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General spherically symmetric solutions in charged dilaton gravity

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The static spherically symmetric metric around a source coupled to an electromagnetically charged massless dilaton is obtained in the general case. The appearance of naked singularities is displayed.

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Nowadays there is a growing body of literature about the gravitational field of string matter coupled to an electromagnetically charged dilaton field. Black hole solutions in dilaton gravity were first analyzed in some generality by Gibbons and Maeda [1]. A family of solutions representing static, spherically symmetric charged black holes was described in a later work by Garfinkle, Horowitz, and Strominger [2] and recently investigated by Kallosh *et al.* [3,4] in the context of supersymmetric theories. The modification of dilaton black holes which result when the dilaton acquires a mass was subsequently analyzed by Gregory and Harvey [5]. In all those papers, however, the dilaton charge is not an independent variable and its particular dependence on the other parameters of the theory necessarily singles out, between all others, only black hole solutions.

In this paper we will obtain, in a simple closed form and with all parameters free, the static spherically symmetric metric around a source coupled to a massless dilaton with both electric and magnetic charge. This general solution covers all the cases suitable to the various regions of the available parameter space and describes, apart from the above-mentioned black hole dilaton, a wider variety of situations where the event horizon shrinks to the pointlike essential singularity, thus forbidding the appearance of black holes.

The equations of motion of the dimensionally reduced superstring theory with the axion field put to a constant are, in the SU(4) version,

$$\begin{gathered}\nabla_\mu(e^{-2\phi}F^{\mu\nu})=0\ ,\quad \nabla_\mu(e^{-2\phi}G^{\mu\nu})=0\ ,\\ \nabla^2\phi-\tfrac{1}{2}e^{-2\phi}F^2-\tfrac{1}{2}e^{-2\phi}G^2=0\ ,\\ R_{\mu\nu}+2\nabla_\mu\phi\nabla_\nu\phi-e^{-2\phi}(2F_{\mu\lambda}F^{\lambda}_{\nu}-\tfrac{1}{2}g_{\mu\nu}F^2)\\ -e^{-2\phi}(2G_{\mu\lambda}G^{\lambda}_{\nu}-\tfrac{1}{2}g_{\mu\nu}G^2)=0\ .\end{gathered} \tag{1}$$

The required line element in four space-time dimensions will be written in the form

$$ds^2=\lambda\,dt^2-\frac{1}{\lambda}dr^2-R^2(d\vartheta^2+\sin^2\vartheta\,d\varphi^2)\ , \tag{2}$$

where λ and R, as well as the dilaton field ϕ, are a function of r only. The Maxwell fields are

$$F=\frac{Q_{\mathrm{elec}}e^{2\phi}}{R^2}dt\wedge dr\ , \tag{3}$$

$$G=P_{\mathrm{magn}}e^{2\phi}d\vartheta\wedge d\varphi\ . \tag{4}$$

For future simplicity the electric charge Q_{elec} of the F field will be put equal to Qe^{ϕ_∞} and the magnetic charge P_{magn} of the G field will be put equal to $Pe^{-\phi_\infty}$, where ϕ_∞ is the asymptotic value of the dilaton field.

In (1) Maxwell equations are automatically satisfied and the other equations become

$$\begin{aligned}(R^2\lambda\phi')'&=\frac{1}{R^2}(Q^2e^{2(\phi-\phi_\infty)}-P^2e^{-2(\phi-\phi_\infty)})\ ,\\ (R^2\lambda')'&=\frac{2}{R^2}(Q^2e^{2(\phi-\phi_\infty)}+P^2e^{-2(\phi-\phi_\infty)})\ ,\\ 2\frac{R''}{R}+\frac{(R^2\lambda')'}{2R^2\lambda}&=-2\phi'^2+\frac{1}{R^2\lambda}(Q^2e^{2(\phi-\phi_\infty)}+P^2e^{-2(\phi-\phi_\infty)})\ ,\\ [\lambda(R^2)']'&=2-\frac{2}{R^2}(Q^2e^{2(\phi-\phi_\infty)}+P^2e^{-2(\phi-\phi_\infty)})\ ,\end{aligned} \tag{5}$$

where a prime denotes a derivative with respect to r. Equations (5) can be combined to give

$$(R^2\lambda)''=2\ , \tag{6}$$

$$R''+R\phi'^2=0\ , \tag{7}$$

$$(2R^2\lambda\phi'+R^2\lambda')'=4Q^2\frac{e^{(2\phi-\phi_\infty)}}{R^2}\ , \tag{8}$$

$$(-2R^2\lambda\phi'+R^2\lambda')'=4P^2\frac{e^{-2(\phi-\phi_\infty)}}{R^2}\ . \tag{9}$$

The solution depends on the five free parameters M, Σ, P, Q, and ϕ_∞, where the mass M of the source and the charge Σ of the massless dilaton are defined, respectively, by the equations $\lambda\sim 1-2M/r$ and $\phi\sim\phi_\infty-\Sigma/r$ at $r\to\infty$.

We will look for the solution of the equations of motion in the following way. Equation (6) gives immediately

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$$R^2\lambda=(r-r_+)(r-r_-)\ . \tag{10}$$

Defining the quantities $f=\lambda e^{2(\phi-\phi_\infty)}$ and $g=\lambda e^{-2(\phi-\phi_\infty)}$, Eqs. (8) and (9) become, respectively,

$$\left[R^2\lambda\frac{f'}{f}\right]'=4Q^2\frac{f}{R^2\lambda}\ , \tag{11}$$

$$\left[R^2\lambda\frac{g'}{g}\right]'=4P^2\frac{g}{R^2\lambda}\ , \tag{12}$$

and can be properly solved with respect to f and g.

The functions f and g have the following expressions in the various regions of parameter space:

$$f=\left[\frac{1-A}{1-Ae^{ah(r)}}\right]^2 e^{ah(r)} \ \text{for}\ (M+\Sigma)^2-2Q^2>0\ , \tag{13a}$$

$$f=[1-\sqrt{2}|Q|h(r)]^{-2} \ \text{for}\ (M+\Sigma)^2-2Q^2=0\ , \tag{13b}$$

$$f=\left[1-\frac{(M+\Sigma)^2}{2Q^2}\right]\left\{1+\tan^2\left[\arctan\frac{1}{\left[\frac{2Q^2}{(M+\Sigma)^2}-1\right]^{1/2}}+\sqrt{2Q^2-(M+\Sigma)^2}h(r)\right]\right\} \ \text{for}\ (M+\Sigma)^2-2Q^2<0\ , \tag{13c}$$

and

$$g=\left[\frac{1-B}{1-Be^{bh(r)}}\right]^2 e^{bh(r)} \ \text{for}\ (M-\Sigma)^2-2P^2>0\ , \tag{14a}$$

$$g=[1-\sqrt{2}|P|h(r)]^{-2} \ \text{for}\ (M-\Sigma)^2-2P^2=0\ , \tag{14b}$$

$$g=\left[1-\frac{(M-\Sigma)^2}{2P^2}\right]\left\{1+\tan^2\left[\arctan\frac{1}{\left[\frac{2P^2}{(M-\Sigma)^2}-1\right]^{1/2}}+\sqrt{2P^2-(M-\Sigma)^2}h(r)\right]\right\} \ \text{for}\ (M-\Sigma)^2-2P^2<0\ , \tag{14c}$$

where

$$h(r)=\begin{cases}\dfrac{1}{r_+-r_-}\ln\left[\dfrac{r-r_+}{r-r_-}\right] & \text{for}\ r_+-r_->0\ ,\\[2ex] -\dfrac{1}{r-r_+} & \text{for}\ r_+-r_-=0\ ,\end{cases} \tag{15}$$

and

$$A=\frac{1-\left[1-\dfrac{2Q^2}{(M+\Sigma)^2}\right]^{1/2}}{1+\left[1-\dfrac{2Q^2}{(M+\Sigma)^2}\right]^{1/2}}\ ,$$

$$B=\frac{1-\left[1-\dfrac{2P^2}{(M-\Sigma)^2}\right]^{1/2}}{1+\left[1-\dfrac{2P^2}{(M-\Sigma)^2}\right]^{1/2}}\ , \tag{16}$$

$$a=2\sqrt{(M+\Sigma)^2-2Q^2}\ ,$$

$$b=2\sqrt{(M-\Sigma)^2-2P^2}\ .$$

Moreover the integration constants in Eq. (10) must obey the condition

$$r_+-r_-=2\sqrt{M^2+\Sigma^2-P^2-Q^2}\ . \tag{17}$$

Physical configurations are therefore obtained in those regions of parameter space where $M^2+\Sigma^2-P^2-Q^2\geq 0$, which is the supersymmetric positivity bound.

The functions f and g chosen, respectively, from Eqs. (13) and from Eqs. (14) are physically acceptable only if the reality condition in Eq. (17) is satisfied, thus forbidding the simultaneous choice of those pairs of functions for which the positivity bound turns out to be violated.

Taking care of this fact Eqs. (1) are therefore solved by

$$\lambda=\sqrt{fg}\ ,$$

$$R^2=\frac{(r-r_+)(r-r_-)}{\sqrt{fg}}\ , \tag{18}$$

$$e^{2(\phi-\phi_\infty)}=\left[\frac{f}{g}\right]^{1/2}\ ,$$

together with the Maxwell fields in Eqs. (3) and (4).

Let us notice that taking the limit $\Sigma\rightarrow 0$ in the line element (2) makes sense, because of the first of Eqs. (5), only if simultaneously $P^2-Q^2\to 0$; accordingly one has the Reissner-Nordström solution when $|P|=|Q|$ and the Schwarzschild solution when $P=Q=0$.

In the general case the metric behaves quite differently in the various regions of parameter space, even if in the asymptotic region its common behavior is

$$ds^2\sim\left[1-\frac{2M}{R}+\frac{P^2+Q^2}{R^2}\right]dt^2 - \frac{dR^2}{\left[1-\frac{2M}{R}+\frac{\Sigma^2+P^2+Q^2}{R^2}\right]} - R^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2)\ , \tag{19}$$

where use has been made of standard coordinates to have a better comparison with classical solutions.

Let us first consider the case when the functions f and g are given by Eqs. (13a) and (14a), respectively, and so $r_+-r_->0$. Because of the form invariance of the metric under shifting r by a constant, we put simply $r_-=0$. The radial coordinate r now is bounded to vary in the range from $r_+=2\sqrt{M^2+\Sigma^2-P^2-Q^2}$ to ∞, and when it reaches its minimum value r_+ one can see that λ vanishes. The standard radial coordinate R behaves differently at $r=r_+$ according to the value of the quantity $2M\Sigma+P^2-Q^2$.

When $2M\Sigma+P^2-Q^2\neq 0$, R tends to zero as r tends to r_+, so the seeming horizon at $r=r_+$ actually shrinks to the essential pointlike singularity at $R=0$ where the invariant scalar curvature $R_{\mu\nu}R^{\mu\nu}$ becomes singular. We cannot speak of a black hole dilaton, because the redshift would become infinitely large for a radius infinitely small. The situation was already considered by us in a previous work [6]; the line element we found there in the case of a scalar modified Schwarzschild spacetime, can now be recovered from the actual metric setting to zero both the electric and the magnetic charges of the dilaton. When $2M\Sigma+P^2-Q^2=0$, which is the choice made in [1–3], R reaches a finite value at $r=r_+$ and one has consequently a black hole dilaton.

Another possible choice for the functions f and g is obtained by selecting either Eqs. (13a) and (14b) or Eqs. (13b) and (14a) and this gives the previous result for the case when $2M\Sigma+P^2-Q^2\neq 0$, i.e., the absence of black holes. When the functions f and g are given, respectively, by Eq. (13b) and by Eq. (14b), which implies $r_+-r_-=0$, it follows also that $2M\Sigma+P^2-Q^2=0$. This is the case of the extreme charged dilaton black holes, which has been extensively treated in Refs. [3,4].

Different conclusions can be drawn when f or g are given by the other admissible combinations of Eqs. (13) and (14), namely, either by Eqs. (13a) and (14c) or by Eqs. (13c) and (14a), and consequently r_+ may be equal or greater than r_- but $2M\Sigma+P^2-Q^2\neq 0$. In each of these cases the radial coordinate r is bounded to vary in the range from r_0 to ∞, where r_0 is the greatest of the values of r for which either the right-hand side of Eq. (14c) or that of Eq. (13c) becomes infinitely large.

One can check that at $r=r_0$, where λ tends to infinity, R tends to zero because r_0 is greater than r_+. Now it happens that there is a blueshift becoming infinitely large for a radius becoming infinitely small. A graphical representation of the above discussion in the (M,Σ) plane of parameter space highlights the existence of continuous regions each with its own spacetime description. Conventional black holes appear only where the condition $2M\Sigma+P^2-Q^2=0$ is exactly verified, otherwise one enters a neighboring region where black holes are not allowed. It is apparent that however small a variation of parameters be in the above equality it forces the surface at $r=r_+$ to suddenly assume the topology of a point.

From the above discussion it follows that in the new solutions we obtained in this paper there is the appearance of naked singularities. Unfortunately definite arguments in support of the stability of these solutions are presently missing, and future investigations into this subject are therefore needed. As a conclusion we nevertheless would like to point out that consequences of charged dilaton gravity for the cosmic censor hypothesis might prove to be of particular interest.

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