PHYSICAL REVIEW D VOLUME 49, NUMBER 4 15 FEBRUARY 1994

New geometrical insight into the anomalies in string theory

Jian-Ge Zhou
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Institute of High Energy Physics, Academia Sinica, P.O. Box 918(4) Beijing 100039, People's Republic of China

Yan-Gang Miao
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, People's Republic of China

Yao-Yang Liu
Department of Modern Physics, University of Science and Technology of China, Hefei 230026, People's Republic of China
(Received 21 July 1993)

We apply the method of coadjoint orbits to evaluate the nonvanishing square of the BRST charge and the central extension of the Virasoro algebra from the Weyl anomaly, which is based on the basic equation satisfied by the BRST operator on a coadjoint orbit associated with string theory.

PACS number(s): 11.25.Pm, 02.40.Dr, 03.65.Fd

Recently, the construction of field theories on coadjoint orbits of infinite-dimensional Lie groups has attracted considerable attention [1–3]. Indeed, the construction of the Wess-Zumino-Novikov-Witten (WZNW) theory and two-dimensional (2D) induced gravity on coadjoint orbits of Kac-Moody and Virasoro groups, respectively, revealed a similarity in their structures, and a natural interpretation for the SL(2,R) current algebra underlying the 2D induced gravity has been found. Later on, the method of coadjoint orbits was applied to construct bosonized actions for anomalous gauge theories in two and four space-time dimensions from the extended Lie algebra generated by the Gauss-law constraints [4]. The anomalous gauge algebra determines the anomalous part of the actions. Otherwise, such a geometrical formulation has also been used to analyze the commutation relations for the Gauss-law operators in anomalous gauge theories [5]. The Becchi-Rouet-Stora-Tyutin (BRST) operator on a coadjoint orbit associated with an anomalous gauge theory satisfies a basic equation, and this equation reproduces the commutation relations for the Gauss-law operators.

In this paper, we apply the method of coadjoint orbits to explore the relationship among the anomalies in the string theory. As is well known, in quantizing relativistic strings at subcritical dimensions one encounters anomalies that appear in different forms: the Weyl anomaly, the nonvanishing square of the BRST charge Q^2, and the Virasoro anomaly. Originally these anomalies were discovered as a result of detailed calculations which involved careful regularization of products of operators. The relationship among these anomalies has been discussed from different schemes, and most of the results can be found in Refs. [6–16]. Here we shall apply the variational principle to derive Eq. (31) which relates the BRST charge with the Weyl anomaly, and exploit the basic equation satisfied by the BRST operator [5] on a coadjoint orbit associated with the string theory to evaluate the square of the BRST charge Q^2 out of the Weyl anomaly. If we include the ghost contribution to the Virasoro anomaly, the central extension of the Virasoro algebra can be obtained from the value of Q^2.

First, let us recapitulate the theory of coadjoint orbits in the symplectic geometry of Lie groups G, laying a special emphasis upon the fundamental equation (22) and its solution, which we will use in our discussion. A coadjoint orbit is a manifold constructed by a coadjoint action of a connected Lie group G on a dual space of the Lie algebra. Let g be a Lie algebra of the group G, and g^* the dual space of g. The coadjoint action of G and g on g^* are defined, respectively, by

$$\langle \mathrm{Ad}^*(x)U, Y\rangle = \langle U, \mathrm{Ad}^{-1}(x)Y\rangle = \langle U, x^{-1}Yx\rangle \ , \tag{1}$$

$$\langle \mathrm{ad}^*(X)U, Y\rangle = -\langle U, \mathrm{ad}(X)Y\rangle = -\langle U, [X,Y]\rangle \ , \tag{2}$$

where $x\in G$, $X,Y\in g$, $U\in g^*$, and $\langle U,X\rangle = U(X)$ is a value of the linear functional $U\in g^*$ on the vector $X\in g$. ad* is the coadjoint representation of the Lie algebra, and satisfies the commutation relation

$$[\mathrm{ad}^*(X), \mathrm{ad}^*(Y)] = \mathrm{ad}^*([X,Y]) \tag{3}$$

which can be obtained from (2).

Let $\{X_a, X_b, \ldots\}$ be a basis of the Lie algebra g with the commutation relations

$$[X_a, X_b] = f_{ab}^c X_c \ , \tag{4}$$

where f_{ab}^c are structure constants, and let $\{\theta^a, \theta^b, \ldots\}$ be a basis of the dual space, which satisfies the orthogonality condition

$$\langle \theta^a, X_b\rangle = \delta_b^a \ . \tag{5}$$

The general point of the coadjoint orbit O_U through a fixed point $U_0 \in g^*$ is defined as

$$U(g) = \mathrm{Ad}^*(g)U_0 \tag{6}$$

which can be expressed in terms of the coordinates $\{u_a, u_b, \ldots\}$ with respect to the dual basis as

$$U(g)=\sum u_a \theta^a . \tag{7}$$

The tangent vector field X_a^* is expressed in terms of the coordinates on the coadjoint orbit as

$$X_a^*=\sum_{b,c} f_{ab}^c u_c \frac{\partial}{\partial u_b} \tag{8}$$

which satisfies the commutation relations (4).

On O_U a natural G-invariant symplectic structure is defined by introducing the symplectic two-form Ω_U at U as

$$\langle \Omega_U; X_U^*, Y_U^* \rangle = \langle U, [X,Y] \rangle . \tag{9}$$

The symplectic structure Ω_U can be written in terms of the one-form $\{v^a, v^b, \ldots\}$ dual to the vector fields $\{X_a^*, X_b^*, \ldots\}$ as

$$\Omega_U = \tfrac{1}{2} u_{ab} v^a \wedge v^b \tag{10}$$

with

$$u_{ab} = f_{ab}^c u_c . \tag{11}$$

The one-forms v^a satisfy the equations

$$du_a = u_{ab} v^b , \tag{12}$$

$$dv^c = -\tfrac{1}{2} f_{ab}^c v^a \wedge v^b , \tag{13}$$

owing to $d\Omega_U=0$ [1,5].

Denote by δ_ϵ the infinitesimal transformation on the orbit by ϵ. As a matter of fact δ_X ($X \in g$) is the Lie derivative with respect to X:

$$\delta_X U = L_X U = \mathrm{ad}^*(X) U , \tag{14}$$

$$\delta_X Y = L_X Y = \mathrm{ad}(X) Y = [X,Y] . \tag{15}$$

Let us consider the infinitesimal transformation on a coadjoint orbit generated by an underlying generator Q which takes values in g. For simplicity, we denote $\delta_Q = \delta$. Then the basic equations are

$$\delta U = \mathrm{ad}^*(Q) U , \tag{16}$$

$$\delta X = \mathrm{ad}(Q) X = [Q,X] , \tag{17}$$

$$\delta v^c = -\tfrac{1}{2} f_{ab}^c v^a \wedge v^b , \tag{18}$$

and

$$\delta u_a = \langle U, [Q, X_a] \rangle = u_{ab} v^b . \tag{19}$$

The solution to (19) is

$$Q = X_a v^a . \tag{20}$$

It should be noted that the solution (20) is not unique, since we may add terms which commute with generators (X_a).

Taking infinitesimal transformations of both sides of Q,

$$\begin{aligned} \delta Q &= \delta X_a v^a + X_a \delta v^a \\ &= [Q, X_a] v^a - \tfrac{1}{2} f_{ab}^c X_c v^a \wedge v^b , \end{aligned} \tag{21}$$

and finally one obtains the basic equation for Q [5]:

$$\delta Q = Q^2 = \tfrac{1}{2}\{Q,Q\} . \tag{22}$$

Equation (22) is the fundamental equation for Q, so that the equation should hold even if there exist central extensions in the algebra. And we will use this basic equation to evaluate the nonvanishing square of the BRST charge from the Weyl anomaly.

Now let us consider the Polyakov string theory described by the action [6]

$$S_0 = -\tfrac{1}{2} \int d^2 Z \sqrt{-g}\, g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu , \tag{23}$$

where $x^\mu (\mu=0,1,\ldots,D-1)$ are the coordinates (in the Minkowski space) of the string; $g^{\alpha\beta}$ $(\alpha,\beta=0,1)$ is the metric tensor of the parameter space; $\partial_0 \equiv \partial_\tau$, $\partial_1 \equiv \partial_\sigma$; $g \equiv \det g_{\alpha\beta}$. To avoid boundary effects, we will only discuss the closed string.

The action S_0 is invariant under a reparametrization and Weyl transformation. And the gauge-fixed action [7,8,13]

$$S = S_0 + S_g \tag{24}$$

is invariant under the BRST symmetry which is defined by

$$\begin{aligned} &\delta g_{\alpha\beta} = \eta^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\beta \eta^\gamma g_{\alpha\gamma} + \partial_\alpha \eta^\gamma g_{\beta\gamma} + \eta_W g_{\alpha\beta} , \\ &\delta x_\mu = \eta^\alpha \partial_\alpha x_\mu , \quad \delta \eta^\alpha = \eta^\beta \partial_\beta \eta^\alpha , \quad \delta \eta_W = \eta^\alpha \partial_\alpha \eta_W , \\ &\delta \bar\eta_\alpha = B_\alpha , \quad \delta \bar\eta_W = B_W , \end{aligned} \tag{25}$$

where $\eta^\alpha(\bar\eta_\alpha)$ stand for the ghost (antighost) fields for reparametrization, $\eta_W(\bar\eta_W)$ for the ghost (antighost) fields for the Weyl symmetry, and B_α, B_W for Nakanishi-Lautrup auxiliary fields. S_g is the sum of the gauge-fixing term and Faddeev-Popov ghost term.

Now we apply the variational principle to the action functional, and obtain

$$\delta S = \int d^2 Z\, \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^i} \delta \phi^i \right] , \tag{26}$$

where we have used the Euler-Lagrange equations. ϕ^i are localized fields.

For the action (24), δS can be expressed as [8]

$$\delta S = \int d^2 Z\, \partial_\alpha \left[\frac{\partial \mathcal{L}_0}{\partial \partial_\alpha x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}_g}{\partial \partial_\alpha \eta^\beta} \delta \eta^\beta + \frac{\partial \mathcal{L}_g}{\partial \partial_\alpha \eta_W} \delta \eta_W \right] , \tag{27}$$

where $\mathcal{L}_0, \mathcal{L}_g$ are the Lagrangian densities of S_0 and S_g, respectively.

On the other hand, the variation of the gauge-fixed action under the BRST transformation (25) is given by [8,14]

$$\delta S = \int d^2 Z\, \partial_\alpha (\eta^\alpha \mathcal{L}_0) . \tag{28}$$

However, because of the Weyl anomaly [6,7], the above equation should be modified as [8,13,14]

$$\delta S = \int d^2 Z\, \partial_\alpha (\eta^\alpha \mathcal{L}_0) + \delta_W S \tag{29}$$

with

$$\delta_W S = \frac{D-26}{48\pi}\int d^2Z\sqrt{-g}\,R(g)\eta_W\ , \tag{30}$$

where we have included the ghost contribution to the Weyl anomaly. $R(g)$ is the curvature for the two-dimensional $g_{\alpha\beta}$.

Combining (27) with (29) and (30) and taking the boundary conditions of the closed string we are led to

$$\delta_W S = dQ \tag{31}$$

with

$$Q=\int_0^{2\pi}\left[p_\mu\delta x^\mu-\eta^0\mathcal{L}_0+\frac{\partial\mathcal{L}_g}{\partial\partial_0\eta^\beta}\delta\eta^\beta+\frac{\partial\mathcal{L}_g}{\partial\partial_0\eta_W}\delta\eta_W\right]\ . \tag{32}$$

Equation (31) is similar to the descent equation (3.4) in Ref. [5].

Under the rotation back to the Minkowski space $\delta_w S$ picks up the factor $-i$ [14]. And (31) can be expressed as

$$(-i)\frac{D-26}{48\pi}\int^\tau d^2Z\sqrt{-g}\,R(g)\eta_W=Q\ , \tag{33}$$

where the variable τ should be integrated out but this operation will be carried out below.

By virtue of the BRST transformation, Q can be written as

$$Q=\int_0^{2\pi}\left[\frac{1}{4}(p+x')^2\bar\eta^+ +\frac{1}{4}(p-x')^2\bar\eta^- +q_c\right] \tag{34}$$

with

$$\bar\eta^\pm=\bar\eta^0\pm\bar\eta^1,\quad \bar\eta_0=-\frac{1}{\sqrt{-g}\,g^{00}}\eta^0,\quad \bar\eta^1=\eta^1-\frac{g^{01}}{g^{00}}\eta^0\ . \tag{35}$$

The explicit form of q_c depends on S_g, which consists of the ghost fields. As a matter of fact Q is nothing but the BRST operator [8,13].

According to the basic equation (22) satisfied by Q, one has

$$\{Q,Q\}=2\delta Q=i\frac{26-D}{24\pi}\int^\tau d^2Z\,\delta[\sqrt{-g}\,R(g)\eta_W]\ . \tag{36}$$

Under the BRST transformation (25), $\delta(\sqrt{-g}\,R\,\eta_W)$ can be put into the form [13,14,16]

$$\delta(\sqrt{-g}\,R\,\eta_W)=\partial_\alpha(\eta^\alpha\eta_W\sqrt{-g}\,R+\sqrt{-g}\,g^{\alpha\beta}\eta_W\partial_\beta\eta_W)\ . \tag{37}$$

By making use of the boundary condition of closed strings and integrating out the variable τ, we obtain

$$\{Q,Q\}=i\frac{26-D}{24\pi}\int_0^{2\pi}d\sigma(\,\eta^0\eta_W\sqrt{-g}\,R +\sqrt{-g}\,g^{0\beta}\eta_W\partial_\beta\eta_W)\ . \tag{38}$$

In the orthonormal gauge, the above result agrees with that in Refs. [14,16]. To see this we insert the orthonormal gauge conditions ($g_{\alpha\beta}=\eta_{\alpha\beta}$) and the "safe" equation of motion which are not affected by the Weyl anomaly [16],

$$\eta_W=-\partial_\alpha\eta^\alpha,\quad \partial_\mp\eta^\pm=0\ ,$$
$$(\partial_\pm=\partial_\tau\pm\partial_\sigma,\quad \eta^\pm=\eta^0\pm\eta^1)\ , \tag{39}$$

into (38) to obtain the known expression [14,16]

$$\{Q,Q\}=i\frac{26-D}{24\pi}\int_0^{2\pi}d\sigma(\eta^+\partial^3\eta^+-\eta^-\partial^3\eta^-)\ . \tag{40}$$

In the orthonormal gauge, the BRST operator becomes

$$Q=\int_0^{2\pi}d\sigma(L_+\eta^+ +L_-\eta^- -\bar\eta^0\eta^\alpha\partial_\alpha\eta^0-\bar\eta^1\eta^\alpha\partial_\alpha\eta^1)\ ,$$
$$[L_+=\tfrac{1}{4}(p+x')^2, L_-=\tfrac{1}{4}(p-x')^2]\ , \tag{41}$$

where we have chosen $q_c=-\bar\eta^0\eta^\alpha\partial_\alpha\eta^0-\bar\eta^1\eta^\alpha\partial_\alpha\eta^1$ as in Ref. [8].

Inserting (39) into (41), the BRST operator can be rewritten as

$$Q=\int_0^{2\pi}d\sigma(\,L_+\eta^+ +L_-\eta^- +\bar\eta^+\partial_1\eta^+\eta^+ -\bar\eta^-\partial_1\eta^-\eta^-)\ . \tag{42}$$

This expression is of the same form as that in Ref. [9]. And the ghost fields η^+, η^- satisfy the fundamental canonical anticommutation relations [8,9]

$$\{\bar\eta^\pm(\sigma),\eta^\pm(\sigma')\}=i\delta(\sigma-\sigma')\ ,$$
$$\{\bar\eta^\pm(\sigma),\eta^\mp(\sigma')\}=0\ . \tag{43}$$

In order to derive the Virasoro anomaly from the value of Q^2, we should Fourier decompose the BRST operator. And $L_\pm$ just become

$$L_+(\tau,\sigma)=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}L_n e^{-i(\sigma+\tau)n}\ ,$$
$$L_-(\tau,\sigma)=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\tilde L_n e^{-i(\sigma-\tau)n}\ . \tag{44}$$

For the ghost variables, we get

$$\eta^\pm(\tau,\sigma)=\sum_{n=-\infty}^{+\infty}\eta_n^\pm e^{-i(\sigma\pm\tau)n}\ ,$$
$$\bar\eta^\pm(\tau,\sigma)=\frac{i}{2\pi}\sum_{n=-\infty}^{+\infty}\bar\eta_n^\pm e^{-i(\sigma\pm\tau)n}\ , \tag{45}$$

with

$$\{\bar\eta_n^\pm,\eta_m^\pm\}=\delta_{n,-m},\quad \{\bar\eta_n^\pm,\eta_m^\mp\}=0\ . \tag{46}$$

Using (44) and (45), we get a decomposition of Q:

$$Q=\sum_n(\eta_{-n}^+L_n+\eta_{-n}^-\tilde L_n) -\frac{1}{2}\sum_{n,m}(n-m)[\bar\eta_{n+m}^+\eta_{-n}^+\eta_{-m}^+ +\bar\eta_{n+m}^-\eta_{-n}^-\eta_{-m}^-]\ . \tag{47}$$

Under Fourier decomposition, the right-hand side (RHS) of (40) becomes

$$\{Q,Q\}=\frac{D-26}{12}\sum_{n,m} n^3\delta_{n,-m}(\eta^{+}_{-n}\eta^{+}_{-m}-\eta^{-}_{-n}\eta^{-}_{-m}) . \tag{48}$$

Inserting (47) into the LHS of (48), we obtain

$$\begin{aligned}
[L_n,L_m]&=(n-m)L_{n+m}+\frac{D-26}{12}n^3\delta_{n,-m} ,\\
[\tilde{L}_n,\tilde{L}_m]&=(n-m)\tilde{L}_{n+m}-\frac{D-26}{12}n^3\delta_{n,-m} ,\\
[L_n,\tilde{L}_m]&=0 ,
\end{aligned} \tag{49}$$

where we have included the ghost contribution to the central extension of the Virasoro algebra, since we do not consider the normal ordering of the ghost part in (47).

Note that redefining L_0 and $\tilde{L}_0$ as

$$\begin{aligned}
L_0'&=L_0+\frac{1}{24}(D-26) ,\\
\tilde{L}_0'&=L_0-\frac{1}{24}(D-26)
\end{aligned} \tag{50}$$

leaves the structure of the Virasoro algebra (49) unchanged, but the central extension is modified to the standard form

$$\frac{1}{12}(D-26)(n^3-n)\delta_{n,-m} . \tag{51}$$

This result shows that the piece of the Virasoro anomaly proportional to n^3 is topological in origin and cannot be eliminated.

In summary, we have evaluated the value of Q^2 and the central extension of the Virasoro algebra out of the Weyl anomaly by virtue of the basic equation satisfied by the BRST operator, which can be derived from the method of coadjoint orbits.

This work was partially supported by the National Nature Science Foundation of China under the Grant No. 19275040.

[1] A. Alekseev and S. Shatashvili, Nucl. Phys. **B323**, 719 (1988); Commun. Math. Phys. **128**, 197 (1990).
[2] S. Aoyama, Phys. Lett. B **228**, 355 (1989); H. Aratyn, E. Nissimov, S. Pacheva, and S. Solomon, *ibid.* **234**, 307 (1990); H. Aratyn, E. Nissimov, S. Pacheva, and A. H. Zimerman, *ibid.* **240**, 127 (1990).
[3] P. B. Wiegman, Nucl. Phys. **B323**, 330 (1989); S. Aoyama, Phys. Lett. B **235**, 79 (1990).
[4] D. Bar-Moshe, M. S. Marinov, and Y. Oz, Phys. Lett. B **254**, 115 (1991).
[5] R. Kubo, Phys. Rev. D **47**, 2591 (1993).
[6] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).
[7] K. Fujikawa, Phys. Rev. D **25**, 2584 (1982).
[8] M. Kato and K. Ogawa, Nucl. Phys. **B212**, 443 (1983).
[9] D. S. Hwang, Phys. Rev. D **28**, 2614 (1983).
[10] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1983).
[11] D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271**, 93 (1986).
[12] K. Fujikawa, Phys. Lett. B **188**, 583 (1987); J. Louis, B. A. Ovrut, R. Garreis, and J. Wess, *ibid.* **199**, 57 (1987).
[13] K. Fujikawa, Nucl. Phys. **B291**, 583 (1987).
[14] P. Mansfield, Ann. Phys. (N.Y.) **180**, 330 (1987); I. Kani and P. Mansfield, Phys. Lett. B **207**, 135 (1988).
[15] K. Fujikawa, T. Inagaki, and H. Suzuki, Phys. Lett. B **213**, 279 (1988); Nucl. Phys. **B332**, 499 (1990).
[16] T. Fujiwara, Y. Igarashi, and J. Kubo, Nucl. Phys. **B387**, 447 (1992); **B391**, 211 (1993).