Helicity conservation in the Aharonov-Bohm scattering of Dirac particles

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We show that the helicity operator Λ for a particle in the presence of an infinitely thin magnetic flux tube requires, as the Hamiltonian H, for its complete determination as a self-adjoint operator, the specification of boundary conditions (BC's) that have to be chosen out of a four parameter family of admissible ones. To each value of the parameters there corresponds a self-adjoint operator with eigenfunctions and eigenvalues determined by the associated BC's. For each choice of the dynamics H we investigate under which conditions the corresponding BC is also admissible for the helicity Λ. When this happens, and only when this happens, it is possible for H and Λ to satisfy identical BC's. Although their actions formally commute before specification of boundary conditions, only identical BC's will ensure effective commutativity, in the sense that they will have a complete set of common eigenfunctions and that Λ will be a conserved quantity. We show this to be the case only for a special (but large) class of BC's. Our results imply that helicity conservation, although imposing some restrictions on the choice of the dynamics, does not solve the problem of the indeterminacy in the choice of BC's in the Aharonov-Bohm scattering of Dirac particles. Our results also show that it is possible to choose BC's such that both helicity is conserved and the Aharonov-Bohm symmetry ($\phi \to \phi+1$) is preserved, where ϕ is the magnetic flux in natural units.

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I. INTRODUCTION

The Hamiltonian for a charged Dirac particle of mass $m>0$ in the presence of an infinitely thin magnetic flux tube is

$$H=\left(\mathbf{P}+\frac{e\,\mathbf{A}}{c}\right)\cdot\boldsymbol{\alpha}+\beta m\ , \tag{1.1}$$

where $\mathbf{P}=(P_1,P_2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$,

$$\alpha_1=\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix},\quad \alpha_2=\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix},\quad \beta=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}, \tag{1.2}$$

where $\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices and

$$\frac{e\,\mathbf{A}}{c}=\frac{\phi}{r^2}(-x_2,x_1). \tag{1.3}$$

This Hamiltonian (1.1) decomposes into two uncoupled operators acting on two component spinors, but we shall use the four-component spinor representation, so that we can accommodate the helicity operator given by

$$\Lambda=\left(\mathbf{P}+\frac{e\,\mathbf{A}}{c}\right)\cdot\boldsymbol{\Sigma}\ , \tag{1.4}$$

where $\boldsymbol{\Sigma}$ is the spin operator which is given by $\boldsymbol{\Sigma}=(\Sigma_1,\Sigma_2)$,

$$\Sigma_1=\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix},\quad \Sigma_2=\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix}. \tag{1.5}$$

The two operators H and Λ are *only formally defined* by (1.1) and (1.4) and, at this level, they commute as one easily verifies. One should, however, be warned against hasty conclusions concerning the possibility of their simultaneous diagonalization and of helicity conservation in the dynamics given by H. In fact, *both operators* H and Λ suffer from the same disease: for them to become self-adjoint operators it is necessary to specify boundary conditions (BC's) at the origin to be satisfied by their eigenfunctions, since the usual assumption of regularity at the origin is not compatible with the requirement of self-adjointness if ϕ is not an integer. In contrast, for integer ϕ, regularity at the origin is not only acceptable but is the unique BC leading to self-adjoint operators H and Λ. For the Hamiltonian H, this fact was first recognized by Gerbert [1] who at the same time provided a description of a class of admissible boundary conditions to be imposed on the eigenfunctions, so that for each one of them H becomes a self-adjoint operator. In fact, he completely solved this problem with the extra restriction that the resulting operator remains decomposable as a pair of uncoupled operators acting on two-component spinors (see also Ref. [2] for a complementary discussion).

In this paper we first show that, indeed, the helicity operator Λ also requires a specification of BC's for it to be realized as a self-adjoint (SA) operator. Next, we describe the class of admissible BC's for this to happen. It turns out that both H and Λ have each a *four-parameter* family of self-adjoint realizations in one-to-one correspondence with BC's to be satisfied by the eigenfunctions at the origin.

The next relevant question is, then, "Is it possible to choose the *same* boundary conditions for both H and Λ so that the corresponding SA operators will have common eigenfunctions and helicity will be conserved?" This question has been considered before in the literature as follows.

A natural procedure, proposed in Refs. [3,4], to choose BC's for H and Λ consists in replacing the "thread of flux" by a fictitious tube of radius R, with the magnetic field concentrated on the surface of the tube, i.e., $\mathbf{A}$ is replaced in (1.1) and in (1.4) by

$$\frac{e\,\mathbf{A}_R}{c}=\begin{cases}\dfrac{\phi}{r^2}(-x_2,x_1)\ , & r\geq R\ ,\\ 0\ , & r<R\ .\end{cases} \tag{1.6}$$

With this replacement the operators H_R and Λ_R so obtained are essentially self-adjoint, since regularity at the origin is the only admissible BC for their eigenfunctions. Self-adjoint realizations of H and Λ are then obtained by taking the limit $R\rightarrow 0$ on the wave functions of H_R and Λ_R. Let us call H_0 and Λ_0 the SA operators, so obtained. Notice that $[H_R,\Lambda_R]=0$, i.e., H_R and Λ_R do commute for $R>0$ and this implies that the two limiting SA operators H_0 and Λ_0 will have common BC's and will commute with the usual implications. In particular, the helicity operator Λ_0 is a conserved quantity in the dynamics defined by H_0. This was actually implicit in [5,6], where it was also implied that "helicity conservation" would remove the indeterminancy in the choice of BC's for H and select the BC assigned to H_0 as the physically relevant one.

We revisit this problem and consider the following questions.

(i) Is every admissible BC for H also admissible for Λ? The answer is no: There are BC's that are admissible for H and not for Λ, and therefore in the dynamics associated to these BC's there is no way of getting helicity conservation.

(ii) Is the BC associated with H_0 the only one which is also admissible for Λ, in other words, is this the only dynamics compatible with helicity conservation? The answer is no: There is a large class of BC's for H which are also admissible for Λ, and for each one of them it is possible to have helicity conservation if we take for Λ and H the same BC. Therefore, helicity conservation cannot by itself be used to select the physical BC.

In particular, also compatible with helicity conservation are the boundary conditions obtained in Ref. [2] through a variant of the above-described limiting procedure, where a constant potential V_R is introduced inside the tube:

$$V_R(r)=u_R\ ,\quad r<R\ ,\quad V_R(r)=0\ ,\quad r>R \tag{1.7}$$

and u_R is suitably fine tuned as a function of R. In this case helicity conservation takes place only in the limit $R\rightarrow 0$ since for $R>0$, the two operators do not commute.

If, however, one is given an *impenetrable* tube of radius $R>0$ with a magnetic flux ϕ confined to its interior [as given for instance by (1.3) or (1.6)] then both the Hamiltonian and the helicity operators require the specification of boundary condition at the surface of the tube. (This should not be confused with the fictitious penetrable tube introduced above as a mathematical tool devised to control the limit $R\rightarrow 0$, in which case both the Hamiltonian and helicity operators are uniquely defined for $R>0$.) The two questions raised in (i) and (ii) above are again relevant and the respective answers turn out to be exactly the same. This shows that the relevant physical feature of this problem is its topology which is preserved in the limit $R\rightarrow 0$ with fixed ϕ.

This paper is organized as follows. In the next section we give a precise description of the admissible BC for the helicity operator Λ; they are parametrized by 2×2 unitary matrices. We make explicit this dependence and analyze the possibility of common BC's for H and Λ. The more technical discussion of how to obtain these BC's is postponed to the Appendix. In Sec. III we discuss the impenetrable tube of finite radius and some of the implications of our results.

II. THE ADMISSIBLE BOUNDARY CONDITIONS

We begin by noticing that there exists a unitary transformation U given by

$$U=\frac{1}{\sqrt{2}}\begin{bmatrix} I & -i\sigma_3 \\ I & i\sigma_3\end{bmatrix}\ , \tag{2.1}$$

which connect the two formal operators (1.1) and (1.4):

$$\Lambda=U(H-\beta m)U^{-1}\ . \tag{2.2}$$

This follows from

$$U\alpha_j U^{-1}=\Sigma_j\ ,\quad j=1,2\ .$$

It is important to notice that both H and Λ commute with the total angular momentum operator

$$J_3=\Sigma_3/2+l_3\ ,$$

where

$$\Sigma_3=\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}$$

and

$$l_3=(X_1P_2-X_2P_1)\ .$$

As a consequence, and it may be also checked by an explicit computation, the operator U leaves the subspace of total angular momentum $n+\frac{1}{2}$ invariant.

The unitarity equivalence (2.2) substantially simplifies the analysis, since the solution of the problem of boundary conditions for Λ can be read off the corresponding problem for $(H-\beta m)$, and the latter is almost identical to the same problem for H.

In polar coordinates (r,φ), after separation of variables,

$$\psi(r,\varphi)=\begin{bmatrix}\chi_1(r)\\ \chi_2(r)e^{i\varphi}\\ \chi_3(r)\\ \chi_4(r)e^{i\varphi}\end{bmatrix}e^{in\varphi}$$

the eigenvalue equation for $H-\beta m$ reads

$$(H-\beta m)_\nu\chi=\begin{bmatrix}h_\nu & 0\\ 0 & h_\nu\end{bmatrix}\chi(r)=E_\chi(r)\ , \tag{2.3}$$

where

$$\chi(r)=\begin{bmatrix}\chi_1(r)\\ \chi_2(r)\\ \chi_3(r)\\ \chi_4(r)\end{bmatrix}\ , \tag{2.4}$$

$$h_\nu=\begin{bmatrix}0 & -i\left[\partial_r+\frac{\nu+1}{r}\right]\\ -i\left[\partial_r-\frac{\nu}{r}\right] & 0\end{bmatrix} \tag{2.5}$$

and $\nu=n+\phi$, $n+\frac{1}{2}$=total angular momentum.

Analogously, the Schrödinger equation for H after the same separation reads

$$H_\nu=\begin{bmatrix}h_\nu-\sigma_3 m & 0\\ 0 & h_\nu+\sigma_3 m\end{bmatrix}\chi(r)=E\chi(r)\ . \tag{2.6}$$

As shown in the Appendix, if $-1<\nu<0$, both Eqs. (2.5) and (2.6) require the specification of boundary conditions to be satisfied by the eigenfunctions $\chi(r)$ in the limit $r\rightarrow 0$. [It is, in particular, impossible to stick to the usual regularity assumption of the eigenfunctions at the origin, as this condition is incompatible with self-adjointness of the operators H_ν and $\Lambda_\nu=U(H-\beta m)_\nu U^{-1}$.] These boundary conditions are to be chosen among a class of admissible ones as follows. Let us introduce the quantities

$$\begin{aligned}
x&=\tfrac{1}{2}\Gamma(1+\nu)\left[\frac{m}{\sqrt{2}}\right]^{-(1+\nu)}\lim_{r\rightarrow 0}r^{-\nu}\chi_1(r)\ ,\\
y&=\tfrac{1}{2}\Gamma(-\nu)\left[\frac{m}{\sqrt{2}}\right]^{\nu}\lim_{r\rightarrow 0}r^{(1+\nu)}\chi_2(r)\ ,\\
z&=\tfrac{1}{2}\Gamma(1+\nu)\left[\frac{m}{\sqrt{2}}\right]^{-(1+\nu)}\lim_{r\rightarrow 0}r^{-\nu}\chi_3(r)\ ,\\
w&=\tfrac{1}{2}\Gamma(-\nu)\left[\frac{m}{\sqrt{2}}\right]^{\nu}\lim_{r\rightarrow 0}r^{(1+\nu)}\chi_4(r)\ .
\end{aligned} \tag{2.7}$$

In terms of these variables the admissible boundary conditions for helicity operator are parametrized by a 2×2 unitary matrix V,

$$V=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\ , \tag{2.8}$$

through the equations

$$\begin{aligned}
&(1+\overline{v_{11}}+i\overline{v_{12}})w+(1-\overline{v_{11}}+i\overline{v_{12}})z\\
&\qquad+(1+\overline{v_{11}}-i\overline{v_{12}})y+(1-\overline{v_{11}}-i\overline{v_{12}})x=0\ ,\\
&(i+\overline{v_{21}}+i\overline{v_{22}})w+(-i-\overline{v_{21}}+i\overline{v_{22}})z\\
&\qquad+(-i+\overline{v_{21}}-i\overline{v_{22}})y+(i-\overline{v_{21}}-i\overline{v_{22}})x=0\ .
\end{aligned} \tag{2.9}$$

For the Hamiltonian, we are not going to consider the most general possible boundary conditions, and we shall restrict our considerations to a special class of boundary conditions that, although not exhausting the class of admissible BC's includes all previously discussed boundary conditions in the literature. They are parametrized by two angles δ_1 and δ_2 through

$$\begin{aligned}
&x\cos\delta_1-iy\sin\delta_1=0\ ,\\
&z\cos\delta_2-iw\sin\delta_2=0\ .
\end{aligned} \tag{2.10}$$

Notice that the boundary conditions of type (2.10) have the special feature of preserving the original symmetry of the Hamiltonian that allowed it to be represented by two independent operators acting on two-component spinors. These are the BC's of the type considered in Ref. [1], and in the notation of that paper $\delta_1=\pi/4+\theta_1/2$ and $\delta_2=\pi/4+\theta_2/2$. A discussion with the most general admissible boundary conditions will be presented elsewhere [8].

If the V-boundary condition (2.9) for the helicity operator Λ is to be the same as the (δ_1,δ_2)-boundary condition (2.10) for the Hamiltonian H, then the two systems of Eqs. (2.9) and (2.10) must have the same solution set. The necessary and sufficient condition for this to happen is that the following set of equations is satisfied:

$$\begin{aligned}
&e^{i\delta_2}v_{11}-ie^{-i\delta_2}v_{12}=-e^{-i\delta_2}\ ,\\
&e^{i\delta_1}v_{11}-ie^{-i\delta_1}v_{12}=-e^{-i\delta_1}\ ,\\
&-ie^{-i\delta_2}v_{22}+e^{i\delta_2}v_{21}=ie^{i\delta_2}\ ,\\
&ie^{-i\delta_1}v_{22}+e^{i\delta_1}v_{21}=-ie^{-i\delta_1}\ .
\end{aligned} \tag{2.11}$$

Notice that the two pairs decouple, and that the system will have solutions if and only if

$$1+\cos 2(\delta_1-\delta_2)\neq 0 \tag{2.12}$$

and under this assumption, the unique solution is

$$\begin{aligned}
v_{11}&=-\frac{2}{e^{2i\delta_2}+e^{2i\delta_1}}\ ,\\
v_{12}&=-i\frac{e^{2i\delta_2}-e^{2i\delta_1}}{e^{2i\delta_2}+e^{2i\delta_1}}\ ,\\
v_{21}&=i\frac{e^{2i\delta_2}-e^{2i\delta_1}}{e^{2i\delta_2}+e^{2i\delta_1}}\ ,\\
v_{22}&=-\frac{2}{e^{2i\delta_2}+e^{2i\delta_1}}\ .
\end{aligned} \tag{2.13}$$

If we now impose the unitarity condition on the matrix V given by (2.13) we obtain the condition

$$\delta_1-\delta_2=2\pi k \ , \tag{2.14}$$

where k is an integer. Equation (2.14) is the necessary and sufficient condition for the (δ_1,δ_2)-boundary condition (2.10) to be also admissible for the helicity operator Λ. In other words, it is the necessary and sufficient condition for the dynamics defined by the (δ_1,δ_2)-boundary condition to conserve the suitably chosen (i.e., with the same BC) helicity operator. A particular way of satisfying condition (2.14) is to take $\delta_1=\delta_2$, i.e., equal boundary conditions for the two pair of operators acting on two-component spinors. This class includes the boundary conditions obtained through the limiting procedure of Refs. [3,4] but includes also much more. It includes also those ϕ-dependent boundary conditions discussed in Ref. [2] which unlike the boundary conditions of Refs. [3,4] preserve the Aharonov-Bohm symmetry $\phi\to\phi+1$. This means that preservation of the Aharonov-Bohm symmetry is compatible with helicity conservation. The BC obtained in Ref. [2] through the alternative limiting procedure (1.3) also satisfies $\delta_1=\delta_2$, and so they are also compatible with helicity conservation.

III. PHYSICAL IMPLICATIONS

A few words should now be said about the possible physical relevance of the results in this paper and of a previous one [2] by the authors.

(1) If one wishes to assume that there is no infinitely thin magnetic flux tube, then the following options are available.

(a) One may admit an *impenetrable* tube of radius $R>0$, as prevalently assumed in the literature describing the Aharonov-Bohm effect. In this case, as pointed out in Ref. [2], one will have to choose among the admissible boundary conditions that one that best suits the device at hand. (The reader should be reminded that at least one of the components of the Dirac spinor is necessarily nonzero at the surface of the tube.) In this case, it is straightforward to repeat step by step the derivation of Sec. II replacing x, y, z, and w of Eq. (2.7) by x_R, y_R, z_R, and w_R given by

$$\begin{aligned}
x_R&=\tfrac{1}{2}\Gamma(1+\nu)\left[\frac{m}{\sqrt{2}}\right]^{-(1+\nu)}R^{-\nu}\chi_1(R)\ ,\\
y_R&=\tfrac{1}{2}\Gamma(-\nu)\left[\frac{m}{\sqrt{2}}\right]^{\nu}R^{(1+\nu)}\chi_2(R)\ ,\\
z_r&=\tfrac{1}{2}\Gamma(1+\nu)\left[\frac{m}{\sqrt{2}}\right]^{-(1+\nu)}R^{-\nu}\chi_3(R)\ ,\\
w_R&=\tfrac{1}{2}\Gamma(-\nu)\left[\frac{m}{\sqrt{2}}\right]^{\nu}R^{(1+\nu)}\chi_4(R)\ .
\end{aligned}\tag{3.1}$$

Choosing for the Hamiltonian the BC(δ_1,δ_2) given by

$$\begin{aligned}
&x_R\cos\delta_1-iy_R\sin\delta_1=0\ ,\\
&z_R\cos\delta_2-iw_R\sin\delta_2=0\ ,
\end{aligned}\tag{3.2}$$

we obtain the same conclusions concerning helicity conservation: Eq. (2.14) is again a necessary and sufficient condition. Furthermore, the low-energy asymptotics of the resulting model will be described by the model with $R\to 0$ with BC's given by (2.10).

(b) Alternatively one may prefer a *penetrable* tube. Then, one will have to deal with the very delicate question involving the choice of a *physically acceptable* dynamics suppressing penetration, as much as possible, of the electron to the inside of the tube, in the context of the Dirac equation. In this case, as opposed to the impenetrable tube, absent any extra singularity the dynamics and helicity require no additional specification of BC and helicity is generally not conserved for $R>0$. Again, the low-energy asymptotics will be controlled by a zero-range tube, whose boundary conditions are strongly dependent upon this choice. The simplified, should we say caricatural, models described in Ref. [2] (requiring a nongauge, strongly fine-tuned repulsive "vector" potentials) are just examples that illustrate the fact that some helicity breaking dynamics may very well have a low-energy asymptotics controlled by helicity-conserving Hamiltonians. It should also be remarked that if one chooses to control penetration through a nongauge "scalar" potential, then a calculation along the lines of Ref. [2] shows that necessarily a helicity-breaking limiting dynamics emerges [8].

(2) There may be contexts where the idea of an infinitely thin magnetic flux tube at the oustset is not infinitely repugnant, as for instance in the description of cosmic strings [3,9]. These are the cases addressed by the analysis of Sec. II, where no limiting procedure was used. However, in absence of alternative plausible physical models, the helicity conserving boundary conditions of Refs. 3,4 emerge naturally as the most suitable prescription. The results are helicity conserving, but violate periodicity in the flux.

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APPENDIX

Technically speaking, the problem of determining the admissible boundary is identical to the problem of determining the self-adjoint extensions of a densely defined symmetric operator. From a practical point of view to solve these problems we have to begin with the determination of the deficiency spaces of the two operators H_ν and Λ_ν when acting in a common dense domain $\mathcal{D}_0$ of smooth functions of r that vanish at the origin. These deficiency subspaces $D_\pm(H_\nu), D_\pm(\Lambda_\nu)$ are defined by

$$H_\nu^*\varphi_\pm=\pm im\varphi_\pm \quad \text{if } \varphi_\pm\in D_\pm(H_\nu)\ ,$$

$$\Lambda_\nu^*\psi_\pm=\pm i\lambda\psi_\pm \quad \text{if } \psi_\pm\in D_\pm(\Lambda_\nu)\ ,$$

where H_ν^* and Λ_ν^* denote the adjoint operator of H_ν and Λ_ν, respectively.

Next we recall the results of Gerbert [1] as adapted to our four-component spinor formalism. Since this four-component equation decouples into two two-component equations of the type already discussed in Ref. [1] we can immediately determine the subspaces of solutions of

$$H_\nu^* \chi = \pm i m \chi \ . \tag{A1}$$

The subspaces $D_\pm[H_\nu]$ are generated by

$$\chi_+^{(1)} = \begin{bmatrix} \Phi_+ \\ 0 \\ 0 \end{bmatrix}, \quad \chi_+^{(2)} = \begin{bmatrix} 0 \\ 0 \\ \Phi_+ \end{bmatrix},$$

$$\chi_-^{(1)} = \begin{bmatrix} \Phi_- \\ 0 \\ 0 \end{bmatrix}, \quad \chi_-^{(2)} = \begin{bmatrix} 0 \\ 0 \\ \Phi_- \end{bmatrix},$$

where the two component spinors $\Phi_\pm$ are given by

$$\Phi_\pm(r) = \frac{1}{N} \begin{bmatrix} K_\nu(\sqrt{2} m r) \\ \pm e^{\pm i\pi/4} K_{\nu+1}(\sqrt{2} m r) \end{bmatrix},$$

where N is a normalization constant. For $-1 < \nu < 0$ the asymptotic behavior at the origin,

$$\begin{aligned} K_\nu(x) &\approx (\tfrac{1}{2})\Gamma(-\nu)(\tfrac{1}{2}x)^\nu \ , \\ K_{\nu+1}(x) &\approx (\tfrac{1}{2})\Gamma(1+\nu)(\tfrac{1}{2}x)^{-(1+\nu)} \ , \end{aligned} \tag{A2}$$

implies the crucial fact that the singular functions $\chi_\pm^1$ and $\lambda_\pm^2$ are square integrable:

$$\int r \chi^\dagger(r) \chi(r) dr < \infty \ , \tag{A3}$$

this implying that the starting operator H_ν was not essentially self-adjoint, thus the need for BC specification.

REMARK. If we look at H_ν, as in Ref. [1], as a pair of uncoupled operators acting on two-spinors then $d_\pm(H_\nu)=1$ for each operator of the pair.

In analogous way we determine the deficiency subspaces of $(H-\beta m)_\nu$, by solving the equation

$$(H-\beta m)_\nu^* \varphi = \pm i \lambda \varphi \ . \tag{A4}$$

It follows that the deficiency subspaces $D_\pm[(H-\beta m)_\nu]$ are generated by the normalized functions

$$\begin{aligned} \varphi_+^{(1)} &= \begin{bmatrix} \phi_+ \\ 0 \\ 0 \end{bmatrix}, \quad \varphi_+^{(2)} = \begin{bmatrix} 0 \\ 0 \\ \phi_+ \end{bmatrix}, \\ \varphi_-^{(1)} &= \begin{bmatrix} \phi_- \\ 0 \\ 0 \end{bmatrix}, \quad \varphi_-^{(2)} = \begin{bmatrix} 0 \\ 0 \\ \phi_- \end{bmatrix}, \end{aligned} \tag{A5}$$

where $\phi_\pm$ are the two-component spinors given by

$$\phi_\pm(r) = \frac{1}{N} \begin{bmatrix} K_\nu(\lambda r) \\ \pm K_{\nu+1}(\lambda r) \end{bmatrix} . \tag{A6}$$

Therefore, the deficiency indices $d_\pm(H-\beta m)_\nu$, i.e., the dimensions of $D_\pm(H-\beta m)_\nu$, are both equal to 2, if $-1<\nu<0$. The unitary equivalence (2.2) implies that the deficiency indices for the helicity operator Λ are also $d_\pm(\Lambda)=2$. Moreover, $D_\pm(\Lambda)$ is generated by $U\varphi_\pm^{(1)}$ and $U\varphi_\pm^{(2)}$.

Now, from the general theory (see Ref. [7], Vol. II), the admissible boundary conditions for a given symmetric operator A to be expanded to a self-adjoint operator with equal deficiency indices $d_+(A)=d_-(A)=n$ are obtained by the following procedure. Let f_+^j, f_-^j, $j=1,\ldots,n$ be normalized and mutually orthogonal vectors in $D_+(A)$ and $D_-(A)$, respectively, i.e., $A^* f_\pm^j = \pm i f_\pm^j$. The admissible boundary conditions for the vectors Ψ in the domain of the self-adjoint extensions of A are parametrized by $n\times n$ unitary matrices T through the set of equations

$$\left\langle A^* \left[f_+^j + \sum_{k=1}^n U_{jk} f_-^k \right], \Psi \right\rangle - \left\langle \left[f_+^j + \sum_{k=1}^n U_{jk} f_-^k \right], A^* \Psi \right\rangle = 0, \quad j=1,\ldots,n \ . \tag{A7}$$

Therefore, the admissible boundary conditions for H_ν can be parametrized by a 2×2 unitary matrix W,

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

through the formulas

$$\begin{aligned} &\lim_{r\to 0} [\chi_+^1(r) + w_{11}\chi_-^1(r) + w_{12}\chi_-^2(r)]^\dagger \alpha_1 \psi(r) = 0 \ , \\ &\lim_{r\to 0} r[\chi_+^2(r) + w_{21}\chi_-^1(r) + w_{22}\chi_-^2(r)]^\dagger \alpha_1 \psi(r) = 0 \ . \end{aligned} \tag{A8}$$

If we take W to be a diagonal matrix,

$$W(\omega_1,\omega_2) = \begin{bmatrix} \exp i\omega_1 & 0 \\ 0 & \exp i\omega_2 \end{bmatrix}, \tag{A9}$$

and use the asymptotic behavior of $\chi_\pm^i, i=1,2$ together with (2.7) we obtain the special class of (δ_1,δ_2) BC's described by (2.10) where

$$\tan\delta_i = \frac{\Gamma(-\nu)}{2^{\nu+1}\Gamma(1+\nu)} \frac{1}{\tan(\omega_i/2)-1} \ , \quad i=1,2 \ . \tag{A10}$$

These are the BC's described in Ref. [1]; more general BC's for the Hamiltonian will be discussed elsewhere [8].

In an analogous way we parametrize the admissible BC for $(H-\beta m)_\nu$ by 2×2 unitary matrices V through the equations:

$$\begin{aligned} &\lim_{r\to 0} r[\varphi_+^1(r) + v_{11}\varphi_-^1(r) + v_{12}\varphi_-^2(r)]^\dagger \alpha_1 \psi(r) = 0 \ , \\ &\lim_{r\to 0} r[\varphi_+^2(r) + v_{21}\varphi_-^1(r) + v_{22}\varphi_-^2(r)]^\dagger \alpha_1 \psi(r) = 0 \ . \end{aligned} \tag{A11}$$

Using the unitary equivalence (2.2) it translates into the admissible boundary conditions for the eigenfunctions of the helicity operator Λ_ν:

$$\lim_{r\to 0} r[U\varphi^1_+(r)+v_{11}U\varphi^1_-(r)+v_{12}U\varphi^2_-(r)]^\dagger\Sigma_1\psi(r)=0\ ,$$
$$\lim_{r\to 0} r[U\varphi^2_+(r)+v_{21}U\varphi^1_-(r)+v_{22}U\varphi^2_-(r)]^\dagger\Sigma_1\psi(r)=0\ . \tag{A12}$$

In order to simplify the comparison of the BC's for H and Λ we take $\lambda=\sqrt{2}m$. Using then the asymptotic behavior of $\varphi^i_\pm, i=1,2$ the explicit action of U given by (2.1) and (2.7) we obtain the BC of formula (2.9) which exhausts the class of admissible BC's for the helicity operator Λ.

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