Perturbative analysis for Kaplan's lattice chiral fermions

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Perturbation theory for lattice fermions with domain wall mass terms is developed and is applied to investigate the chiral Schwinger model formulated on the lattice by Kaplan's method. We calculate the effective action for gauge fields to one loop, and find that it contains a longitudinal component even for anomaly-free cases. From the effective action we obtain gauge anomalies and Chern-Simons currents without ambiguity. We also show that the current corresponding to the fermion number has a nonzero divergence and it flows off the wall into the extra dimension. Similar results are obtained for a proposal by Shamir, who used a constant mass term with free boundaries instead of domain walls.

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I. INTRODUCTION

Construction of chiral gauge theories is one of the long-standing problems of lattice field theories. Because of the fermion doubling phenomenon, a naively discretized lattice fermion field yields 2^d fermion modes, half of one chirality and half of the other, so that the theory is nonchiral. Several lattice approaches have been proposed to define chiral gauge theories, but so far none of them have been proven to work successfully.

Kaplan has proposed a new approach [1] to this problem. He suggested that it may be possible to simulate the behavior of massless chiral fermions in $2k$ dimensions by a lattice theory of massive fermions in $2k+1$ dimensions if the fermion mass has a shape of a domain wall in the $(2k+1)$th dimension. He showed for the weak gauge coupling limit that massless chiral states arise as zero modes bound to the $2k$-dimensional domain wall while all doublers can be given large gauge invariant masses. If the chiral fermion content that appears on the domain wall is anomalous the $2k$-dimensional gauge current flows off the wall into the extra dimension so that the theory cannot be $2k$ dimensional. Therefore he argued that this approach possibly simulates the $2k$-dimensional chiral fermions only for anomaly-free cases.

His idea was tested for smooth external gauge fields. Jansen [2] showed numerically that in the case of the chiral Schwinger model in two dimensions with three fermions of charge 3, 4, and 5 the anomalies in the gauge currents cancel on the wall. The Chern-Simons current far away from the domain wall was calculated in Ref. [3]. It is shown that the $(2k+1)$th component of the current is nonzero in the positive mass region and zero in the negative mass region such that the derivative of the current cancels the $2k$-dimensional gauge anomaly on the wall, as was argued in Ref. [1].

In the continuum perturbation theory Frolov and Slavnov [4] proposed a gauge invariant regularization of the standard model through an infinite tower of regulator fields. Some similarity between their proposal and the Kaplan's approach was pointed out by Narayanan and Neuberger [5]. It has been also shown that the chiral fermion determinant can be nonperturbatively defined as an overlap of two vacua [6], which can be extended to lattice theories. Using the Narayanan-Neuberger point of view, we observed in Ref. [7] that nongauge (chiral) anomalies are correctly reproduced within the Frolov-Slavnov regularization method.

The results above provide positive indications that Kaplan's method for chiral fermion may work. There exists, however, several potential problems in his approach. Since the original $(2k+1)$-dimensional model is vectorlike, there always exists an antichiral mode, localized on an antidomain wall formed by periodicity of the extra dimension. If the chiral mode and the antichiral mode are paired into a Dirac mode, this approach fails to simulate chiral gauge theories. Without dynamical gauge fields, the overlap between the chiral mode and the antichiral mode is suppressed as $O(e^{-L})$ where L is the size of the extra dimension. If gauge fields become dynamical, the overlap depends on the gauge coupling. In the original paper [1] the strong coupling limit of the gauge coupling in the extra dimension was proposed to suppress the overlap. However, a mean-field calculation [8] in this limit indicated that the chiral mode disappears and the model becomes vectorlike.

More recently Distler and Rey [9] pointed out that the Kaplan's method may have a problem in reproducing fermion number nonconservation expected in the standard model. Using the two-dimensional chiral Schwinger model they argued that either the two-dimensional fermion number current is exactly conserved or the light degree of freedom flows off the wall into the extra dimension so that the model can not be two-dimensional.

In this paper we carry out a detailed perturbative analysis of the Kaplan's proposal for smooth background gauge fields on a finite lattice taking the chiral Schwinger model in two-dimensions as a concrete example. In Sec. II, we formulate the lattice perturbation theory for the Kaplan's method with the periodic boundary condition. Since translational invariance is violated by domain wall mass terms, usual Feynman rules in the momentum space cannot be used except in the regions far away from the domain wall [3]. To perform perturbative calculations near or on the wall, we use the Feynman rules in real space of the extra dimension, as proposed in Ref. [5]. We calculate the fermion propagator for the periodic boundary condition, which reproduces the fermion propagator in Ref. [5] near the origin of the extra dimension. A similar calculation is also made for the constant fermion mass with *free* boundaries in the $(2k+1)$th dimension. As shown by Shamir [10] the constant mass term with this boundary condition can also produce the chiral zero mode on the $2k$-dimensional boundary. The results are similar but simpler than those by the Kaplan's method. In Sec. III, using the Feynman rules of Sec. II we calculate a fermion one-loop effective action for the U(1) gauge field of the chiral Schwinger model simulated by the Kaplan's method. We find that the effective action contains the longitudinal component as well as parity-odd terms, and that this longitudinal component, which breaks gauge invariance, remains nonzero even for anomaly-free cases. This result is compared with those of the conventional Wilson fermion formulation of this model [11]. In Sec. IV we derive gauge anomalies as well as the Chern-Simons current from the effective action without ambiguity. Then we show that the current corresponding to the fermion number has a nonzero divergence and the fermion number current flows off the walls into the extra dimension. In Sec. V, we give our conclusions and discussions.

II. ACTION, FERMION PROPAGATOR, AND CHIRAL ZERO MODES

A. Lattice action

We consider a vector gauge theory in $D = 2k+1$ dimensions with a domain wall mass term. For later convenience we use the notation of Ref. [5], where the fermionic action is written in terms of a $d = 2k$ dimensional theory with infinitely many flavors. Our action is denoted as

$$S = S_G + S_F. \tag{1}$$

The action for gauge field S_G is given by

$$S_G = \beta \sum_{n,\mu>\nu} \sum_{s} \mathrm{Re}\{\mathrm{Tr}[U_{\mu\nu}(n,s)]\} + \beta_D \sum_{n,\mu} \sum_{s} \mathrm{Re}\{\mathrm{Tr}[U_{\mu D}(n,s)]\}, \tag{2}$$

where μ, ν run from 1 to d, n is a point on a d-dimensional lattice, and s a coordinate in the extra dimension; β is the inverse gauge coupling for plaquettes $U_{\mu\nu}$ and β_D that for plaquettes $U_{\mu D}$. In general we can take $\beta \neq \beta_D$. The fermionic part of the action S_F is given by

$$S_F = \frac{1}{2} \sum_{n,\mu} \sum_{s} \bar{\psi}_s(n)\gamma_\mu[U_{s,\mu}(n)\psi_s(n+\mu) - U^\dagger_{s,\mu}(n-\mu)\psi_s(n-\mu)] + \sum_{n} \sum_{s,t} \bar{\psi}_s(n)[M_0 P_R + M_0^\dagger P_L]_{st}\psi_t(n) + \frac{1}{2} \sum_{n,\mu} \sum_{s} \bar{\psi}_s(n)[U_{s,\mu}(n)\psi_s(n+\mu) + U^\dagger_{s,\mu}(n-\mu)\psi_s(n-\mu) - 2\psi_s(n)], \tag{3}$$

where s, t are considered as flavor indices, $P_{R/L} = (1 \pm \gamma_{2k+1})/2$,

$$(M_0)_{st} = U_{s,D}(n)\delta_{s+1,t} - a(s)\delta_{st}, \tag{4a}$$

$$(M_0^\dagger)_{st} = U^\dagger_{s-1,D}(n)\delta_{s-1,t} - a(s)\delta_{st}, \tag{4b}$$

and $U_{s,\mu}(n)$, $U_{s,D}(n)$ are link variables for gauge fields. We consider the above model with periodic boundaries in the extra dimension, so that s, t run from $-L$ to $L-1$, and we take

$$a(s) = 1 - m_0[\mathrm{sgn}(s+\tfrac{1}{2}) \cdot \mathrm{sgn}(L-s-\tfrac{1}{2})] = \begin{cases} 1-m_0, & -\frac{1}{2} < s < L-\frac{1}{2}, \\ 1+m_0, & -L-\frac{1}{2} < s < -\frac{1}{2}, \end{cases} \tag{5}$$

for $-L \leq s < L$. It is easy to see [5] that S_F above is identical to the Kaplan's action in $D = 2k+1$ dimensions [1] with the Wilson parameter $r = 1$. In fact the second term in Eq. (3) can be rewritten as

$$\frac{1}{2}\bar{\psi}_s\gamma_D[U_{s,D}\psi_{s+1} - U_{s-1,D}\psi_{s-1}] + \frac{1}{2}\bar{\psi}_s[U_{s,D}\psi_{s+1} + U_{s-1,D}\psi_{s-1} - 2\psi_s] + M(s)\bar{\psi}_s\psi_s, \tag{6}$$

with $M(s) = m_0[\mathrm{sgn}(s+1/2) \cdot \mathrm{sgn}(L-s-1/2)]$. Note that our action is slightly different from that of Ref. [5]: We have the Dth component of the link variable $U_{s,D}(n)$ and all link variables have s dependence. With the gauge fixing condition $U_{s,D}(n) = 1$ for all s and n [9], our action becomes almost identical to that of Ref. [5], but still the s dependence exists in our link variables in d dimensions. The model in Ref. [5] corresponds to our model at $\beta_D = \infty$, where s dependences of gauge fields are completely suppressed, and the model at $\beta_D = 0$ was investigated by the mean-field method [8].

B. Chiral zero modes

We now consider chiral zero modes of the action S_F in the weak coupling limit, i.e., $\forall U_{s,\mu} = \forall U_{s,D} = 1$. According to Ref. [5], the right-handed zero modes are given by zero modes of the operator M and the left-handed zero modes by those of the operator $M^\dagger$, where

$$(M)_{st} = (M_0)_{st} + \frac{\nabla(p)}{2}\delta_{st},$$
$$(M^\dagger)_{st} = (M_0^\dagger)_{st} + \frac{\nabla(p)}{2}\delta_{st}, \tag{7}$$

with $\nabla(p) \equiv \sum_{\mu=1}^{d} 2[\cos(p_\mu a)-1]$ in momentum space of d dimensions. It is noted that $0 \le -\nabla(p) \le 4d$ and zero modes exist if and only if $-\nabla(p) \le 2m_0$ [3]. Hereafter we only consider the case that $0 < m_0 < 2$. In this range of m_0, there is only one right-handed zero mode u_R satisfying $Mu_R = 0$, which is given by

$$u_R(s) = \begin{cases} [1-\nabla(p)/2 - m_0]^s C_0^{-1} & \text{for } s \ge 0, \\ [1-\nabla(p)/2 + m_0]^s C_0^{-1} & \text{for } s < 0, \end{cases} \tag{8}$$

where the d-dimensional momentum p has to be restricted to $0 \le m_0 + \nabla(p)/2$ so that $(1-\nabla/2-m_0) \le 1$. The normalization constant C_0 takes the value

$$\frac{1-[1-\nabla(p)/2-m_0]^L}{m_0+\nabla(p)/2} + \frac{1-[1-\nabla(p)/2+m_0]^{-L}}{m_0-\nabla(p)/2}. \tag{9}$$

This zero mode is localized around $s = 0$. On a finite lattice (i.e., $L \ne \infty$) with the periodic boundary condition, there exists another zero mode u_L with the opposite chirality satisfying $M \cdot u_L = 0$, which is given by $u_L(s) = u_R(L-t-1)$ and is localized around $s = L$. The overlap between the two zero modes vanishes exponentially as $L \to \infty$:

$$\sum_{s=-L}^{L-1} u_R(s)u_L(s) = C_0^{-2} L \left(1 - \frac{\nabla(p)}{2} - m_0\right)^L \times \left(1 - \frac{\nabla(p)}{2} + m_0\right)^{-L} \longrightarrow 0, \tag{10}$$

We illustrate the shape of the two zero modes u_R and u_L at $m_0 = 0.1$ and 0.5 for $p = 0$ in Fig. 1.

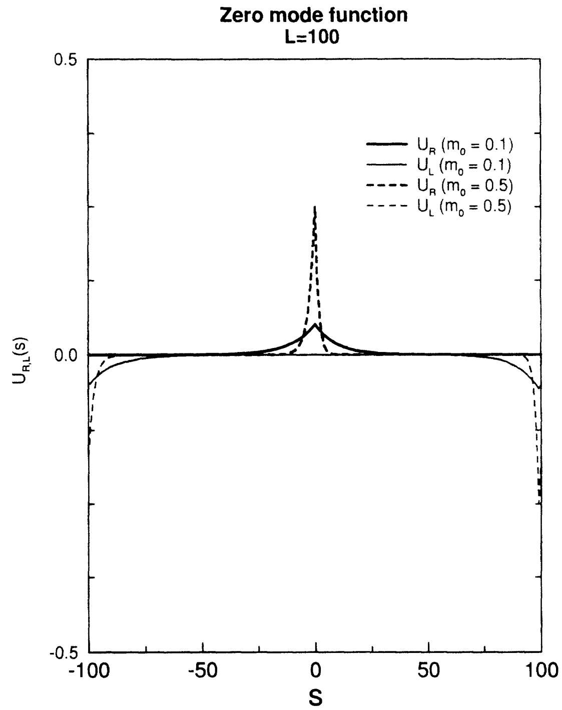

FIG. 1. Two zero modes u_L and u_R as a function of s at $m_0 = 0.1$ and 0.5 for $p_1 = p_2 = 0$. We take $L = 100$.

C. Fermion propagator and zero modes

The fermion propagator in d-dimensional momentum space and in real Dth space has been obtained in Ref. [5] for the infinite Dth space (i.e., $L = \infty$). It is not difficult to obtain the fermion propagator for a finite lattice with periodic boundaries. We have

$$S_F(p)_{st} = -\left\{ \left[\left(i\sum_\mu \gamma_\mu \bar{p}_\mu + M\right) G_L(p)\right]_{st} P_L + \left[\left(i\sum_\mu \gamma_\mu \bar{p}_\mu + M^\dagger\right) G_R(p)\right]_{st} P_R \right\}, \tag{11}$$

where

$$G_L(p) = \frac{1}{\bar{p}^2 + M^\dagger M}, \quad G_R(p) = \frac{1}{\bar{p}^2 + MM^\dagger}, \tag{12}$$

with $\bar{p}_\mu = \sin(p_\mu a)$. Explicit expressions for G_L and G_R are complicated in general, but they become simple for large L where we neglect terms of order $O(e^{-cL})$ with $c > 0$. We obtain

$$G_L(p)_{st} = \begin{cases} Be^{-\alpha_+|s-t|} + (A_L - B)e^{-\alpha_+(s+t)} + (A_R - B)e^{-\alpha_+(2L-s-t)} & (s,t \ge 0), \\ A_L e^{-\alpha_+ s + \alpha_- t} + A_R e^{-\alpha_+(L-s)-\alpha_-(L+t)} & (s \ge 0,\ t \le 0), \\ A_L e^{\alpha_- s - \alpha_+ t} + A_R e^{-\alpha_-(L+s)-\alpha_+(L-t)} & (s \le 0,\ t \ge 0), \\ Ce^{-\alpha_-|s-t|} + (A_L - C)e^{\alpha_-(s+t)} + (A_R - C)e^{-\alpha_-(2L+s+t)} & (s,t \le 0), \end{cases} \tag{13}$$

$$G_R(p)_{st} = \begin{cases} Be^{-\alpha_+|s-t|} + (A_R - B)e^{-\alpha_+(s+t+2)} + (A_L - B)e^{-\alpha_+(2L-s-t-2)} & (s,t \geq -1), \\ A_R e^{-\alpha_+(s+1)+\alpha_-(t+1)} + A_L e^{-\alpha_+(L-s-1)-\alpha_-(L+t+1)} & (s \geq -1,\ t \leq -1), \\ A_R e^{\alpha_-(s+1)-\alpha_+(t+1)} + A_L e^{-\alpha_-(L+s+1)-\alpha_+(L-t-1)} & (s \leq -1,\ t \geq -1), \\ Ce^{-\alpha_-|s-t|} + (A_R - C)e^{\alpha_-(s+t+2)} + (A_L - C)e^{-\alpha_-(2L+s+t+2)} & (s,t \leq -1), \end{cases} \tag{14}$$

where

$$a_\pm = 1 - \frac{\nabla(p)}{2} \mp m_0, \tag{15a}$$

$$\alpha_\pm = \operatorname{arccosh}\left[\frac{1}{2}\left(a_\pm + \frac{1+\bar{p}^2}{a_\pm}\right)\right] \geq 0, \tag{15b}$$

$$A_L = \frac{1}{a_+e^{\alpha_+} - a_-e^{-\alpha_-}}, \qquad A_R = \frac{1}{a_-e^{\alpha_-} - a_+e^{-\alpha_+}}, \tag{15c}$$

$$B = \frac{1}{2a_+\sinh\alpha_+}, \qquad C = \frac{1}{2a_-\sinh\alpha_-}. \tag{15d}$$

For $|s|,|t| \ll L$ the propagator above coincides with that of Ref. [5]. From the form of A_L, A_R, B, and C, it is easy to see [5] that singularities occur only in A_L at $p = 0$:

$$A_L \longrightarrow \frac{m_0(4 - m_0^2)}{4p^2a^2}, \qquad p \to 0. \tag{16}$$

Therefore the propagator G_L describes a massless right-handed fermion around $s,t = 0$ and G_R a massless left-handed fermion around $|s|,|t| = L$, which correspond to the two zero modes in the previous subsection. Later we use the above forms of the fermion propagator to calculate fermion one-loop diagrams. It is also noted that the fermion propagator away from the two domain walls approaches the Wilson fermion propagator with a *constant* mass term, i.e.,

$$S_F(p) \to \int \frac{dp_D}{2\pi} \frac{e^{iap_D(s-t)}}{i\gamma\cdot p + i\gamma_D p_D \pm m_0 - \nabla(p) + 1 - \cos(p_D a)} \tag{17}$$

for $1 \ll |s|,|t|,|L-s|,|L-t|$ with $st > 0$, where $+m_0$ is taken for $s,t > 0$ and $-m_0$ for $s,t < 0$. Therefore the calculation in Ref. [3] is valid in this region of s and t.

Before closing this subsection, we give the form of propagator for the Shamir's free boundary fermions [10]. This is again given by Eq. (11) with $M_{st} = \delta_{s+1,t} - a_+\delta_{s,t}$ and $M^\dagger_{st} = \delta_{s-1,t} - a_+\delta_{s,t}$. For large L, it becomes [10]

$$G_L(p)_{st} = Be^{-\alpha_+|s-t|} + A'_L e^{-\alpha_+(s+t)} + A'_R e^{\alpha_+(s+t-2L)}, \tag{18}$$

$$G_R(p)_{st} = Be^{-\alpha_+|s-t|} + A'_R e^{-\alpha_+(s+t)} + A'_L e^{\alpha_+(s+t-2L)}, \tag{19}$$

where

$$A'_L = B\frac{1 - a_+e^{-\alpha_+}}{a_+e^{\alpha_+} - 1}, \qquad A'_R = -Be^{-2\alpha_+}, \tag{20}$$

and now $0 \leq s,t \leq L$. Singularities occur only in A'_L at $p = 0$ such that

$$A'_L \to \frac{m_0(2 - m_0)}{p^2a^2}. \tag{21}$$

Thus, the propagator describes a right-handed massless fermion around one boundary at $s,t = 0$ and a left-handed massless around the other boundary at $s,t = L$.

D. Fermion Feynman rules

In this subsection, we write down the lattice Feynman rules for fermions relevant for fermion one-loop calculations, which will be performed in the next section. We first choose the axial gauge fixing $U_{s,D} = 1$. Although the full gauge symmetries in D dimensions are lost, the theory is still invariant under gauge transformations independent of s [9]. We consider the limit of small d-dimensional gauge coupling, and take

$$U_{s,\mu}(n) = \exp[iagA_\mu(s, n + \mu/2)], \tag{22}$$

where a is the lattice spacing, and $g \propto 1/\sqrt{\beta}$ is the gauge coupling constant whose mass dimension is $2-D/2$ (mass dimension of the gauge fields A_μ is $D/2 - 1$). It is noted that the other gauge coupling $g_s \propto 1/\sqrt{\beta_s}$ is not necessarily small and can be made arbitrarily large. We consider Feynman rules in momentum space for the physical d dimensions but in real space for the extra dimension.

The fermion propagator is given by

$$\langle \psi_s(-p)\bar{\psi}_t(p)\rangle = S_F(p)_{st}, \tag{23}$$

where S_F has been given in Eq. (11) with Eqs. (13,14) for the Kaplan's fermions or with Eqs. (18,19) for the Shamir's fermions.

The fermion vertex coupled to a single gauge field is given by

$$ag\bar{\psi}_s(q)\partial_\mu\left[S_F^{-1}\left(\frac{q+p}{2}\right)\right]_{ss}A_\mu(s,p-q)\psi_s(-p). \quad (24)$$

Here $\partial_\mu S_F^{-1}(q) = \frac{\partial S_F^{-1}(q)}{\partial(q_\mu a)} = iC_\mu(q)\gamma_\mu + S_\mu(q)$ with $C_\mu(q)=\cos(q_\mu a)$ and $S_\mu(q)=\sin(q_\mu a)$. From this form of the vertex it is easy to see that the fermion tadpole diagram for an external gauge field vanishes identically.

The fermion vertex with two gauge fields is given by

$$-a^2\frac{g^2}{2}\bar{\psi}_s(q)\left[\partial_\mu^2 S_F^{-1}\left(\frac{q+p}{2}\right)\right]_{ss}A_\mu^2(s,p-q)\psi_s(-p), \quad (25)$$

where $A_\mu^2(s,p)=A_\mu(s,p-p_1)A_\mu(s,p_1)$ with p and p_1 fixed.

III. PERTURBATIVE CALCULATIONS FOR THE CHIRAL SCHWINGER MODEL

In the following two sections we analyze the chiral Schwinger model formulated via the Kaplan's method for lattice chiral fermions. Using the Feynman rules of the previous section for $D=3$, we calculate the effective action for external gauge fields, from which we derive gauge anomalies, Chern-Simons current, and anomaly of the fermion number current. We perform the calculations for the Shamir's method in parallel with those for the Kaplan's method.

A. Effective action at fermion one-loop level

Since $[A_\mu, A_\nu]=0$ for U(1) gauge fields, all diagrams with odd number of external gauge fields vanishes identically. Furthermore diagrams with four or more external gauge fields are all convergent. Therefore only the diagrams with two external gauge fields are potentially divergent. The effective action for two external gauge fields is denoted by

$$S_{\rm eff}^{(2)} \equiv -\frac{g^2}{2}\sum_{p,s,t}A_\mu(s,p)A_\nu(t,-p)I^{\mu\nu}(p)_{st}$$
$$= -\frac{g^2}{2}\sum_{p,s,t}A_\mu(s,p)A_\nu(t,-p)[I_a^{(2)}+I_b^{(2)}]_{st}^{\mu\nu}, \quad (26)$$

where

$$[I_a^{(2)}]_{st}^{\mu\nu} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^2q}{(2\pi)^2}\mathrm{tr}\left\{\left[\partial_\mu S_F^{-1}\left(q+\frac{p}{2}\right)S_F(q+p)\right]_{st}\times\left[\partial_\nu S_F^{-1}\left(q+\frac{p}{2}\right)S_F(q)\right]_{ts}\right\}a^2 \quad (27)$$

and

$$[I_b^{(2)}]_{st}^{\mu\nu} = -\delta_{st}\delta_{\mu\nu}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^2q}{(2\pi)^2}\mathrm{tr}[\partial_\mu^2 S_F^{-1}(q)\cdot S_F(q)]_{ss}\cdot a^2, \quad (28)$$

with tr meaning trace over spinor indices.

B. Evaluation of zero mode contributions

To evaluate $I^{\mu\nu}(p)$ we decompose it into two parts as

$$I^{\mu\nu}(p) = I_0^{\mu\nu}(p) + [I^{\mu\nu}(p) - I_0^{\mu\nu}(p)], \quad (29)$$

where $I_0^{\mu\nu}$ is the contribution of zero modes and $I^{\mu\nu}-I_0^{\mu\nu}$ is the remaining contribution. For $I_0^{\mu\nu}$ we replace the integrand of $I^{\mu\nu}$ with that in the $a\to 0$ limit, and we obtain

$$I_0^{\mu\nu}(p)_{st} = \sum_X\int_{-\nabla(q)\le 2m_0}\frac{d^2q}{(2\pi)^2}a^2 \times\mathrm{tr}\{i\gamma_\mu[-i\gamma_\alpha(q+p)_\alpha a]G_X^0(q+p)_{st}P_X \times i\gamma_\nu[-i\gamma_\beta(q+p)_\beta a]G_X^0(q+p)_{ts}P_X\}, \quad (30)$$

with $X=L$ for $|s|,|t|\approx 0$, or $X=R$ for $|s|,|t|\approx L$. The zero mode propagators G_X^0 are given by

$$G_X^0(q)_{st} = \lim_{a\to 0}G_X(q)_{st} = \frac{1}{q^2a^2}F_X(s,t), \quad (31)$$

where $F_X(s,t)=F_X(t,s)$ and

$$F_L(s,t) = \frac{m_0(4-m_0^2)}{4}\times\begin{cases}(1-m_0)^{s+t} & \text{for } s,t\ge 0\ ,\\ (1-m_0)^s(1+m_0)^t & \text{for } s\ge 0 \text{ and } t<0\ ,\\ (1+m_0)^{s+t} & \text{for } s,t\le 0\ ,\end{cases} \quad (32)$$

$$F_R(s,t) = \frac{m_0(4-m_0^2)}{4}\times\begin{cases}(1-m_0)^{2L-s-t-2} & \text{for } s,t\ge 0\ ,\\ (1-m_0)^{L-s-1}(1+m_0)^{-L-t-1} & \text{for } s\ge 0 \text{ and } t<0\ ,\\ (1+m_0)^{-2L-s-t-2} & \text{for } s,t\le 0,\end{cases} \quad (33)$$

for the Kaplan's fermion with the domain wall mass terms, and

$$F_L(s,t) = m_0(2-m_0)(1-m_0)^{s+t}, \quad (34a)$$
$$F_R(s,t) = m_0(2-m_0)(1-m_0)^{2L-s-t}, \quad (34b)$$

for the Shamir's fermion with the constant mass terms and free boundaries.

We evaluate $I_0^{\mu\nu}(p)$ in the $a \to 0$ limit. In this limit,

$$\lim_{a\to 0}\int_{-\nabla(q)\le 2m_0}\frac{d^2q}{(2\pi)^2}\mathrm{tr}[P_X\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta]\frac{(q+p)_\alpha q_\beta}{(q+p)^2q^2}$$
$$=\int_{-\infty}^{\infty}\frac{d^2q}{(2\pi)^2}\mathrm{tr}[P_X\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta]\frac{(q+p)_\alpha q_\beta}{(q+p)^2q^2}$$
$$=\frac{1}{2\pi}\left[\delta_X i\epsilon^{\mu\alpha}\frac{p_\nu p_\mu}{p^2}+(\delta^{\mu\nu}-\frac{p_\mu p_\nu}{p^2})-\frac{\delta^{\mu\nu}}{2}\right]; \quad (35)$$

therefore, we obtain

$$\lim_{a\to 0}I_0^{\mu\nu}(p)=\sum_X\frac{1}{2\pi}\left[\delta_X i\epsilon^{\mu\alpha}\frac{p_\nu p_\mu}{p^2}+\left(\delta^{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)-\frac{\delta^{\mu\nu}}{2}\right]F_X(s,t)^2, \quad (36)$$

where $\delta_L = 1$ and $\delta_R = -1$. It is noted that F_X satisfies

$$\sum_t F_X(s,t)^2 = F_X(s,s), \qquad \sum_{s,t}F_X(s,t)^2 = 1. \quad (37)$$

C. Evaluation of remaining contributions

We consider the remaining terms in $I^{\mu\nu}$. Since the combination $I^{\mu\nu}(p) - I_0^{\mu\nu}(p)$ is infrared finite, we can change the integration variable from q to qa and take the $a \to 0$ limit in the integrand. Thus we obtain

$$\lim_{a\to 0}I^{\mu\nu}(p)-I_0^{\mu\nu}(p)=I^{\mu\nu}(0)=\frac{1}{2\pi}[i\epsilon^{\mu\nu}\Gamma_{\mathrm{CS}}+\delta^{\mu\nu}K], \quad (38)$$

where

$$\Gamma_{\mathrm{CS}}(s,t)=\frac{\epsilon^{\mu\nu}}{i}2\pi\int\frac{d^2q}{(2\pi)^2}\mathrm{tr}\Big\{[\partial_\mu S_F^{-1}(q)S_F(q)]_{st}\times[\partial_\nu S_F^{-1}(q)S_F(q)]_{ts}\Big\} \quad (39)$$

and

$$K(s,t)=2\pi\int\frac{d^2q}{(2\pi)^2}(\mathrm{tr}\{[\partial_\mu S_F^{-1}(q)S_F(q)]_{st}\times[\partial_\mu S_F^{-1}(q)S_F(q)]_{ts}\}-\delta_{st}\mathrm{tr}[\partial_\mu^2 S_F^{-1}(q)S_F(q)]_{ss}). \quad (40)$$

Here no summation over μ, ν is implied.

The parity-odd term Γ_{CS} is the coefficient function of a three-dimensional Chern-Simons term in the axial gauge [9], which satisfies $\Gamma_{\mathrm{CS}}(s,t) = -\Gamma_{\mathrm{CS}}(t,s)$. It is easy to show that

$$\sum_t\Gamma_{\mathrm{CS}}(s,t)=-\frac{\epsilon^{\mu\nu}}{i}\int\frac{d^2q}{2\pi}\mathrm{tr}\left\{\partial_\mu S_F^{-1}(q)\partial_\nu S_F(q)\right\}_{ss}=-2\frac{\epsilon^{\mu\nu}}{i}\left[\int\frac{dq_\mu}{2\pi}\mathrm{tr}\left\{\partial_\mu S_F^{-1}(q)S_F(q)\right\}_{ss}\right]_{q_\nu=\epsilon}^{\pi}.$$

This would be zero if there were no infrared singularities in S_F. However, because of the contribution from zero modes, S_F is singular at $q = 0$. Therefore,

$$\sum_t\Gamma_{\mathrm{CS}}(s,t)=-\sum_t\Gamma_{\mathrm{CS}}(t,s)=4\lim_{\epsilon\to 0}\left[\int_\epsilon^{\pi/2}\frac{dq_\mu}{2\pi}\sum_X\delta_X[S_\nu(q)C_\mu(q)G_X^0(q)]_{ss}\right]_{q_\nu=\epsilon}^{\pi/2}$$
$$=-4\sum_X\delta_X F_X(s,s)\lim_{\epsilon\to 0}\int_\epsilon^{\pi/2}\frac{dq_\mu}{2\pi}\frac{\epsilon}{q_\mu^2+\epsilon^2}$$
$$=-\sum_X 2\delta_X\frac{F_X(s,s)}{\pi}\left[\arctan\frac{q}{\epsilon}\right]_\epsilon^{\pi/2}$$
$$=-\sum_X\frac{\delta_X}{2}F_X(s,s), \quad (41)$$

where $F_X(s,t)$ is given in the previous subsection.

Since S_F becomes the Wilson fermion propagator with constant mass term for $1 \ll |s|, |t|, |L-s|, |L-t|$ with $st > 0$ [see Eq. (17)], it becomes

$$\sum_{s,t}\Gamma_{\mathrm{CS}}^{\mu\nu}(s,t)A_\mu(s,p)A_\mu(t,-p)\longrightarrow -\epsilon^{\mu\nu}\int dp_3A_\mu(p_3,p)p_3A_\mu(-p_3,-p)\int\frac{d^3q}{(2\pi)^2}\mathrm{tr}\left\{[\partial_\mu S_F^{-1}S_F][\partial_3 S_F^{-1}S_F][\partial_\nu S_F^{-1}S_F]\right\}, \quad (42)$$

which coincides with the result of Ref. [3]. This is a good check of our calculation. From Ref. [3] we obtain

$$= \epsilon^{\mu\nu} \int dp_3 A_\mu(p_3,p) p_3 A_\nu(-p_3,-p) \times \begin{cases} 1 & \text{for } +m_0, \\ 0 & \text{for } -m_0. \end{cases} \tag{43}$$

The parity-even term K satisfies $K(s,t) = K(t,s)$ and

$$\begin{aligned} \sum_t K(s,t) &= \sum_t K(t,s) \\ &= -\int \frac{d^2q}{(2\pi)^2} \text{tr}\{[\partial_\mu S_F^{-1} \partial_\mu S_F]_{ss} \\ &\quad + [\partial_\mu^2 S_F^{-1} S_F]_{ss}\} \\ &= \sum_X \frac{1}{2} F_X(s,s). \end{aligned} \tag{44}$$

The derivation of the last equality is similar to that of Eq. (41).

D. Total contributions

Combining the above contributions we finally obtain

$$\begin{aligned} S_{\text{eff}}^{(2)} = -\frac{g^2}{4\pi} \sum_{s,t} \int d^2x \Bigg\{ & \sum_X F_X(s,t)^2 \left[A_\mu(s,x) \left(\delta^{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \right) A_\nu(t,x) \right] \\ & + \left[K(s,t) - \sum_X \frac{1}{2} F_X(s,t)^2 \right] A_\mu(s,x) A_\mu(t,x) \\ & + \sum_X i\delta_X F_X(s,t)^2 \left[\frac{\partial_\mu}{\Box} A_\mu(s,x) \epsilon^{\alpha\nu} \partial_\alpha A_\nu(t,x) \right] + i\Gamma_{\text{CS}}(s,t) \epsilon^{\mu\nu} A_\mu(s,x) A_\nu(t,x) \Bigg\}. \end{aligned} \tag{45}$$

This is the main result of this paper. It is noted that the above formula is valid for both the Kaplan and the Shamir methods. The following consequences can be drawn from Eq. (45) above.

The parity-odd terms, which are proportional to $\epsilon^{\alpha\nu}$, are unambiguously defined, contrary to the case of the continuum regularization for anomaly-free chiral gauge theories [4] which only regulates the parity even terms [5,7]. These parity-odd terms break gauge invariance in the two-dimensional sense.

For *anomalous* chiral Schwinger model, the parity-odd term with $X = R$ is localized around $s = 0$ and that for $X = L$ is localized around $s = L$. The effective action above for anomalous chiral Schwinger model via Kaplan's method or Shamir's variation is different from the one via the usual Wilson fermion in two dimensions [11]: The term proportional to Γ_{CS}, which cannot be evaluated analytically for s-dependent gauge fields, is special for chiral fermions from three-dimensional theories, and the presence of this term prevents us from concluding whether the anomalous chiral Schwinger model can be consistently defined via Kaplan's (Shamir's) method.

For anomaly-free cases such that $\sum_R g_R^2 = \sum_L g_L^2$, the parity-odd terms are exactly canceled *locally* in s space. Here $g_{R(L)}$ is the coupling constant of a fermion with positive (negative) m_0 which generate a right-handed (left-handed) zero mode around $s = 0$. The simplest but nontrivial example is a Pythagorean case, $g_R = 3, 4$ and $g_L = 5$ [9]. Even for these anomaly-free cases, the longitudinal term, whose coefficient is $K - F^2/2$, remains nonzero in the effective action, so that gauge invariance in the *two-dimensional* sense is violated. In this regard the form of the effective action via Kaplan's (Shamir's) method is similar to the one via the usual Wilson fermion [11].

Let us consider the effective action for s-independent gauge fields as in Ref. [5]. Since $\sum_{\mu,\nu} \epsilon^{\mu\nu} A_\mu(x) A_\nu(x) = 0$ for s-independent gauge fields, the Chern-Simons term vanishes. The other parity-odd term is canceled between the two zero modes since $\sum_{X,s,t} \delta_X F_X(s,t)^2 = 0$. The longitudinal term also vanishes due to the identity

$$\sum_t \left[K(s,t) - \frac{F_X(s,t)^2}{2} \right] = 0 \tag{46}$$

[see Eqs. (37) and (44)]. Therefore the effective action becomes

$$S_{\text{eff}}^{(2)} = -2\frac{g^2}{4\pi} \int d^2x \left[A_\mu(x) \left(\delta^{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \right) A_\nu(x) \right]. \tag{47}$$

This effective action is transverse and thus gauge invariant in the two-dimensional sense. Both zero modes around $s = 0$ and $s = L$ equally contribute so that a factor 2 appears in the above result. This is consistent with the general formula derived in Ref. [6]. The anomalous chiral Schwinger model cannot be simulated by Kaplan's (Shamir's) method with the s-independent gauge fields, since the gauge fields see *both* of the zero modes so that it fails to reproduce the parity-odd term, expected to exist [12].

IV. ANOMALIES IN THE CHIRAL SCHWINGER MODEL

A. Currents and their divergence

From the effective action obtained in the previous section, we can calculate the vacuum expectation values of various currents in the presence of background gauge fields. Let us define the fermion number current as

$$\langle J_\mu^g(s,x)\rangle = \frac{\delta S_{\rm eff}^{(2)}}{g\delta A_\mu(s,x)}, \tag{48}$$

where the index g in the current explicitly shows the charge of the fermion. From Eq. (45) we obtain

$$\begin{aligned} J_\mu^g(s,x) &= i\frac{g}{4\pi}\sum_t\left[\sum_X \delta_X F_X(s,t)^2(\epsilon^{\alpha\nu}\partial_\mu+\epsilon^{\alpha\mu}\partial_\nu)\frac{\partial_\alpha}{\Box}A_\nu(t,x)-2\Gamma_{\rm CS}\epsilon^{\mu\nu}A_\nu(t,x)\right]\\ &\quad -\frac{g}{4\pi}\sum_t\left[2\sum_X F_X(s,t)^2\left(\delta^{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)A_\nu(t,x)+\left(2K(s,t)-\sum_X F_X(s,t)^2\right)A_\mu(t,x)\right]\\ &\equiv J_\mu^{g,{\rm odd}}+J_\mu^{g,{\rm even}}, \end{aligned} \tag{49}$$

where $J_\mu^{g,{\rm odd}}$ is a parity-odd current (the first two terms) and $J_\mu^{g,{\rm even}}$ is a parity-even current (the last two terms). Hereafter all J_μ should be understood as vacuum expectation values, though $\langle\quad\rangle$ is suppressed. From the fermion number current the gauge current for a fermion with charge g is easily constructed as $J_\mu^G(s,x)\equiv gJ_\mu^g(s,x)$.

Divergences of the parity-odd and parity-even currents become

$$\partial_\mu J_\mu^{g,{\rm odd}}(s,x) = i\frac{g}{4\pi}\sum_t\left[\sum_X \delta_X F_X(s,t)^2-2\Gamma_{\rm CS}(s,t)\right]\times\epsilon^{\mu\nu}\partial_\mu A_\nu(t,x), \tag{50}$$

$$\partial_\mu J^\mu_{g,{\rm even}}(s,x) = \frac{g}{4\pi}\sum_t\left[\sum_X F_X(s,t)^2-2K(s,t)\right]\times\partial_\mu A_\mu(t,x). \tag{51}$$

B. Gauge invariance

As mentioned in Sec. II, the action in $A_3=0$ gauge is invariant under s-independent gauge transformation. This invariance implies $\sum_s\partial_\mu J_\mu^G(s,x)=0$. This identity is satisfied in our calculation of the effective action, since

$$\sum_s\left[\sum_X\delta_X F_X(s,t)^2-2\Gamma_{\rm CS}(s,t)\right] = \sum_X\delta_X[F_X(t,t)-F_X(t,t)]=0, \tag{52}$$

$$\sum_s\left[\sum_X F_X(s,t)^2-2K(s,t)\right] = \sum_X[F_X(t,t)-F_X(t,t)]=0, \tag{53}$$

from Eqs. (37,41,44).

C. Gauge anomalies

The gauge anomaly for a fermion with a charge g, denoted T_g, is defined by $T_g=g\partial_\mu J_\mu^{g,{\rm odd}}$, and it becomes

$$T_g(s,x)=\sum_t g^2C(s,t)T^0(t,x), \tag{54}$$

where

$$T^0(t,x)=i\frac{1}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(t,x) \tag{55}$$

is the gauge anomaly of a *two-dimensional* theory, and

$$C(s,t)=\sum_X\delta_X F_X(s,t)^2-2\Gamma_{\rm CS}(s,t) \tag{56}$$

represents the spread of the gauge anomaly over the third direction due to the spread of zero modes. This spread of the anomaly has been observed in a numerical computation [2]. It is noted that the divergence of the gauge current J_μ^G also contains parity-even contributions, given by

$$\sum_t D(s,t)\frac{g^2}{4\pi}\partial_\mu A_\mu(t,x), \tag{57}$$

where $D(s,t)=\sum_X F_X(s,t)^2-2K(s,t)$.

The one-loop integral (39) defining $\Gamma_{\rm CS}$ is too complicated to calculate analytically. For t-independent gauge fields $A_\mu(t,x)=A_\mu(x)$ there is a considerable simplification and we obtain

$$T_g(s,x)=g^2C(s)T^0(x), \tag{58}$$

where

$$T^0(x)=i\frac{1}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(x) \tag{59}$$

and

$$C(s) = \sum_t C(s,t) = 2\sum_X \delta_X F_X(s,s)$$
$$= \frac{m_0(4-m_0^2)}{2} \times \begin{cases} \left[(1-m_0)^{2s} - (1-m_0)^{2(L-s-1)}\right] & \text{for } s \geq 0, \\ \left[(1+m_0)^{2s} - (1+m_0)^{-2(L+s+1)}\right] & \text{for } s \leq 0, \end{cases} \tag{60}$$

for Kaplan's method. We plot $C(s)$ as a function of s at $m_0 = 0.1$ and 0.5 in Fig. 2. For Shamir's method, we obtain

$$C(s) = 2m_0(2-m_0)\left[(1-m_0)^{2s} - (1-m_0)^{2(L-s)}\right] \tag{61}$$

and plot this in Fig. 3. It is noted that there is no parity-even contribution in $\partial_\mu J_\mu^G$ since $\sum_t D(s,t) = 0$ in this case.

D. Chern-Simons current

From the three-dimensional point of view, the gauge anomaly should be canceled in such a way that $T_g + \partial_3 g J_3^{\rm CS}(s,x) = 0$ [1,3], where $J_3^{\rm CS}$ is the third component of the Chern-Simons current for the three-dimensional vector gauge theory. With our gauge fixing the effective action does not depend on A_3, and it is difficult to calculate $J_3^{\rm CS}$ analytically except in the region away from domain walls [3]. However, for t-independent gauge fields, we can obtain the Chern-Simons current everywhere via the relation $\partial_3 g J_{g,\rm odd}^3(s,x) = -T_g$, which becomes

$$J_3^{\rm CS}(s+\tfrac{1}{2},x) - J_3^{\rm CS}(s-\tfrac{1}{2},x) = g^2 C(s) \times T^0(x). \tag{62}$$

Taking $J_3^{\rm CS}(s+\frac{1}{2},x) = g^2 I(s) T^0(x)$, we obtain

$$I(s+\tfrac{1}{2}) - I(s-\tfrac{1}{2}) = C(s). \tag{63}$$

We have to solve this equation with the boundary condition $I(s) \to -2$ as $s \to +\infty$ [3]. For a finite s space $s \to +\infty$ means $1 \ll s \ll L$.

For Kaplan's method we obtain

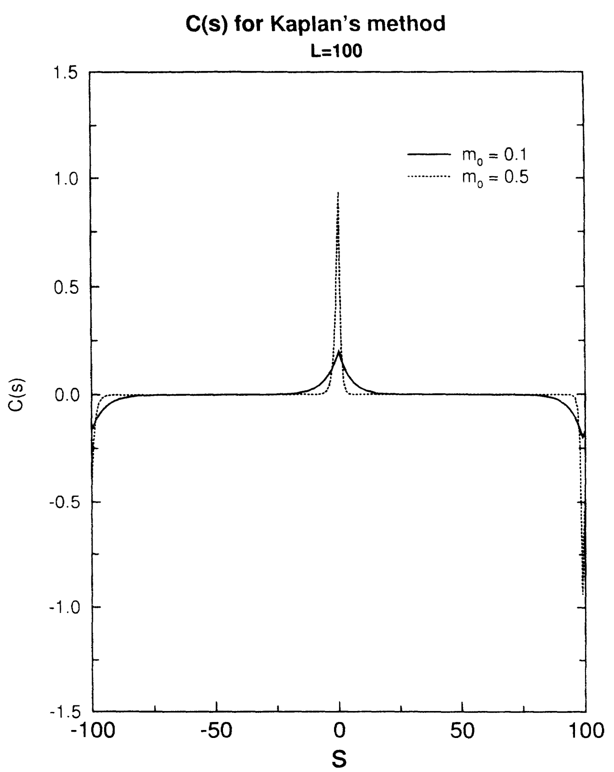

FIG. 2. The coefficient of the anomaly $C(s)$ for Kaplan's method as a function of s at $m_0 = 0.1$ and 0.5 for $L = 100$.

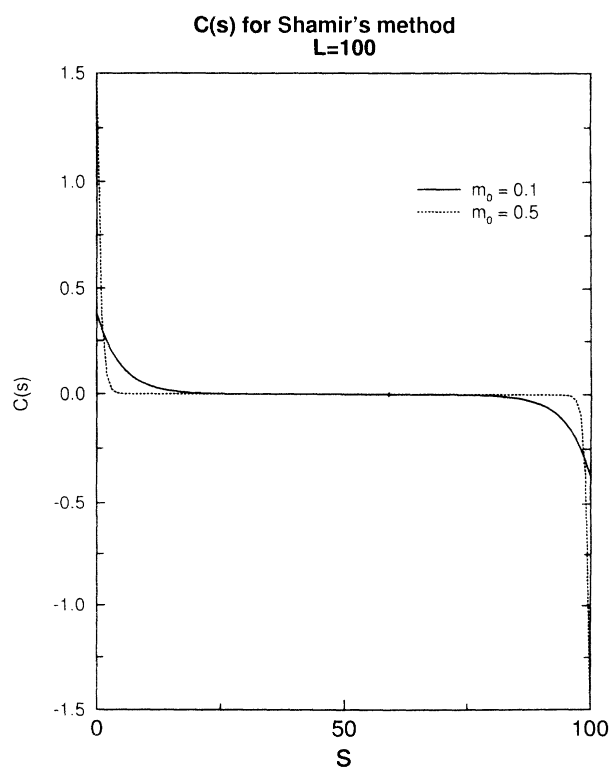

FIG. 3. The coefficient of the anomaly $C(s)$ for Shamir's method as a function of s at $m_0 = 0.1$ and 0.5 for $L = 100$.

$$I(s-\tfrac{1}{2}) = \begin{cases} \dfrac{2+m_0}{2}[(1-m_0)^{2s}+(1-m_0)^{2(L-s)}]-2, & 0\le s\le L, \\ -\dfrac{2-m_0}{2}[(1+m_0)^{2s}+(1+m_0)^{-2(L+s)}], & -L\le s\le 0. \end{cases} \quad (64)$$

This solution automatically satisfies the other boundary condition that $I(s)\to 0$ as $\to -\infty$ [3]. Again $\to -\infty$ means $1 \ll -s \ll L$ for a finite s space. We plot $I(s)$ as a function of s at $m_0 = 0.1$ and 0.5 in Fig. 4. For Shamir's method we obtain

$$I(s-\tfrac{1}{2}) = 2[(1-m_0)^{2s}+(1-m_0)^{2(L-s+1)}]-2 \quad \text{for } 0\le s\le L-1, \quad (65)$$

which is plotted in Fig. 5.

E. Pythagorean chiral Schwinger model and anomaly in fermion number current

Let us consider the Pythagorean chiral Schwinger model [9]. In this model there are two right-handed fermions with charges g_1 and g_2, and one left-handed fermion with charge g_3. Formulation of this model via Kaplan's method has already been discussed in Ref. [1,2,9] (an extension to Shamir's method is straightforward). We assign $+m_0$ for fermions with charge g_1 and g_2, and $-m_0$ for a fermion with charge g_3. The value of $|m_0|$ should be equal for all fermions, as will be seen below.

The theory has a $U(1)^3$ symmetry [9] corresponding to independent phase rotations of three fermions. The corresponding currents are

$$\begin{aligned} J_\mu^G &= g_1 J_\mu^{g_1} + g_2 J_\mu^{g_2} + g_3 J_\mu^{g_3}, \\ J_\mu^R &= g_2 J_\mu^{g_1} - g_1 J_\mu^{g_2}, \\ J_\mu^F &= J_\mu^{g_1} + J_\mu^{g_2} + J_\mu^{g_3}. \end{aligned} \quad (66)$$

The first one is the gauge current, whose divergence becomes

$$\partial_\mu J_\mu^G(s,x) = (g_1^2+g_2^2-g_3^2)\sum_t C(s,t)T^0(t,x). \quad (67)$$

Therefore, if $g_1^2+g_2^2=g_3^2$ (Pythagorean relation) is satisfied and if all fermions have the same value of $|m_0|$ to give the same $C(s,t)$, this current is conserved and there is no gauge anomaly for any background gauge fields. The second current is nonanomalous, since

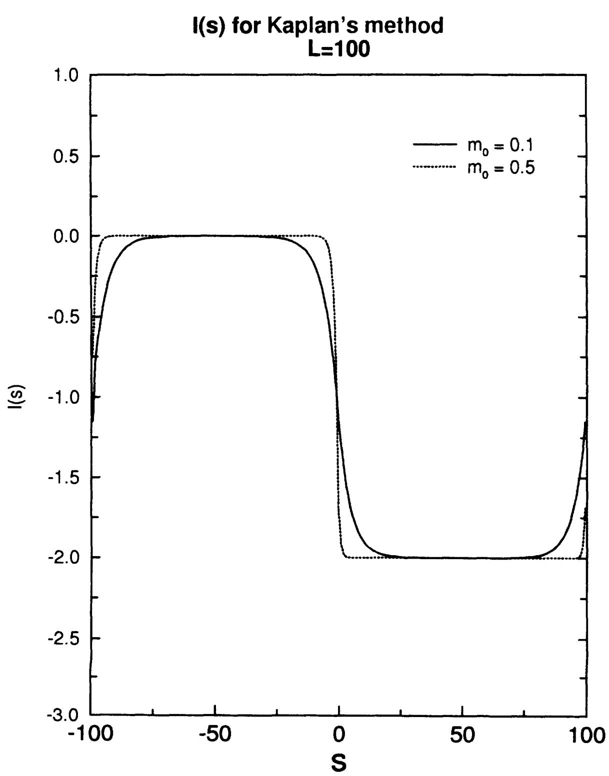

FIG. 4. The coefficient of the Chern-Simons current $I(s)$ for Kaplan's method as a function of s at $m_0 = 0.1$ and 0.5 for $L = 100$.

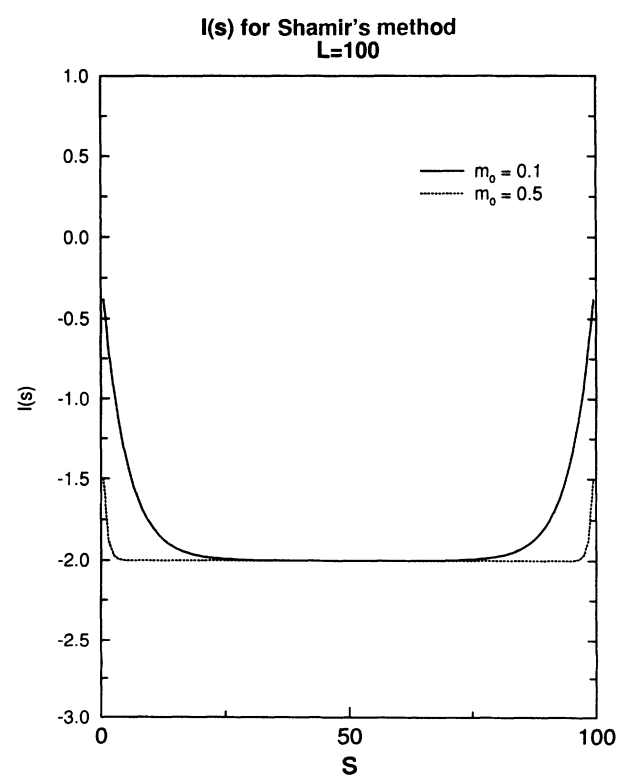

FIG. 5. The coefficient of the Chern-Simons current $I(s)$ for Shamir's method as a function of s at $m_0 = 0.1$ and 0.5 for $L = 100$.

$$\partial_\mu J_\mu^R(s,x) = (g_2 g_1 - g_1 g_2) \sum_t C(s,t) T^0(t,x) = 0. \quad (68)$$

The third current, which corresponds to the fermion number of the theory, is anomalous, since

$$\partial_\mu J_\mu^F(s,x) = (g_1 + g_2 - g_3) \sum_t C(s,t) T^0(t,x). \quad (69)$$

Kaplan's method as well as Shamir's successfully gives a nonzero divergence for the fermion number current, though the coefficient $C(s,t)$ has a finite width. For t-independent gauge fields, this anomaly becomes

$$(g_1 + g_2 - g_3) C(s) T^0(x), \quad (70)$$

where $C(s)$ is almost localized at $s = 0$ and at $s = L$ as seen in Figs. 2 and 3. Since the fermion number is conserved in the three-dimensional theory, the third component of the fermion number current should satisfy $\partial_3 J_3^F + (g_1 + g_2 - g_3) C(s) T^0(x) = 0$ [9]. Therefore we obtain

$$J_3^F(s,x) = (g_1 + g_2 - g_3) I(s) T^0(x). \quad (71)$$

V. CONCLUSIONS

In this paper we have formulated a lattice perturbative expansion for Kaplan's chiral fermion theories, extending the suggestion by Narayanan and Neuberger [5]. Applying our perturbative technique to the chiral Schwinger model formulated via Kaplan's or Shamir's method, we have calculated the fermion one-loop effective action for gauge fields. The effective action contains parity-odd terms and longitudinal terms, both of which break two-dimensional gauge invariance, and the anomaly of the gauge current is obtained from the effective action. The gauge anomaly is calculable in Kaplan's (Shamir's) method if the perturbative expansion is carefully formulated. For the anomaly-free Pythagorean chiral Schwinger model, the fermion number current is anomalous. To obtain this anomaly the fermion number current should not be summed over s, in contrast to the case of the continuum calculation [7], where the anomaly comes from an infinite summation over s.

The main conclusions drawn from the results are as follows.

(1) Anomaly of the fermion number current is shown to be nonzero in this method, though the current flows off walls into the extra dimension. Since the current is external we feel that this does not affect the dynamics of the model and therefore does not spoil the two-dimensional nature of the chiral zero mode. The three-dimensional nature of Kaplan's (Shamir's) formulation manifests itself only in the nonconservation of the fermion number, which is expected to occur in nature.

(2) Two-dimensional gauge invariance at low energy cannot be assured by Kaplan's (Shamir's) method, except for s-independent gauge fields, even for anomaly-free cases. This is similar to the situation with lattice chiral gauge theories formulated with the ordinary Wilson mass term [11]. In this point Kaplan's (Shamir's) method does not seem better than the conventional approaches. At this moment it is not clear whether this violation of gauge invariance spoils the whole program of this method. In particular the effect of the longitudinal component of gauge fields has to be analyzed further. It may be better to use s-independent gauge fields instead of s-dependent ones since the problematic longitudinal term automatically disappears in this case.

(3) If the theory is anomaly free and gauge fields are s independent [5], the gauge invariance as a two-dimensional theory can be maintained. However, the gauge fields feel both of the zero modes even in the $L \to \infty$ limit, and the fermion loop contribution to the effective action is twice as large as the one expected from a single chiral fermion. Therefore we have to take a square root of the fermion determinant to obtain the correct contribution. For fermion quantities such as the fermion number current, however, it seems possible to separate the contribution of the chiral zero mode at $s = 0$ from that of the antichiral zero mode at $s = L$, as seen in the previous section. Finally it should be mentioned that the antichiral mode at $s = L$ is a consequence of the finiteness of the extra dimension. If L is strictly infinite from the beginning [5], only a chiral zero mode exists; hence, the theory becomes *chiral*. Therefore, it is interesting to study this model in detail.

Perturbative calculations performed in this paper can be extended to (4+1)-dimensional theories. Of course actual calculations become much more complicated and difficult because of severe ultraviolet divergences in 4+1 dimensions than in 2+1 dimensions. Work in this direction is in progress.

Note added. After finishing this work, a new paper by Narayanan and Neuberger [13] appeared. In their paper the gauge anomaly for the chiral Schwinger model was calculated semianalytically via the overlap formula of Ref. [6].

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