Coarse-grained entropy and stimulated emission in curved space-time

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We study entropy generation and particle production in scalar quantum field theory in expanding space-times with many-particle mixed initial states. The recently proposed coarse-grained entropy approach by Brandenberger *et al.* is applied to systems which may have a nonzero initial entropy. We find that although the particle production is amplified as a result of boson statistics, the (coarse-grained) entropy generation is *attenuated* when initial particles are present.

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One of the interesting features of quantized fields in a curved space-time [1] is that the concept of particles becomes very observer dependent. For instance, in an expanding universe spontaneous particle creation can occur. One defines generally a vacuum state such that all inertial observers in the past region agree that the space-time looks empty of particles. As a result of the expansion of the Universe, the above vacuum state looks full of particles using modes natural to inertial observers in the far future region. Stated differently, a no-particle initial state can evolve to a many-particle state. However, since one starts with a pure state, one ends with a pure state. Thus there must be subtle correlations between the particles in the final state. In particular, there is no entropy production in this process even if lots of particles are produced. But it may be that some of these subtle correlations are very difficult to detect and/or that they may be quite sensitive to interactions between the produced particles. One may then consider such information about the system to be "less relevant" and either discard it altogether or apply some kind of a "statistical averaging" procedure to it. This way one can try to associate a "coarse-grained" entropy to the final state of the system, hopefully in as natural way as possible. There has been a lot of work in this direction by Hu, Kundrup, and coworkers [2].

Recently, such novel approaches have been proposed. Brandenberger, Mukhanov, and Prokopec (BMP) discussed in Refs. [3,4], among other issues, a coarse-graining procedure based on averaging over the so-called squeeze angles which appear in the S matrix of particle production. On the other hand, Gasperini and Giovannini [5], together with Veneziano (GGV) [6] related entropy generation to an increased dispersion of a superfluctuant operator. Both groups were especially interested in the entropy generation related to the production of gravitational waves and density fluctuations in inflationary universe models.

In this Brief Report, we study the coarse-graining procedure based on averaging over the squeeze angles, which we shall call the BMP approach. We investigate the entropy generation starting not from an initial vacuum state with zero entropy, but allowing the system to be initially in some generic many-particle (mixed) state with nonzero entropy. If one starts with many bosons it is known [7] that the particle production will be amplified as a result of boson statistics, as one would expect. So, in general, one can ask whether or not the entropy generation (in the coarse-grained sense) would also be amplified. Indeed, as a consistency check it is necessary to investigate if definitions of coarse-grained entropy will lead to a growing entropy even if the initial state is allowed to be an arbitrary many-particle state with initial entropy. In Refs. [5,6] the GGV entropy was shown to be growing at least in certain classes of initial states. Interestingly, it was found that their entropy generation did not depend at all on the number of particles or entropy of the initial state. Here we will attempt to investigate the BMP entropy in similar situations. At least in the case of an initial density matrix where particles appear as pairs of opposite momenta and the initial entropy depends on the average occupation number per mode, we can show that the BMP entropy grows, though the entropy generation is *attenuated.* The BMP entropy *does* depend on the initial number of particles in a nontrivial way. In the end we comment briefly on the case of an initial thermal density matrix.

A scalar field in a D-dimensional curved space-time is described by an action

$$S=\int d^Dx\sqrt{-g}\,\frac{1}{2}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-[m^2+\xi R(x)]\phi^2\}\ , \qquad (1)$$

where $R(x)$ is the Ricci scalar curvature of the metric and ξ is a coupling constant. Assume that the metric depends explicitly on time and that it is asymptotically flat in the far past and far future: $g_{\mu\nu}(\vec{x},t)\to C_\pm\eta_{\mu\nu}$, as $t\to\pm\infty$. In this case there are two natural ways to quantize the field ϕ in the Heisenberg picture [1]. One can either use modes which look like plane waves in the far past region or modes which look like plane waves in the far future region, respectively. One then associates two sets of annihilation and/or creation operators to these modes, the "in" and "out" operators. These in turn define two vacua: one for the "in" annihilation operators and one for the "out" operators.

The "in" and "out" modes can be related via a Bogoliubov transformation, which can be given in terms of annihilation and/or creation operators as

$$a_{\vec{k}}^{\text{in}}=\alpha_{\vec{k}\vec{p}}^{*}a_{\vec{p}}^{\text{out}}-\beta_{\vec{k}\vec{p}}^{*}a_{\vec{p}}^{\dagger\text{out}} . \tag{2}$$

This transformation is generated by an S matrix

$$a_{\vec{k}}^{\text{in}}=\mathbf{S}a_{\vec{k}}^{\text{out}}\mathbf{S}^{-1} , \tag{3}$$

which has the explicit form [8]

$$\mathbf{S}=\frac{1}{\sqrt{\det(\alpha_{\vec{k}\vec{p}})}}:\exp\{\tfrac{1}{2}[\alpha^{-1}\beta^{*}]_{\vec{k}\vec{k}'}a_{\vec{k}}^{\dagger\text{out}}a_{\vec{k}'}^{\dagger\text{out}} +[\alpha^{-1}-1]_{\vec{k}\vec{k}'}a_{\vec{k}}^{\dagger\text{out}}a_{\vec{k}'}^{\text{out}} -\tfrac{1}{2}[\alpha^{-1}\beta]_{\vec{k}\vec{k}'}a_{\vec{k}}^{\text{out}}a_{\vec{k}'}^{\text{out}}\}: . \tag{4}$$

The factor $1/\sqrt{\det(\alpha_{\vec{k}\vec{p}})}$ is the in-out vacuum amplitude. We use the convention of Ref. [9] where the coefficients α are taken to be real. The S matrix is known to generate a unitary transformation between the "in" and "out" representations if the gravitational field has a compact support [10]. For Robertson-Walker-type universes the in-out vacuum amplitude is zero and the "in" and "out" representations are thus unitarily inequivalent.

The S matrix relates the in- and out-vacuum states in the following way:

$$|0,\text{in}\rangle=\mathbf{S}|0,\text{out}\rangle =\frac{1}{\sqrt{\det(\alpha)}}\exp\{\tfrac{1}{2}[\alpha^{-1}\beta]_{\vec{k}\vec{k}'}a_{\vec{k}}^{\dagger\text{out}}a_{\vec{k}'}^{\dagger\text{out}}\}|0,\text{out}\rangle . \tag{5}$$

This is the statement that an inertial observer in the far future region sees the in-vacuum state as full of out particles. Similarly, the density matrix of the system expanded using in modes ($\equiv\rho_i$) can be related to an expression using out modes ($\equiv\rho_f$) as follows:

$$\begin{aligned}\rho_i&=\prod_{\vec{k}}\sum_{n_{\vec{k}}=0}^{\infty}f(n_{\vec{k}})|n_{\vec{k}},\text{in}\rangle\langle\text{in},n_{\vec{k}}|\\&=\prod_{\vec{k}}\sum_{n_{\vec{k}}=0}^{\infty}f(n_{\vec{k}})\mathbf{S}|n_{\vec{k}},\text{out}\rangle\langle\text{out},n_{\vec{k}}|\mathbf{S}^{-1}\\&\equiv\mathbf{S}\rho_f\mathbf{S}^{-1} ,\end{aligned} \tag{6}$$

using (3) and (5). Suppose now that the system is initially in an arbitrary many-particle state. In this state the average occupation number per mode (using in modes) is given by

$$\bar{n}_{\vec{k}}^{i}=\frac{1}{\text{Tr}\rho_i}\text{Tr}(\rho_i a_{\vec{k}}^{\dagger\text{in}}a_{\vec{k}}^{\text{in}}) . \tag{7}$$

In the far future region an inertial observer sees the average occupation number per mode using out modes as

$$\bar{n}_{\vec{k}}^{f}=\frac{1}{\text{Tr}\rho_f}\text{Tr}(\mathbf{S}\rho_f\mathbf{S}^{-1}a_{\vec{k}}^{\dagger\text{out}}a_{\vec{k}}^{\text{out}}) . \tag{8}$$

Using the cyclicity of the trace and the properties of the Bogoliubov transformation one can derive the relation between $\bar{n}_{\vec{k}}^{f}$ and $\bar{n}_{\vec{k}}^{i}$ to be

$$\bar{n}_{\vec{k}}^{f}=|\alpha_{\vec{k}\vec{p}}|^2\bar{n}_{\vec{p}}^{i}+|\beta_{\vec{k}\vec{p}}|^2(1+\bar{n}_{\vec{p}}^{i}) . \tag{9}$$

This is the formula for "stimulated emission" [7]. It tells us that even if the spontaneous creation of particles is weak, $|\beta_{\vec{k}\vec{p}}|^2\ll 1$, the particle production $\bar{n}_{\vec{k}}^{f}-\bar{n}_{\vec{k}}^{i}$ can become arbitrarily large, if the initial average occupation number per mode is arbitrarily large. This amplification of particle production is a result of the boson statistics of the particles. For fermions the particle production would be attenuated [11].

Let us now discuss for simplicity metrics of the form $ds^2=dt^2-a^2(t)d\vec{x}^2$, where $a(t)$ is a scale parameter of the Universe. We again just require that $a(t)\to a_{\pm}$ asymptotically as $t\to\pm\infty$. As a result of the invariance under spatial translations, the Bogoliubov coefficients can be written as

$$\begin{aligned}\alpha_{\vec{k}\vec{p}}&=\alpha_{\vec{k}}\delta_{\vec{k},\vec{p}}\equiv\cosh r_{\vec{k}}\delta_{\vec{k},\vec{p}} ,\\\beta_{\vec{k}\vec{p}}&=\beta_{\vec{k}}\delta_{\vec{k},-\vec{p}}\equiv e^{i\theta_{\vec{k}}}\sinh r_{\vec{k}}\delta_{\vec{k},-\vec{p}} .\end{aligned} \tag{10}$$

The parameters $r_{\vec{k}}$ and $\theta_{\vec{k}}$ are called squeeze parameter and squeeze angle in the quantum optics literature, and the S matrix is called a two-mode squeeze operator [12]. If one starts with a vacuum state, the final state (5) is called a squeezed vacuum. In the initial vacuum case, if one expands the corresponding $\mathbf{S}\rho_f\mathbf{S}^{-1}$ in the "out" basis of energy eigenstates, one finds [4] that the off-diagonal components of $\mathbf{S}\rho_f\mathbf{S}^{-1}$ have an oscillatory dependence of the angles $\theta_{\vec{k}}$. In the BMP coarse-grained entropy approach it is assumed that these angles represent irrelevant information about the system (e.g., in the sense that they would be very difficult to measure) and they are therefore averaged over. After the averaging only the diagonal elements of $\mathbf{S}\rho_f\mathbf{S}^{-1}$ survive and one then defines a coarse-grained entropy with the resulting reduced density matrix ρ_{red} with the usual formula $S=-k_B\text{Tr}(\rho_{\text{red}}\ln\rho_{\text{red}})$. The result is [3,4]

$$\begin{aligned}S^{f,0}&\equiv\sum_{\vec{k}}s_{\vec{k}}^{f,0}\equiv k_B\sum_{\vec{k}}(\cosh^2 r_{\vec{k}}\ln\cosh^2 r_{\vec{k}}-\sinh^2 r_{\vec{k}}\ln\sinh^2 r_{\vec{k}})\\&=k_B\sum_{\vec{k}}[(\bar{n}_{\vec{k}}^{f,0}+1)\ln(\bar{n}_{\vec{k}}^{f,0}+1)-(\bar{n}_{\vec{k}}^{f,0})\ln(\bar{n}_{\vec{k}}^{f,0})] ,\end{aligned} \tag{11}$$

where the notation $\bar{n}_{\vec{k}}^{f,0}$ means the left-hand side (LHS) of (9) in the case of an initial vacuum state. Now we turn to consider initial density matrices ρ_i, which can describe generic many-particle states with nonzero entropy. Let us assume that the initial density matrix has the form

$$\rho_i=\prod_{\vec{k},(k_z>0)}\sum_{n=0}^{\infty}f_{\vec{k}}(n)|n_{\vec{k}},n_{-\vec{k}},\text{in}\rangle\langle\text{in},n_{\vec{k}},n_{-\vec{k}}| , \tag{12}$$

where the coefficients $f_{\vec{k}}(n)$ are of the form

$$f_{\vec{k}}(n)=(\bar{n}_{\vec{k}}^{i})^n/(\bar{n}_{\vec{k}}^{i}+1)^{n+1} .$$

That is, we start with a many-particle state where particles appear in pairs of opposite momenta, with an initial average occupation number spectrum $\bar{n}^{i}_{\vec{k}}=\bar{n}^{i}_{-\vec{k}}$ and with (ordinary) entropy given by $-k_B\mathrm{Tr}(\rho_i \ln \rho_i)$. Writing the initial entropy in a more explicit form, it is

$$S^i \equiv \sum_{\vec{k},(k_z>0)} s^i_{\vec{k}}$$
$$= k_B \sum_{\vec{k},(k_z>0)} [(\bar{n}^{i}_{\vec{k}}+1)\ln(\bar{n}^{i}_{\vec{k}}+1)-(\bar{n}^{i}_{\vec{k}})\ln(\bar{n}^{i}_{\vec{k}})] \ . \quad (13)$$

When we expand the resulting final density matrix $\mathbf{S}\rho_f\mathbf{S}^{-1}$ (6) in "out" energy eigenstates, we find that, in this case, also, the off-diagonal elements have an oscillatory dependence on the angles $\theta_{\vec{k}}$. Therefore, following the BMP approach and averaging over the angles, only the diagonal elements will survive. Thus the reduced density matrix of the final state has the form

$$\rho_{\mathrm{red}} = \prod_{\vec{k},(k_z>0)} \sum_{n=0}^{\infty} \bar{f}_{\vec{k}}(n)|n_{\vec{k}},n_{-\vec{k}},\mathrm{out}\rangle\langle \mathrm{out},n_{\vec{k}},n_{-\vec{k}}| \ , \quad (14)$$

where

$$\bar{f}_{\vec{k}}(n) = \langle n_{\vec{k}},n_{-\vec{k}},\mathrm{out}|\mathbf{S}\rho_f\mathbf{S}^{-1}|\mathrm{out},n_{\vec{k}},n_{-\vec{k}}\rangle \ .$$

After some effort, one can show that the coefficients have the form

$$\bar{f}_{\vec{k}}(n) = (\bar{n}^{f}_{\vec{k}})^n/(\bar{n}^{f}_{\vec{k}}+1)^{n+1} \ , \quad (15)$$

where $\bar{n}^{f}_{\vec{k}}$ is the LHS of (9) with the $\bar{n}^{i}_{\vec{k}}$ of (12). Thus, the final coarse-grained entropy is

$$S^f \equiv \sum_{\vec{k},(k_z>0)} s^f_{\vec{k}}$$
$$= k_B \sum_{\vec{k},(k_z>0)} [(\bar{n}^{f}_{\vec{k}}+1)\ln(\bar{n}^{f}_{\vec{k}}+1)-(\bar{n}^{f}_{\vec{k}})\ln(\bar{n}^{f}_{\vec{k}})] \ . \quad (16)$$

The entropy depends only on the occupation number spectrum of particles in the final state. This result is in agreement with a similar formula given in Ref. [3] by a more heuristic argument to define entropy of a statistical system with a definite spectrum which is valid both in and far out of thermodynamical equilibrium. Let us now compare the entropy generation per mode in the initial vacuum and initial many-particle cases. Denote

$$\Delta_0 s_{\vec{k}} \equiv s^{f,0}_{\vec{k}} - 0 \ , \quad (17)$$

$$\Delta s_{\vec{k}} \equiv s^{f}_{\vec{k}} - s^{i}_{\vec{k}} \ , \quad (18)$$

where (17) applies to the former case and (18) to the latter case. As a first consistency check, we find that $\Delta s_{\vec{k}} \geq 0$, so the coarse graining led to a growing entropy in our many-particle case. However, as we compare (18) and (17) we find that

$$\Delta s_{\vec{k}} \leq \Delta_0 s_{\vec{k}} \ ; \quad (19)$$

i.e., the entropy generation is *attenuated* if one starts with many particles present in the mode $\vec{k}$. The equality holds if and only if $\bar{n}^{i}_{\vec{k}}=0$. This result is easiest to see in the following way. Consider the difference $\Delta s_{\vec{k}} - \Delta_0 s_{\vec{k}}$. Substitute (16) and (13) to (18), and (11) to (17). Then substitute $\bar{n}^{f}_{\vec{k}}$ as a function of $\bar{n}^{i}_{\vec{k}}$ and $|\beta_{\vec{k}}|^2 \equiv \sinh^2 r_{\vec{k}}$ by using (9) and (10). The difference $\Delta s_{\vec{k}} - \Delta_0 s_{\vec{k}}$ depends then symmetrically on $\bar{n}^{i}_{\vec{k}}$ and $|\beta_{\vec{k}}|^2$. By drawing a three-dimensional (3D) plot one can see that it is always nonpositive and it decreases monotonically as either variable increases. Further, as both variables approach infinity,

$$\Delta s_{\vec{k}} - \Delta_0 s_{\vec{k}} \to \ln 2 - 1 \approx -0.31 \quad (20)$$

asympotically. This finite value is the maximum difference between the generated entropies per mode. Thus, unlike the GGV entropy, the BMP entropy generation is *not* independent of the number of particles in the initial state, but has some "memory" about the initial occupation numbers. Since entropy is a measure of loss of information, it would appear that more information about the initial state of the system is conveyed to the final coarse-grained state when stimulated emission dominates the spontaneous particle production (since the entropy generation is attenuated).

The next case to be investigated would be an initial thermal density matrix

$$\rho_i = \prod_{\vec{k}} Z_{\vec{k}}^{-1} \exp(-\beta\omega^{\mathrm{in}}_{\vec{k}} a^{\dagger\mathrm{in}}_{\vec{k}} a^{\mathrm{in}}_{\vec{k}}) \ .$$

This situation is somewhat trickier to deal with for the following reason. Initially, the particles of opposite momenta are uncorrelated. However, in the expansion of the Universe the particles are produced in pairs of opposite momenta. This induces correlations between the opposite momenta in the final density matrix. It would have the form

$$\mathbf{S}\rho_f\mathbf{S}^{-1} = \prod_{\vec{k},(k_z>0)} \sum_{n,m,n',m'=0}^{\infty} f_{\vec{k}}(n,m;n',m')|n_{\vec{k}},m_{-\vec{k}},\mathrm{out}\rangle\langle \mathrm{out},n'_{\vec{k}},m'_{-\vec{k}}| \ , \quad (21)$$

where

$$f_{\vec{k}}(n,m;n',m')$$
$$= \left\langle n_{\vec{k}},m_{-\vec{k}},\mathrm{out} \left| \frac{1}{Z_{\vec{k}}Z_{-\vec{k}}} \exp\{-\beta\omega^{\mathrm{in}}_{\vec{k}}[(\alpha_{\vec{k}}a^{\dagger\mathrm{out}}_{\vec{k}} - \beta_{\vec{k}}a^{\mathrm{out}}_{-\vec{k}})(\alpha_{\vec{k}}a^{\mathrm{out}}_{\vec{k}} - \beta^{*}_{\vec{k}}a^{\dagger\mathrm{out}}_{-\vec{k}}) + (\vec{k} \to -\vec{k})]\} \right| \mathrm{out},n'_{\vec{k}},m'_{-\vec{k}} \right\rangle \ . \quad (22)$$

Again, one can see that the $n \neq n'$ or $m \neq m'$ components have an oscillatory dependence of the squeeze angles and they vanish in the coarse graining. However, the diagonal coefficients (those of the reduced density matrix) will have a form $f_{\vec{k}}(n,m)$ where the dependence on n and m does not factorize. Hence the opposite momenta have acquired correlations through the particle production and the reduced density matrix is not of the same type as the initial one. As advocated in Ref. [6], one would like to ignore the correlations between different modes. Further, one should not do this by replacing the two-mode squeeze operator by a one-mode squeeze operator, since the particles are then not created in the correct way as pairs of opposite momenta. We would like to propose that the correlations between opposite momenta could be ignored by proceeding to define $\bar{f}_{\vec{k}}(n)=\sum_m f_{\vec{k}}(n,m)$ and $\bar{f}_{-\vec{k}}(m)=\sum_n f_{\vec{k}}(n,m)$. Then we would define the final reduced density matrix to be

$$\rho_{\text{red}}=\prod_{\vec{k},(k_z>0)} \sum_{n,m=0}^{\infty} \bar{f}_{\vec{k}}(n)\bar{f}_{-\vec{k}}(m)|n_{\vec{k}},m_{-\vec{k}},\text{out}\rangle \times \langle \text{out},n_{\vec{k}},m_{-\vec{k}}| \ , \tag{23}$$

which is of the same type as the initial density matrix. Now the final entropy would be given by

$$S^f=-k_B\sum_{\vec{k}}\sum_n \bar{f}_{\vec{k}}(n)\ln\bar{f}_{\vec{k}}(n) \ . \tag{24}$$

Unfortunately, at the present we do not have explicit formulas for the coefficients $\bar{f}_{\vec{k}}(n)$ or the final entropy. It would be very interesting to see if the resulting expressions could depend on the final average occupation number in the same fashion as in the earlier case. We hope to be able to return to this question in the future.

Finally, let us clarify that even if we found a different result as in the GGV approach, that the entropy generation depends on the number of particles in the initial state, we are not arguing that it would mean that the BMP approach is "better" than the GGV approach. As stated in Ref. [2], it is good to have different definitions of entropy, corresponding to loss of different information about the system. Both the BMP and GGV approaches have the virtue of giving the correct average occupation number of particles in the final state. Otherwise the GGV approach appears to discard information about the system a bit more generously, since it leads to a greater growth of entropy.

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[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[2] B. Hu and D. Pavon, Phys. Lett. B **180**, 329 (1986); H. Kandrup, J. Math. Phys. **28**, 1398 (1987); Phys. Rev. D **37**, 3505 (1988); B. Hu and H. Kandrup, *ibid.* **35**, 1776 (1987); S. Habib and H. Kandrup, Ann. Phys. (N.Y.) **191**, 335 (1989).
[3] R. Brandenberger, V. Mukhanov, and T. Prokopec, Phys. Rev. D **48**, 2443 (1993); Phys. Rev. Lett. **69**, 3606 (1992).
[4] T. Prokopec, Class. Quantum Grav. **10**, 2295 (1993).
[5] M. Gasperini and M. Giovannini, Phys. Lett. B **301**, 334 (1993); Class. Quantum Grav. **10**, L133 (1993).
[6] M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Rev. D **48**, R439 (1993).
[7] L. Parker, Phys. Rev. **183**, 1057 (1969); R. M. Wald, Phys. Rev. D **13**, 3176 (1976); J. Audretsch, in *Quantum Mechanics in Curved Space-Time*, edited by J. Audretsch and V. de Sabbata (Plenum, New York, 1990).
[8] The explicit formula for the S matrix can be found (in the case of a Robertson-Walker universe), e.g., in J. Ambjørn, R. J. Hughes, and N. K. Nielsen, Ann. Phys. (N.Y.) **150**, 92 (1983). Note that there are minor differences in conventions used in that paper and in this work.
[9] B. DeWitt, Phys. Rep. **19**, 297 (1975).
[10] R. M. Wald, Ann. Phys. (N.Y.) **118**, 490 (1979).
[11] L. Parker, Phys. Rev. D **3**, 346 (1971).
[12] B. Schumaker, Phys. Rep. **135**, 317 (1986). The relations between entropy, correlations, and squeezing have also been studied in quantum optics, see, e.g., S. M. Barnett and S. J. D. Phoenix, Phys. Rev. A **40**, 2404 (1989); S. Abe, Phys. Lett. A **166**, 163 (1992).