}{\omega(k)}\frac{N}{(2\pi)^3}\int\frac{d^3p}{\omega(p)}f(p)(k\cdot p)^2=\frac{4}{3}\omega(k)\epsilon\ , \tag{A12}$$

where, by definition, ϵ is the energy density associated with a single Weyl degree of freedom in thermal equilibrium:

$$\epsilon=\frac{N}{(2\pi)^3}\int d^3p\, f(p)\omega(p)=\frac{7\pi^2}{240}T^4\ . \tag{A13}$$

Substitution of (A12) into (A11) yields

$$D=\frac{8G_F^2}{6\pi}\left[\left[\frac{1}{2}[C_\alpha^2+C_\beta^2]-C_\alpha C_\beta\right]+\frac{1}{3}\left[\frac{1}{2}[D_\alpha^2+D_\beta^2-D_\alpha D_\beta\right]\right]\epsilon\langle\omega\rangle\frac{n_e}{2}\ , \tag{A14}$$

where, again by definition, $\langle\omega\rangle$ is the average energy of a Fermion in thermal equilibrium:

$$\langle\omega\rangle=\frac{\Omega}{(2\pi)^3}\int d^3k\,\omega(k)f(k)=\left[\frac{7\zeta(4)}{2\zeta(3)}\right]T\ . \tag{A15}$$

In the instance of $\nu_e\leftrightarrow\nu_\mu$ [$\nu_e\leftrightarrow\nu_s$] oscillations (A14) becomes $D=3F_0n_e$ [$D=F_0(4X^2+3X+\frac{3}{4})n_e$] where

$$F_0=\frac{4}{9\pi}G_F^2\epsilon\langle\omega\rangle\ , \tag{A16}$$

which is equivalent to the earlier definition (60). In a similar manner all the other QRE's parameters can be calculated for the four possible oscillation schemes. The results, which agree with those previously calculated [3,39], appear in Table I. It is important to note that the earlier calculations of D essentially convoluted the cross section, flux factor, and momentum distribution, by hand. In contrast, the present development simply requires inserting the appropriate matrix elements into the general expression derived from the QKE's.

Finally, let us conclude this appendix with a brief example of how the effective potential can be directly calculated from the QRE's and/or QRE's formalism. For simplicity, we shall ignore the $O(E/M_Z)^2$ corrections and simply reproduce the usual MSW result for active-active oscillations. The weak-interaction matrix elements (A4) and (A5) can be explicitly evaluated in the rest frame of an unpolarized medium [9] to yield

$$\begin{aligned}V_e&=V(\nu_e(k),e(p)|\nu_e(k'),e(p'))=\frac{G_F(C_e+D_e)}{\sqrt{2}\Omega}\ ,\\ V_\mu&=V(\nu_\mu(k),e(p)|\nu_\mu(k'),e(p'))=\frac{G_F(C_\mu+D_\mu)}{\sqrt{2}\Omega}\ .\end{aligned} \tag{A17}$$

Substitution into the defining expression (35), and using the definition (54), then gives

$$\begin{aligned}V_{ee}(k)-V_{\mu\mu}(k)&=\frac{NG_F\Omega g_e}{(2\pi)^3}\int d^3p\,(V_e-V_\mu)n_e(p)\\&=\sqrt{2}G_FN_e\ .\end{aligned} \tag{A18}$$

Thus, similarly to the $O(G_F^2)$ contributions, the QRE's and/or QKE's formalism *automatically* reproduces the $O(G_F)$ forward-scattering effective potential.

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