Lyapunov spectra in SU(2) lattice gauge theory

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We develop a method for calculating the Lyapunov characteristic exponents of lattice gauge theories. The complete Lyapunov spectrum of SU(2) gauge theory is obtained and Kolmogorov-Sinai entropy is calculated. Rapid convergence with lattice size is found.

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In Ref. [1] we have studied chaos in lattice gauge systems by obtaining the largest Lyapunov exponents. The method used there, though straightforward, has two drawbacks. First the results have large error bars because the exponential divergences of trajectories have fluctuations, which results in an uncertainty in the determination of the exponential rate of divergence between trajectories in phase space. The second drawback of the method is that only the largest Lyapunov exponent can be obtained, but not the whole Lyapunov spectrum.

There is a well developed method for calculating Lyapunov spectra of systems with many degrees of freedom, which is explained in Ref. [2] and briefly outlined here. Given initially a point $q(0)$ in the phase space and ν_L vectors $v_i, i = 1, ..., \nu_L$ in the tangent space $T_{q(0)}$, we can integrate the equations of motion in phase space and simultaneously the evolution equations for small perturbations in tangent space to obtain $q(t)$ and $v_i(t) \in T_{q(t)}$. At regular time intervals $k\tau$, the Gram-Schmidt orthonormalization is applied to the tangent vectors v_i. The scaling factors s_i obtained by this procedure determine the Lyapunov exponents as

$$\lambda_i = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\ln s_i^k}{\tau}, \tag{1}$$

where n is the number of iterations performed. The time needed to obtain the largest Lyapunov exponent depends on how fast the exponents converge with increasing n.

One necessary condition in the above procedure is that we are able to identify tangent spaces at different points of phase space. When the phase space is Euclidean, the natural identification is used and the problem is trivial. For lattice gauge theories, where the phase space is not Euclidean, we must confront this problem explicitly. In this Brief Report two different approaches are proposed to obtain the Lyapunov spectrum in lattice gauge theories. In the first approach, we work in phase space and avoid the problem by embedding the curved phase space into a larger dimensional Euclidean space. This approach is most useful for SU(2) gauge theory. The method is first tested on a 10^3 lattice to calculate the two largest Lyapunov exponents and then applied to obtain the complete Lyapunov spectrum on lattices of size 1^3, 2^3, and 3^3, where the scaling behavior is observed. A second more general approach is to construct the tangent space. The results of this method for SU(2) are not as satisfactory as the first one. The advantage of this second method, however, is that it can be directly applied to other gauge theories.

The SU(2) lattice gauge theory is defined by the Hamiltonian [3,4]

$$H = \sum_l \frac{1}{2} E_l^a E_l^a + 4 \sum_p \left(1 - \frac{1}{2}\mathrm{tr}\, U_p\right), \tag{2}$$

where E_l are electric fields and U_p are plaquette variables which are ordered products of four link variables U_l. The lattice spacing as well as the coupling constant have been scaled to unity, $a = g = 1$, and the only parameter of the system is the scaled average energy per plaquette $g^2 Ea$, which we will refer to as E in the following. The temporal axial gauge adopted here does not fix the gauge completely and the system is invariant under arbitrary time-independent gauge transformations. The total phase space is the direct product of phase space for each link. The latter itself is the direct product $R^3 \times \mathrm{SU}(2)$, where the two subspaces are for E_l and U_l, respectively. The SU(2) group manifold is isomorphic to a three-dimensional sphere, which can easily be embedded in a four-dimensional Euclidean space. This is easily achieved if we use the quaternion representation, where a link variable U_l is represented by a four-vector $(u_{0l}, \mathbf{u}_l)$ in Euclidean space with the condition of unit norm, i.e., $u_0^2 + \mathbf{u}^2 = 1$. Then, for each link, phase space forms a six-dimensional subspace of a seven-dimensional Euclidean space. One way to implement the rescaling method is to study the evolution of vectors in the corresponding seven-dimensional tangent space. In order to incorporate the condition of unit norm of U_l, the initial conditions must be carefully chosen. In the following, we choose a slightly different way to implement the method.

Instead of following the evolution of ν_L vectors in the tangent space, we study $\nu_L + 1$ trajectories $z_i, i = 0, ..., \nu_L$ in phase space. The trajectory $z_0(t)$ is called the reference trajectory. At regular time intervals $t = k\tau$, ν_L vectors are formed from these trajectories by

$$d_i = z_i(t) - z_0(t), \quad i = 1, ..., \nu_L. \tag{3}$$

These vectors are treated as vectors in Euclidean space. The normal Gram-Schmidt method is then used to obtain the Lyapunov numbers λ_i as in (1). A small difference here is that we do not scale the norm of the vectors $D_i \equiv |d_i|$ to unity, but to a chosen small value D_0. The scale factor is referred to as s_i. After this procedure, all

the trajectories except the reference trajectory assume their new positions in phase space. Although time evolution itself conserves the norm of U_l, Gauss' law, as well as the gauge condition, the Gram-Schmidt rescaling procedure violates them slightly, because they form nonlinear constraints. The condition of unit norm is easily imposed by hand. The violation of Gauss' law is expected not to be serious, the reason for which is as follows. If the values D_i are small, then the violation of Gauss' law is of second order in D_i. If we limit ourselves to sufficiently small D_i, then the violation of Gauss' law in each rescaling step is negligible. Remembering also that the evolution of the system respects Gauss' law, the violation does not increase with time. On the other hand, the next rescaling decreases the previous violation of Gauss' law by a scale factor s_i, and so the violations do not accumulate. The same argument applies to small changes in the choice of gauge.

To test our method, we have applied it to the SU(2) theory and measured the two largest Lyapunov exponents. We indeed find that the violation of Gauss' law remains of the order of 10^{-6}, which gives us confidence in the method. The result of a typical run for the Lyapunov exponents is shown in Fig. 1 for a configuration with scaled energy $E = 4.06$. The solid line corresponds to the largest exponent and the dotted line to the second largest one. They converge at $t \approx 100$. From our previous study [1] we know that the time scale for saturation of the distance $D(t)$ between two configurations in the case without rescaling is about 30 at the same energy. The result obtained with our new, improved method, $\lambda_1 = 0.667$, is very close to, but slightly lower than, our previous result $E/6 = 4.06/6 = 0.68$. We note that the result for the Lyapunov exponent generally converges from above; i.e., the Lyapunov exponents are overestimated when the trajectories are not followed for sufficiently long times. We also observe that λ_2 is almost identical to λ_1. The reason is that, as we are going to show next, there exists a whole Lyapunov spectrum which forms a continuous curve in the large volume limit.

This method in principle can be used to obtain the Lyapunov spectrum of a SU(2) lattice gauge system of an arbitrary size. Practically, the computing requirements limit us to rather small lattices. We have numerically studied lattices of size N^3, with $N = 1, 2, 3$. Fortunately, as we will show, the spectrum starts to scale as early as at size $N = 3$, which permits us to extrapolate the results to the thermodynaimc limit without actually going to a larger lattice.

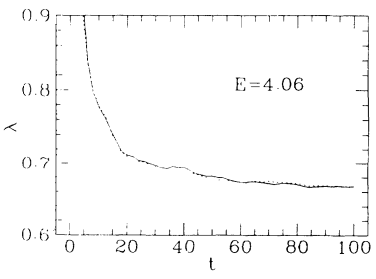

FIG. 1. The two largest Lyapunov exponents for SU(2) determined by the rescaling method. The average of the logarithmic scaling factors $s_i^{(k)}$ approaches the limit from above.

We have obtained the complete Lyapunov spectrum for systems on 1^3 and 2^3 lattices. On a N^3 lattice the phase space dimension is $3^2N^3 \times 2 = 18N^3$, because there are three space directions and three color directions at each site for magnetic and electric fields. Hence on a 2^3 lattice there are 144 Lyapunov exponents which are shown in Fig. 2. Measurements are performed at different times and convergence in time is observed. In Fig. 2, two measurements at $t = 200$ (the crosses) and $t = 1000$ (the solid triangles) are shown. The latter has a smaller fluctuaction. We can see that the spectrum is divided into three equal parts. The first one-third of Lyapunov exponents are positive, while the second one-third are all zero (they are not exactly vanishing at $t = 1000$, but they are clearly seen to converge to zero). The last one-third of exponents are approximately the negative of the first one-third. The vanishing Lyapunov exponents correspond to the conservation of charge (Gauss' law) and the gauge degrees of freedom at each lattice site. Thus our results confirm the general properties of Lyapunov spectra [2]. Our result also implies that in addition to the total energy and static color charge at each lattice site, there is no other conservative quantity in the system. The results for a 1^3 lattice are basically the same, but the Lyapunov spectrum consists only of 18 numbers. For a run with energy $E = 2.632$, the value of the largest Lyapunov exponent is 0.44. We can compare this result with the result obtained earlier for the model of spatially constant Yang-Mills potentials [5], where it is found $\lambda_1 = hE^{1/4}$, with $h \approx 0.38$. Inserting $E = 2.632$ we get $\lambda_1 = 0.48$. Taking into account the uncertainties in h, we think the results obtained in these two entirely different approaches are surprisingly close to each other.

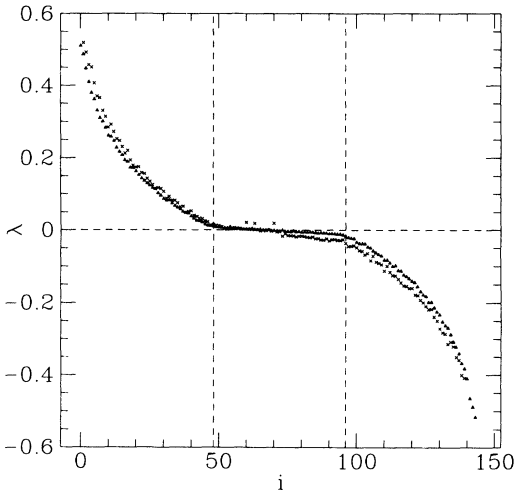

FIG. 2. Complete spectrum of 144 Lyapunov exponents for SU(2) gauge theory on a 2^3 lattice. The trajectories were followed up to time $t/a = 200$ (crosses) and $t/a = 1000$ (triangles). The central third fraction of Lyapunov exponents (enclosed between the vertical dashed lines) corresponds to the unphysical degrees of freedom that describe gauge transformations and deviations from Gauss' law. These exponents converge to zero in the limit $t \to \infty$.

In Fig. 3 we show the scaling of the Lyapunov spectrum, where we compare the results from lattices of different sizes. The solid line is for a 3^3 lattice, the dotted line is for a 2^3 lattice, and the solid squares are for a 1^3 lattice. In the 3^3 case we only calculated the positive Lyapunov exponents. In each case, the initial configurations were chosen as entirely random magnetic fields and zero initial electric fields. In order to observe the scaling of Lyapunov spectrum with respect to lattice size, the Lyapunov exponents are scaled with respect to the largest exponent λ_1. The indices for the Lyapunov numbers are scaled to the total number of Lyapunov numbers, i.e., 18 for a 1^3 lattice, 144 for a 2^3 lattice, and 486 for a 3^3 lattice. The solid squares do not yet scale very well with the lines, but results for $N = 2$ and $N = 3$ coincide nicely, exhibiting an early scaling behavior. The scaled Kolmogorov-Sinai entropy, i.e., the sum over all positive Lyapunov exponents, is

$$\alpha = \frac{\sum_i \lambda_i}{N^3 \lambda_1} \approx 2.0 \quad \text{for } N = 2, 3. \tag{4}$$

The reason of this early scaling behavior has not been fully understood yet.

Now we study the energy dependence of the Lyapunov spectrum. For initial conditions, we chose the link variables $U_l = \cos(\rho/2) - i\mathbf{n} \cdot \tau \sin(\rho/2)$ randomly and the total energy is varied by selecting δ, which limits the range of the parameter ρ to $(0, 2\pi\delta)$. In the top part of Fig. 4 the Lyapunov spectra corresponding to three different energies on a 3^3 lattice are plotted in the scaled form. The solid triangles correspond to an average plaquette energy of $E = 4.25$, the squares correspond to $E = 3.21$, and the crosses correspond to $E = 1.67$. We see that at this small lattice size, the spectrum does not yet scale with energy, or, in other words, the scaled summation α is a function of E. In the above three cases, $\alpha = 2.0, 1.7, 1.3$, respectively. The energy dependence of α on energy is shown in the lower part of Fig. 4, where we see α increases with energy. In the previous study [1], we found that if the lattice is large enough ($N \geq 6$), the largest Lyapunov exponent λ_1 depends linearly on energy, $\lambda_1 a \approx \frac{1}{6} g^2 E a$; i.e., λ_1 does not depend on the lattice cutoff a. (Here and in the remainder of this paragraph, we keep g and a explicitly in order to make connections with physics in the continuum limit.) If we assume that in the large lattice limit the other Lyapunov exponents are also independent of the lattice cutoff a, from dimensional consideration, they can only depend linearly on $g^2 E$. Then in the large volume limit we expect that α is independent of energy and is a universal number, which is approximately 2. We obtain the Kolmogorov-Sinai entropy density in the thermodynamic limit

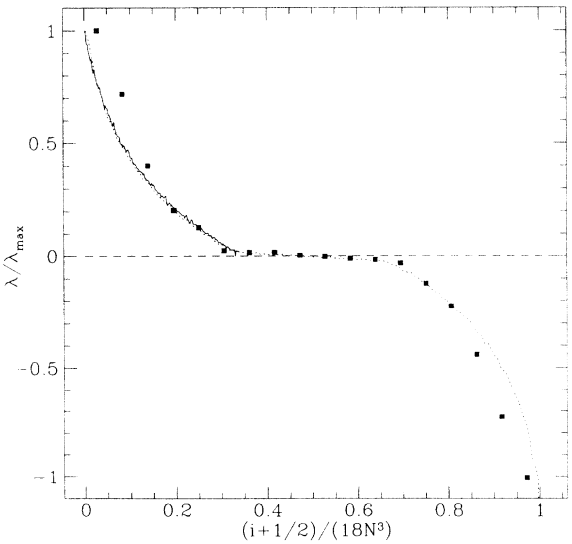

FIG. 3. Scaling of the Lyapunov spectrum with lattice size N. The solid line corresponds to a 3^3 lattice; the dashed line is for a 2^3 lattice. Only the positive Lyapunov exponents are shown. The exponents λ_i are scaled with the maximal Lyapunov exponent $\lambda_{\max} = \lambda_1$ for each lattice size, and the index i is scaled with total number of Lyapunov number $18N^3$.

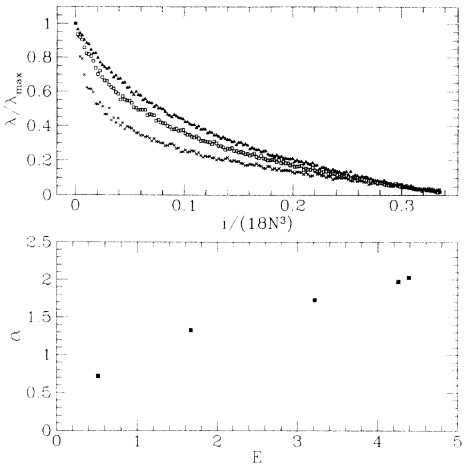

FIG. 4. Energy scaling on a 3^3 lattice. In the top part, the positive part of the spectra for three different energies are shown. The solid triangles correspond to an average plaquette energy of 4.25, the squares correspond to 3.21, and the crosses correspond to 1.67. They do not scale with energies. In the lower part, the scaled Kolmogorov-Sinai entropy α is shown as a function of energy. In the large-N limit, we expect it to be a horizontal line.

$$\dot{\sigma} = \frac{1}{(Na)^3} \sum_i \lambda_i = \alpha \lambda_1 \approx \frac{1}{9} g^2 \varepsilon, \tag{5}$$

where $\varepsilon = 3E/a^3$ is the energy density. To represent the entropy growth rate in terms of temperature, we make use of the thermodynamic relation $\sigma T = \varepsilon + P$. Using the fact that the partition function $Z(T, g, a)$ in the classical limit of the lattice gauge theory depends only on the combination of $g^2 T a$, it is easy to prove $\varepsilon = 3P$ [6]. Thus we have $\sigma T = \frac{4}{3}\varepsilon$, and we find that the characteristic entropy growth rate for SU(2) is given by

$$\frac{\dot{\sigma}}{\sigma} \approx \frac{1}{9} g^2 \frac{\varepsilon}{\sigma} = \frac{1}{12} g^2 T. \tag{6}$$

Its inverse gives an estimation of the thermalization time for highly excited SU(2) gauge fields.

The above method is quite successful for SU(2) gauge

theory, but it is not obvious how to apply it to SU(3) gauge fields. The reason is that this method relies on the quaternion representation, which is quite special for the SU(2) group. In this final part, we propose a more general method which can be used to study other gauge theories such as SU(3).

We try to construct a tangent space upon the curved phase space $\{E_l, U_l\}$ and then study the evolution of vectors in this space. The tangent space of E_l is simple, in which a vector is just δE_l. We shall be careful about how to specify a vector in the tangent space on U_l. Here, to be consistent with our definition of the congugate momenta E_l as the left group generators [3,4], we define a vector b_l in the tangent space of U_l as

$$\delta U_l = i b_l U_l. \tag{7}$$

A vector in the complete tangent space is the direct product of b_l and δE_l. The linear evolution equations for b_l and δE_l are derived from the equations of motion of U and E:

$$\frac{d}{dt} b_l = \delta E_l + i[E_l, b_l],$$
$$\frac{d}{dt} \delta E_l^a = \sum_{p(l),m,b} (-\tfrac{1}{2}) \mathrm{tr}\, U_{p,l,m}^{a,b} b_m^b, \tag{8}$$

where $U_{p,l,m}^{a,b}$ is obtained from U_p, which contains link l and m in one of its four positions, by substituting U_l by $\tau^a U_l$ and then U_m by $\tau^b U_m$, where τ^a are Pauli matrices. These equations can be integrated along with the equations of motion of E_l and U_l in the phase space.

We have tried this method on SU(2) gauge theory, where the results can be compared with the results obtained by our first method. Good agreement is found for large positive exponents; but for the smaller ones, the convergence of the new method is not satisfactory. Future work is still required here.

In conclusion, we have developed a method to study the Lyapunov characteristic numbers of lattice gauge theory. The whole Lyapunov spectrum of SU(2) is obtained explicitely and the scaling behavior is observed. The Kolmogorov-Sinai entropy is obtained.

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[6] Here we take g as a parameter which is independent of temperature T and lattice spacing a. It is possible and probably more desirable to think g as an effective coupling which runs with both T and a. In this latter case, the simple energy-pressure relation $\varepsilon = 3P$ no longer holds. But the change shall not be very large, and so our estimate of Kolmogorov-Sinai entropy density (6) still holds to a good approximate.