Catalyzed decay of a false vacuum in four dimensions

M. B. Voloshin
Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455
and Institute of Theoretical and Experimental Physics, Moscow, 117259 Russia
(Received 10 September 1993)

The probability of destruction of a metastable vacuum state by the field of a highly virtual particle with energy E is calculated for a $(3+1)$-dimensional theory in the leading WKB approximation in the thin-wall limit. It is found that the induced nucleation rate of bubbles, capable of expansion, is exponentially small at any energy. The negative exponential power in the rate reaches its maximum at the energy, corresponding to the top of the barrier in the bubble energy, where it is a finite fraction of the same power in the probability of the spontaneous decay of the false vacuum, i.e., at $E=0$.

PACS number(s): 11.10.Jj

A number of problems in statistical physics [1,2] and in cosmology [3,4] involve a consideration of a metastable (false) vacuum state of quantum fields, which corresponds to a local, rather than global minimum of the Hamiltonian. Such a state can spontaneously decay into either the true vacuum or a lower-energy false vacuum due to quantum fluctuations at zero temperature [1–3,5,6] or due to thermal ones [7,8] if the temperature is sufficiently high.

The decay proceeds through nucleation and subsequent expansion of bubbles filled with the lower-energy phase. The expansion is possible only for bubbles of sufficiently large size, for which the gain in the volume energy compensates the energy associated with the surface of the bubble. Thus the problem of the calculation of the decay rate is reduced to a calculation of the probability of nucleation of the critical bubbles, which in the quantum case is a tunneling process [1–3]. The rate of the spontaneous nucleation of critical bubbles due to tunneling is exponentially small in the inverse of the difference ϵ of the energy density between the metastable vacuum and the lower one. Thus it is especially interesting to look for mechanisms that would enhance the decay rate.

If there are particles present in the false vacuum, they can facilitate nucleation of the bubbles, thus catalyzing the decay process. The presence of a massive particle is known [9,10] to enhance the tunneling rate since the tunneling proceeds at an energy equal to the particle mass rather than zero, whereas the problem of the catalysis of the false vacuum decay by collisions of particles thus far has been addressed either only for theories in two dimensions [11–13] or purely phenomenologically [13,14].

In this paper, for a $(3+1)$-dimensional theory, the exponential power $-F(E)$ is calculated in the probability of the nucleation of critical and subcritical bubbles in the presence of a highly virtual field ϕ:

$$|\langle B(E)|\phi|0\rangle|^2 \sim \exp[-F(E)] ,$$

with $|B(E)\rangle$ being a state of a bubble with energy E. The calculations are done within the so-called thin-wall approximation, which assumes that the size of the bubbles is much larger than the thickness of its wall and which is applicable at small ϵ. The result of this calculation is that the induced nucleation rate of critical bubbles is exponentially small in ϵ^{-1} at any energy E. As calculated by the WKB technique, the probability reaches its maximum at the value of energy E_c corresponding to the top of the barrier, which separates the critical and subcritical regions. However, at that point the factor F in the exponent differs only numerically from that at $E=0$. The value of the ratio is found to be

$$F(E_c)/F(0) \approx 0.160 .$$

This behavior is different from the one derived [12,13] for a two-dimensional theory, where the exponential suppression in ϵ^{-1} disappears at and above the top of the barrier, leaving only a possible exponential suppression in the inverse of a coupling constant g in the theory $\exp(-\text{const}/g)$. As will be shown, the leading contribution to the critical bubble nucleation rate at an energy below the top of the barrier is a product of two factors: one being the probability of excitation of a subcritical bubble with energy E and the other given by the tunneling rate at the same energy. At the top of the barrier the suppression due to the tunneling disappears; however, the excitation factor is already exponentially small. The difference with the two-dimensional case arises from the fact that in the two-dimensional problem there is no subcritical region for the bubbles in the thin-wall approximation (the barrier starts at the zero size of the bubble); hence, the excitation factor there is not related to the parameter ϵ^{-1}, but rather, possibly, to g^{-1}.

It should be noted that the exponential decrease of the probability beyond the critical energy E_c is an artifact of the approximation in which only the collective degrees of freedom describing the bubble are taken into account. In reality, if the initial energy is above E_c, it can be reduced down to E_c by perturbative emission of one or few particles at the cost of an additional factor in the probability with the coupling g in a power of order one. Therefore, as far as the exponential factor is concerned, its relevant value at energies above E_c stays constant and equal to $\exp[-F(E_c)]$. At energies below E_c, where the probability of the bubble nucleation is exponentially growing, processes with the emission of additional particles are exponentially suppressed, which justifies ignoring other degrees of freedom of the field rather than those describing the dynamics of a bubble.[1]

The problem under discussion in this paper is closely related to the one of multiparticle production in high-energy collisions in theories with weak interaction (for a recent review see, e.g., Ref. [15]). Like some of the recent papers on that subject [16–20], the present calculation uses the Landau-WKB technique [21,22] for evaluating matrix elements between strongly different states of a quantum system.

The simplest model, in which there is a false vacuum state, is the theory of one real scalar field ϕ with the Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{\lambda}{4}(\phi^2-v^2)^2-a\phi \tag{1}$$

with λ, v, and a being constants. In the limit of the vanishing asymmetry parameter a the field has two degenerate vacuum states, corresponding to $\langle 0|\phi|0\rangle=\pm v$. For small positive a the state ϕ_+ at $+v$ becomes a local minimum (false vacuum) and the one near $-v(\phi_-)$ becomes the true vacuum. The difference ϵ in the energy density between these states is given by

$$\epsilon\equiv\epsilon(\phi_+)-\epsilon(\phi_-)=2av+O(a^2)\ . \tag{2}$$

The bubbles in the false vacuum are droplets of the phase ϕ_- embedded in the phase ϕ_+. The transition region between the phases (bubble wall) is of the thickness $\sim 1/(\sqrt{\lambda}v)$, and throughout this paper only the bubbles whose characteristic size is much larger than this scale will be considered (thin-wall approximation). The energy E of a bubble, as measured in the false vacuum, consists of a negative part proportional to its volume, $-\epsilon V$, and a positive part, associated with the surface energy density μ. For a small asymmetry parameter a the surface density can be taken as that of the domain wall in the symmetrical limit:

$$\mu=\tfrac{2}{3}\sqrt{2\lambda}v^3\ . \tag{3}$$

In the tunneling process the lowest-action path is provided by spherical bubbles, which have the maximal volume to surface ratio. Thus in the leading WKB approximation it is sufficient to consider only spherically symmetrical bubbles whose dynamics in the thin-wall approximation is described in terms of only one collective variable: the radius r. The classical equations of motion are determined by the following relation [3] for the Hamiltonian H:

$$(H+\tilde{\epsilon}r^3)^2-p^2=(\tilde{\mu}r^2)^2\ , \tag{4}$$

where p is the canonical momentum conjugate of r and the notation $\tilde{\epsilon}=(4\pi/3)$ and $\tilde{\mu}=4\pi\mu$ is introduced in order to minimize the appearance of factors of π in subsequent formulas.

According to Eq. (4) the potential energy of a bubble is given by the sum of the (negative) volume term and the (positive) surface term:

$$U(r)\equiv H(r,p=0)=\tilde{\mu}r^2-\tilde{\epsilon}r^3\ . \tag{5}$$

Thus, as shown in Fig. 1, at an energy E such that

$$0<E<E_c=\tfrac{4}{27}(\tilde{\mu}^3/\tilde{\epsilon}^2)$$

there are two classically allowed regions for a bubble with energy E: the subcritical region to the left of the barrier and the critical region to the right of the barrier. The bubbles in the subcritical region oscillate and relatively slowly [23] dissipate their energy by emission of particles. The bubbles in the critical region infinitely expand thus destroying the false vacuum. At an energy above E_c there is no distinction between the subcritical and critical bubbles, and nucleation of a bubble with such energy would automatically imply destruction of the false vacuum.

A semiclassical quantization of the effective theory with the Hamiltonian determined by Eq. (4) enables one to calculate the rate of the spontaneous decay of the false vacuum [3], and the same approach is used in what follows to calculate the matrix elements $\langle B(E)|\phi|0\rangle$ by means of the Landau-WKB technique. According to Landau [21,22], for a system with the coordinates q, the matrix element of an operator $f(q)$ between two strongly different states $|X(E_1)\rangle$ and $|Y(E_2)\rangle$ with energies E_1 and E_2, $\langle Y(E_2)|f|X(E_1)\rangle$, in the leading WKB approximation is given by

$$|\langle Y(E_2)|f|X(E_1)\rangle|\sim\exp\left[\mathrm{Re}\left[i\int_{q_X}^{q^*}p(q;E_q)dq+i\int_{q^*}^{q_Y}p(q;E_2)dq\right]\right]\ , \tag{6}$$

where q^* is the (generally complex) "transition point," i.e., the point of stationary phase of the expression

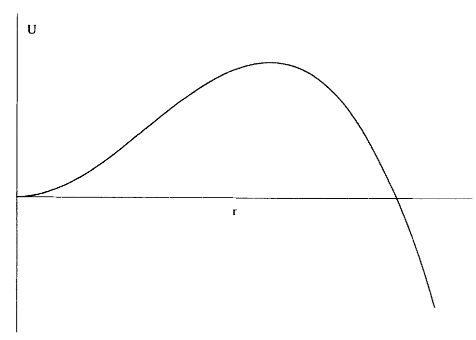

FIG. 1. Potential energy of a bubble vs its radius.

[1] I am thankful to Valery Rubakov for insisting on emphasizing this point.

$$\exp\left[i\int_{q_X}^{q^*} p(q;E_1)dq + i\int_{q^*}^{q_Y} p(q;E_w)dq\right], \qquad (7)$$

$p(q,E_1)$ $[p(q,E_2)]$ are the momenta on the classical (generally complex) trajectory with energy E_1 (E_2), which runs between the points q_X and q^* (q^* and q_Y), and, finally, q_X and q_Y are points, correspondingly chosen somewhere in the classically allowed regions for the states X and Y. The particular choice of each of the latter points in a simply connected domain of the classically allowed region does not affect the real part of the integrals in Eq. (6). The interpretation of the Landau formula (6) is straightforward within the approach consistently pursued in the Landau-Lifshits textbook in connection with the WKB calculation of various transition amplitudes: the amplitude is given by the exponent of the truncated classical action on the trajectory, which runs from the initial state to the final through a (complex) "transition point."

A few remarks are in order in connection with the application of Eq. (6) in the problem discussed here. First is that Eq. (6) is written for the case, relevant to present calculation, when the classical value of the operator f is not exponential at the "transition point" q^*, so that the exponential factor given by Eq. (6) is not sensitive to the specific form of the operator. Second is that Eq. (6) does not require the WKB approximation to be applicable for the wave functions of either of the states X and Y in the classically allowed region, i.e., where these wave functions are large. Thus it can be applied even if the lowest of the two energies, for example, E_1, is small, including the case $E_1=0$. The only condition for applicability of Eq. (6) is that the states X and Y are "strongly different" in the sense that the matrix element, given by this equation, contains a large exponential power, i.e., that it is strongly exponentially suppressed. Third is that the branch of the function $p(q,E)$ in the complex plane is to be chosen so that the exponential power in Eq. (6) is negative. Finally, if there are several "transition points" q^*, only the contribution of the one which gives the maximal transition probability is to be retained. A detailed justification of these statements about the Landau formula (6) can be found in Chapter 7 of the Landau-Lifshits textbook [21].

In the matrix element $\langle B(E)|\phi|0\rangle$, the field operator with zero spatial momentum (c.m. system) translates in the effective theory of the thin-wall bubbles into the operator

$$\int [\phi(x)-\phi_+]d^3x = \tfrac{8}{3}\pi v r^3 . \qquad (8)$$

Thus the whole problem can be reformulated in terms of the effective theory as a calculation of the matrix element $\langle B(E)|r^3|0\rangle$ for a system with the Hamiltonian determined by Eq. (4). Using the Landau formula (6), one can write the exponential estimate for this matrix element as

$$|\langle B(E)|\phi|0\rangle| \sim \exp\left[-\mathrm{Re}\left[\int_0^{r^*}\sqrt{(\tilde\mu r^2)^2-(\tilde\epsilon r^3)^2}dr + \int_{r^*}^{r(E)}\sqrt{(\tilde\mu r^2)^2-(\tilde\epsilon r^3+E)^2}dr\right]\right]$$
$$= \exp\left[-\frac{1}{\xi}\,\mathrm{Re}\left[\int_0^{x^*}\sqrt{x^4-x^6}dx + \int_{x^*}^{x(E)}\sqrt{x^4-(x^3+w)^2}dx\right]\right], \qquad (9)$$

where instead of r and E the dimensionless variables x and w are introduced as $r=x\tilde\mu/\tilde\epsilon$ and $E=w\tilde\mu^3/\tilde\epsilon^2$ and $\xi=\tilde\epsilon^3/\tilde\mu^4$ is the small dimensionless constant in the effective theory of bubbles. In Fig. 2 are shown the classical turning points for bubbles at zero energy and also for an energy $E<E_c$. At $E=0$ the classically allowed domain consists of the region $x>1$ and of the point $x=0$. At a nonzero energy $E<E_c$ the classically allowed domain consists of two finite regions: to the left of barrier, $x<x_1(E)$, corresponding to subcritical bubbles and to the right of the barrier, $x>x_2(E)$, which corresponds to infinitely expanding critical bubbles. Accordingly, the final point $x(E)$ of the transition trajectory in Eq. (9) can be chosen either in the subcritical domain (path I+II in Fig. 2) or in the critical one (path I+III in Fig. 2). The former choice produces the amplitude of the excitation of a subcritical bubble,

$$A_- = \langle B_{\mathrm{subc}}(E)|\phi|0\rangle ,$$

while the latter choice gives the amplitude of production of an infinitely expanding critical bubble $A_+=\langle B_c(E)|\phi|0\rangle$. In either case the transition path starts at the point $x=0$ and with $E=0$, which corresponds to the absence of a bubble in the initial state. Strictly speaking, the thin-wall approximation is not applicable at $r=0$. However, the inaccuracy of the approximation at the values of the radius of the order of the thickness of the wall does not affect the factors $\sim\exp(-\mathrm{const}/\xi)$ which are being considered in this calculation. In other words, the expression in Eq. (9) receives the dominant contribution from the region of large r, and therefore is calculable within the thin-wall approximation. From the paths shown in Fig. 2 it is clear that the amplitudes A_+ and A_- are related as

$$|A_+| = |A_-|\exp[-b(E)/\xi] , \qquad (10)$$

where

$$b(E)/\xi = \int_{r_1(E)}^{r_2(E)} |p(r;E)|dr$$
$$= \frac{1}{\xi}\int_{x_1(E)}^{x_2(E)} \sqrt{x^4-(x^3+w)^2}dx \qquad (11)$$

is the exponential power in the barrier penetration rate at

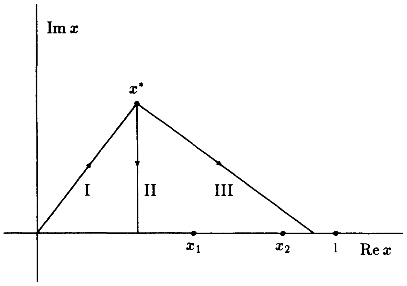

FIG. 2. Classical turning points and the transition path in the Landau formula for the bubbles. $x=1$ is the turning point on the right of the barrier at zero energy. x_1 and x_2 are the turning points on the left and on the right of the barrier at energy E. The transition trajectory starts at $x=0$ and goes with energy $E=0$ to the "transition point" x^* (the link I), then it goes with energy E either to the subcritical region (the link II) or to the critical one (the link III).

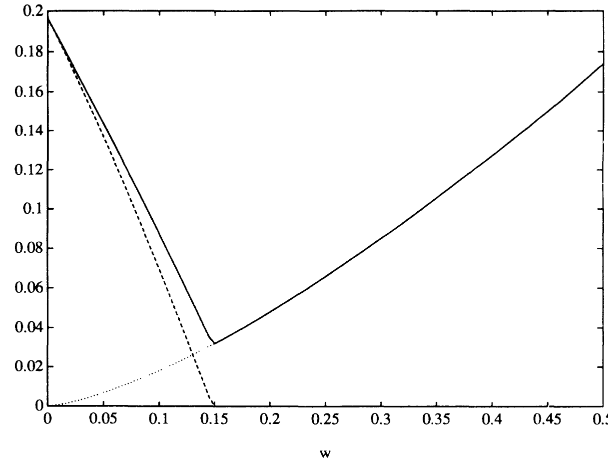

FIG. 3. The barrier penetration function $b(E)$ (dashed), the excitation function $c(E)$ (dotted), and their sum (solid) vs $w=E\bar{\epsilon}^2/\bar{\mu}^3$. At the point $w_c=\frac{4}{27}$ and beyond the barrier disappears, hence $b(E)=0$ and the sum coincides with $c(E)$.

energy E. The relation (10) can thus be interpreted as stating that the production of the critical bubble at $E<E_c$ proceeds through excitation of a subcritical one with subsequent tunneling through the barrier.

According to the expression (7) the "transition point" x^* is determined by the solution of the equation

$$\sqrt{x^4-x^6}-\sqrt{x^4-(x^3+w)^2}=0 \ . \tag{12}$$

The solutions to this equation are given by the values of the cubic root $(-w/2)^{1/3}$. A simple inspection shows that one can choose either of the complex values of the root in the right half plane as the appropriate "transition point" [choosing one instead of another gives the same result after proper redefinition of the branches of the expressions in Eq. (9)]. the integrals in Eq. (9) were evaluated numerically to determine the functions $c(E)$ and $b(E)$, appearing in the amplitudes A_- and A_+:

$$|A_-|\sim\exp\left[-\frac{c(E)}{\xi}\right] ,$$
$$|A_+|\sim\exp\left[-\frac{c(E)+b(E)}{\xi}\right] . \tag{13}$$

The results of the numerical calculation are shown in Fig. 3. At the critical energy E_c, corresponding to the top of the barrier, the barrier penetration term $b(E)$ vanishes. However, the excitation term $c(E)$ at this energy has a finite value

$$c(E_c)\approx 0.0314\approx 0.160b(0) ,$$

where $b(0)=\pi/16$ is the value of the barrier penetration term for the spontaneous false vacuum decay. [In fact, $c(E_c)$ can be found exactly in terms of elliptic integrals, but the final expression for the result is unusually cumbersome.]

The function $c(E)$ can be found analytically in the limit of large w as well as small w. For large w one can neglect x^4 in comparison with x^6 and with $(x^3+w)^2$ in Eq. (9) and thus find

$$c(E)=\frac{3\sqrt{3}}{4}\left|\frac{w}{2}\right|^{4/3} \quad (w\gg 1) \ . \tag{14}$$

For $w\ll 1$ the expression in Eq. (9) is determined by the region of x near the classical turning point $x_1(E)$. In this region one can neglect in Eq. (9) x^6 in comparison with x^4 and also neglect x^3 in comparison with w. Then $c(E)$ can be found as

$$c(E)=\int_0^L x^2dx-\int_{\sqrt{w}}^L\sqrt{x^4-w^2}dx$$
$$=\frac{\sqrt{\pi}}{6}\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}w^{3/2}\approx 0.0874w^{3/2} \quad (w\ll 1) \ , \tag{15}$$

where both integrals run along the real axis and L is a cutoff parameter. $L\gg\sqrt{w}$. The difference of the integrals is determined by the region $x\sim\sqrt{w}$, which substantiates the approximation, leading from Eq. (9) to Eq. (15). The full exponential power in the excitation amplitude A_- for small w is thus given by

$$c(E)/\xi=\text{const}\times E\sqrt{E/\mu} \ ,$$

which coincides with the result for the amplitude of excitation of a bubble with energy E in the case of degenerate vacua obtained in Ref. [20]. (Clearly, in that case only subcritical bubbles exist.) One should, however, keep in mind that the region of small w is limited from below by the condition of applicability of the thin-wall approximation, which implies that the characteristic size of the bubbles in the relevant region $r\sim\sqrt{w}\,\bar{\mu}/\bar{\epsilon}$ is larger than the thickness of the wall. In terms of E this translates into the condition [20] $E\gg\mu^{1/3}$. The latter condition justifies the use of the effective Hamiltonian (4) throughout the calculation, since in this case the contribution of the region of small radius r of the bubble, where the effective Hamiltonian is inapplicable, can be neglected.

It can be also noticed that at a small energy the barrier penetration term

$$b(E)=\frac{\pi}{16}-w+o(w) \tag{16}$$

decreases faster than the $w^{3/2}$ growth of the $c(E)$. Therefore, the probability of the induced decay of the false vacuum grows with energy in this region. As is seen from Fig. 3, this behavior continues up to the top of the barrier, where $b(E)$ vanishes.

The behavior of the induced tunneling amplitude calculated in this paper is similar to the one observed [18,19] in the quantum-mechanical example with the double-well potential $(x^2-1)^2$, where at the top of the barrier the exponential power in excitation probability is a finite fraction, namely, one-half, of that in the tunneling probability at $E=0$. That the ratio of the exponential powers in that case is exactly one half is a consequence of the reflection symmetry of the potential and of the standard relation of the Hamiltonian to the kinetic and the potential energy. Neither of these features hold for the problem discussed here; hence, the particular value of the ratio of the exponential powers is different, and is approximately equal to 0.160.

As a final remark one can note that the Landau formula (6) is not sensitive in the leading exponential approximation to the particular form of the operator $f(q)$, provided that the function $f(q)$ by itself is not exponential in the parameters in the problem. Therefore, though for definiteness the catalysis of the false vacuum decay by the particular operator ϕ has been discussed, the same results should be applicable for destruction of the false vacuum in any few-particle process at energy E. Also, one can notice, the particular form of the Lagrangian in Eq. (1) was used only to give the parameters ϵ and μ a particular expression in terms of the underlying theory. The rest of the calculation is based on the relation (4) for the Hamiltonian of the effective theory, which is a general relation for the dynamics of spherical bubbles in the thin-wall approximation. Therefore, the results of the present calculation are applicable whenever the latter approximation is valid.

This work was supported in part by DOE Grant No. DOE-AC02-83ER40105.

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