Anomaly cancellation in 2+1 dimensions in the presence of a domain wall mass

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A fermion in 2+1 dimensions, with a mass function which depends on one spatial coordinate and passes through a zero (a domain wall mass), in the background of an Abelian gauge field is considered. In this model, originally proposed in a non-Abelian version by Callan and Harvey, the gauge variation of the effective gauge action mainly consists of two terms. One comes from the induced Chern-Simons term and the other from the chiral fermions, bound to the (1+1)-dimensional wall, and they are expected to cancel each other. Though there exist arguments in favor of this, based on the possible forms of the effective action valid far from the wall and some facts about theories of chiral fermions in 1+1 dimensions, a complete calculation is lacking. In this paper we present an explicit calculation of this cancellation at one loop which is valid even close to the wall. We show that integrating out the "massive" modes of the theory does produce the Chern-Simons term, as appreciated previously. In addition, we show that it generates a term that softens the high energy behavior of the (1+1)-dimensional effective chiral theory thereby resolving an ambiguity present in a general (1+1)-dimensional theory.

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I. INTRODUCTION

It was understood some time ago that there exist intimate connections between the Chern-Simons term in an odd dimensional space-time and the chiral anomaly in one lower dimension. After such a connection was understood, Callan and Harvey [1] proposed a model in which the connection was physically realized. They considered a three-dimensional fermion with a domain wall mass (a mass term that depends on one space coordinate, passes through zero at the origin, and goes to a constant with opposite signs at plus and minus infinity) coupled to a gauge theory. Since there are no anomalies in the continuous symmetries in odd dimension, the theory must be gauge invariant. However, in the theory with a domain wall mass, one can show, as we will see later, that there exist effectively two-dimensional massless chiral fermions attached to the domain wall. The resulting two-dimensional chiral theory should have an anomaly in the gauge current. However, since the whole theory has no anomaly, there must be yet another contribution to the current in the whole theory which will cancel the chiral anomaly. It was found that there indeed exist currents on either side of the wall which flow into or away from the wall depending on the sign of the anomaly on the wall. This current can be approximately calculated away from the wall using methods of Goldstone and Wilczek [2]. We will call these currents Goldstone-Wilczek currents.

However, when one investigates the Goldstone-Wilczek currents flowing from the third dimension into the wall and thus accounts for the charge appearance on the wall (the person on the wall considers the charge ap-

pearing as an anomaly), one encounters difficulties. In an Abelian theory, for example, the charge appearing on the wall is twice as much as that predicted from an anomaly in an exclusively (1+1)-dimensional theory [4]. In a non-Abelian theory, the problem is more evident. Here the anomaly in the two-dimensional chiral theory is necessarily gauge noncovariant. This noncovariant form is required by the Wess-Zumino [3] consistency conditions obeyed by the usual definition of the current. Hence the anomaly in this current is also referred to as a *consistent anomaly*. On the other hand, the Goldstone-Wilczek current accounts for the anomaly on the wall which is gauge covariant in its form. This form of the anomaly, which is gauge covariant in its form, is also referred to as the *covariant anomaly*. Thus this Goldstone-Wilczek current alone cannot completely cancel the consistent anomaly on the wall, as one is gauge covariant and the other is not.

When Bardeen and Zumino discuss consistent and covariant anomalies [5], they show how an addition of an extra term to the consistent current can make the anomaly covariant in its form. Thus it seems that there must exist an extra piece of current on the wall that arises naturally and which makes the anomaly in the effective (1+1)-dimensional theory covariant in its form. This term cannot be obtained from the Lagrangian of an exclusively (1+1)-dimensional theory as the consistency conditions would not allow its presence. On the other hand, in our model this extra piece of the current can be induced by the effects of the extra dimension. This problem was addressed by Naculich [4] in which he suggests how a particular form of the Chern-Simons term in the (2+1)-dimensional effective action (produced when you integrate out the massive fermion modes of the theory) can induce this extra piece. In fact, this particular form of the Chern-Simons terms was originally suggested by Callan and Harvey [1]. However, there is no complete derivation for this Chern-Simons term in the effective ac-

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tion, which is valid arbitrarily close to the wall. This is not satisfactory because the calculations suggested to extract the effective Chern-Simons term are valid only far from the wall; on the other hand, the actual questions which are at issue here are related to terms induced on the wall. Also one wonders, if there is one extra effect on the wall other than that of a simple (1+1)-dimensional massless chiral theory, if there may be others that are hidden and unclear until a complete calculation of the effective action is done.

A more clean and simple view of this anomaly cancellation comes from considering an effective action in terms of the gauge fields after integrating out the fermions. This effective action must be gauge invariant. This was the original way Callan and Harvey [1] analyzed the problem. In their paper [1] they argue that integrating out the massive fermion modes (the modes that are not chiral and do not live on the wall) produces a Chern-Simons term in the effective action. They show how the gauge variation of this term can cancel the variation coming from the remaining (1+1)-dimensional effective chiral theory on the wall. This cancellation, when examined in terms of the currents, motivates the above discussion of covariant and consistent anomalies considered first by Naculich [4]. Hence, even in this simpler view, one needs a derivation of the suggested Chern-Simons term in the effective action that is valid close to the wall. Further, one must show that the resulting (1+1)-dimensional theory can be treated as a naive (1+1)-dimensional massless chiral theory with no other effects induced.

These questions are of particular interest in the context of the recent proposal, by Kaplan, to solve the doubling problem on the lattice [6]. The basic model used in this new proposal is the same as the one proposed by Callan and Harvey. In this model one is looking for a theory of massless chiral fermions on the wall. To make the final theory exclusively live on the wall, it is important that the massive modes have very little effect on the wall, as the massive modes are presumed to decouple from the theory on the wall, so that the issues discussed above are important in this context.

Taking all this into account, it seems quite important to understand the structure of the cancellation of the anomaly in this model with a domain wall mass. An explicit calculation would clarify the effects of the massive modes in this cancellation. Also, the calculations given in Refs. [1] and [4] deal mainly with axionic strings apart from mentioning the applicability to domain walls. In view of the recent interest in the domain wall problem [6], we think it makes sense to write down some results explicitly valid for the domain wall case along with some proof for the previously suggested results.

In this paper we study the model in which fermions move in a background of an Abelian gauge field and interact with a domain wall, which is a smooth function of one space coordinate. In general, one could couple the fermion to a non-Abelian gauge field, but here we present a calculation of the anomaly cancellation only for the case of an Abelian gauge field. In Sec. II we study the eigenstates for the free Dirac operator with this domain

wall mass and show how the eigenstates of the theory change as the steepness of the mass function changes. We actually find that, as the mass function becomes smooth, the number of states bound to the wall increases, though only one is chiral. We then pick a particular mass function, which turns out to be easy to analyze and which has only one (chiral) bound state, and derive the complete set of eigenstates for the free Dirac operator with this choice of the mass function. In Sec. III we find the free propagator for this theory including the effects of the space-dependent mass term using the exact eigenstates derived in Sec. II for the particular choice of the mass function. As computations are much easier in Euclidean space, we continue our results to Euclidean space and obtain a closed form expression for the propagator in Euclidean configuration space. In Sec. IV we integrate out the fermion fields using this Euclidean-space propagator and treat the gauge coupling perturbatively to compute the one loop effective action for the gauge fields. When we look at only the terms potentially contributing to the anomaly (i.e., the terms that contain the completely antisymmetric tensor), we find that in the low energy limit the effective gauge action consists of two terms. The first is the old Chern-Simons term, as suggested by Naculich [4], but now without any assumptions and valid arbitrarily close to the wall. The second term is the chiral term, which has contributions not only from the chiral bound states, but also from fully three-dimensional massive states. The extra contribution from the massive states acts to "regulate" the chiral term. Thus we find that the chiral anomaly is generated not by a potentially singular term, but by a well regulated term. The high energy characteristic of the chiral fermions on the wall is "softer" than the usual chiral fermions in two dimensions. Finally, we show how, because of these effects, the gauge variation of the Chern-Simons term and the chiral term in the effective gauge action cancel each other explicitly. Before concluding, we show how this cancellation can be viewed in terms of the currents in Sec. V.

II. STATES OF THE THEORY

We start with a theory defined in Minkowski space to give the theory a physical setting and to be able to analyze the physical states of the theory. The theory can be given in terms of the action

$$\mathcal{S} = \int d^3z \, \overline{\Psi} \{ i\gamma^\mu (\partial_\mu - ieA_\mu) + m(s) \} \Psi , \qquad (1)$$

where the γ^μ's are the Dirac matrices, which obey the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}, \quad \mu, \nu = 0, 1, 2 , \qquad (2)$$

and $\overline{\Psi} = \Psi^\dagger \gamma^0$. We assume that $\mu = 0$ is the "time," $\mu = 1, 2$ the space direction, so that the metric would be $g^{00} = 1$, $g^{11} = -1$, and $g^{22} = -1$. The coordinates of the space-time are labeled by $z^\mu = (t, x, s)$. The mass depends on the second space direction, labeled by s. We want $m(s)$ to be a function with a domain wall shape: i.e.,

$$m(s) = \begin{cases} m_0, & s \to +\infty, \\ -m_0, & s \to -\infty, \\ 0, & s = 0. \end{cases} \qquad (3)$$

We first want to solve for the states of the theory to see the effects of the mass function. In order to proceed further, we assume a specific form of the mass function which is sufficiently general to be able to study its various limiting forms. We choose

$$m(s) = m_0 \tanh \mu_0 s .$$ (4)

To be concrete we choose a *chiral* representation for the Dirac matrices defined by

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma^2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} .$$ (5)

Having fixed the theory, let us ask what the eigenstates of the theory are. We will treat the gauge coupling as a perturbation and think of the remaining theory as a *free* theory and solve the equations of motion. However, keeping in mind that we would like to solve for the propagator of the theory too, we will try to solve the eigenvalue equation

$$\gamma^0 \{ i\gamma^\mu \partial_\mu + m(s) \} \Psi_\lambda = \lambda \Psi_\lambda$$ (6)

or in the matrix form this can be written as

$$\begin{bmatrix} i\partial_t - i\partial_x & -\partial_s + m(s) \\ \partial_s + m(s) & i\partial_t + i\partial_x \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \lambda \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} .$$ (7)

Clearly, the eigensolutions for $\lambda = 0$ will also be the solutions for the equations of motion. The propagator for the theory, $S(z,z')$, can then be constructed from the eigensolutions above by

$$S(z,z') = \sum_\lambda \frac{\Psi_\lambda(z) \overline{\Psi}_\lambda(z')}{\lambda} .$$ (8)

When we construct the propagator, we must remember to take the Feynman prescription, which can be derived using the usual canonical commutation relations. However, let us first solve for the eigenvalues λ and the corresponding eigenfunctions Ψ_λ.

To do this, first observe that the eigenfunctions can be written in the form $\Psi_\lambda = \Phi_{\lambda,\mathbf{k}}(s) e^{-ik_0 t} e^{-ik_1 x}$, where we characterize the eigenvalues λ by $\mathbf{k} = (k_0, k_1, k_2)$. Actually, we can look at the asymptotic behavior of Eq. (7) and fix the eigenvalues. We then find that for a given k_0, k_1, k_2 we get two values of λ given by

$$\lambda_\pm = k_0 \pm \omega_k, \quad \omega_k = (k_1^2 + k_2^2 + m_0^2)^{1/2} .$$ (9)

Using these facts, one can write the differential equations obeyed by the two components of $\Phi_{\lambda,\mathbf{k}}$, ϕ_1 and ϕ_2, as

$$\begin{bmatrix} k_0 - k_1 & -\partial_s + m(s) \\ \partial_s + m(s) & k_0 + k_1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \lambda_\pm \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} .$$ (10)

One can rewrite the above two coupled equations for ϕ_1 and ϕ_2 as decoupled second order equations given below:

$$\left[\frac{\partial^2}{\partial s^2} + \left[k_2^2 + m_0^2 - m^2(s) + \frac{\partial m(s)}{\partial s} \right] \right] \phi_1 = 0 ,$$

$$\left[\frac{\partial^2}{\partial s^2} + \left[k_2^2 + m_0^2 - m^2(s) - \frac{\partial m(s)}{\partial s} \right] \right] \phi_2 = 0 .$$ (11)

If we solve for ϕ_1 or ϕ_2, we can substitute it in (10) and obtain the other by solving a simple algebraic equation. Substituting $m(s) = m_0 \tanh \mu_0 s$ in (11), we get

$$\left[\frac{\partial^2}{\partial s^2} + k_2^2 + \alpha(\alpha+1) \frac{\mu_0^2}{\cosh^2 \mu_0 s} \right] \phi_1 = 0 ,$$

$$\left[\frac{\partial^2}{\partial s^2} + k_2^2 + \alpha(\alpha-1) \frac{\mu_0^2}{\cosh^2 \mu_0 s} \right] \phi_2 = 0 ,$$ (12)

where $\alpha = m_0 / \mu_0$.

We wish to solve (12). In fact, these two equations describe quantum mechanical scattering off a *modified Pöschl-Teller potential* and the energy eigenvalues and functions are known; see, for example, [7]. The potential can be written generically as $\beta(\beta+1)\mu_0^2/\cosh^2 \mu_0 s$ and is known in the literature as the *modified Pöschl-Teller potential*. It is clear from (12) that the only difference between ϕ_1 and ϕ_2 is that $\beta = \alpha$ for ϕ_1 and $\beta = \alpha - 1$ for ϕ_2. It is known that given β, the number of bound states equals the largest integer less than $\beta + 1$. Clearly, ϕ_1 has always one more bound state than ϕ_2. This is the chiral bound state since it can be shown that if this ϕ_1 is substituted back in (10) it gives $\phi_2 = 0$. This is the chiral state which is responsible for the anomaly in [1]. However, we also see here that as the mass function becomes less steep, i.e., as m_0/μ_0 becomes large, the number of bound states increases, though none of these are chiral except the one discussed above. If $0 < m_0/\mu_0 \leq 1$, the only bound state is the chiral bound state. As this is the case of most interest at present, we will assume $m_0 = \mu_0$, so that the mass function becomes $m_0 \tanh m_0 s$. The reason for doing this is that then the eigenstates of the theory can be written in closed form. Substituting that $\alpha = m_0/\mu_0 = 1$ in (12), we get

$$\left[\frac{\partial^2}{\partial s^2} + k_2^2 + \frac{2\mu_0^2}{\cosh^2 \mu_0 s} \right] \phi_1 = 0 ,$$

$$\left[\frac{\partial^2}{\partial s^2} + k_2^2 \right] \phi_2 = 0 .$$ (13)

It is now evident how the choice of α above simplifies things. We can solve (13) for ϕ_2 in the above equation and substitute it back in Eq. (10) to obtain ϕ_1. These will be the scattering eigenstates, given by

$$\Phi_{\lambda_\pm, \mathbf{k}} = \begin{bmatrix} ik_2 + m(s) \\ \pm \omega_k + k_1 \end{bmatrix} e^{-ik_2 s} ,$$ (14)

where $m(s) = m_0 \tanh m_0 s$. [We will hereafter assume $m(s) = m_0 \tanh m_0 s$ wherever we use $m(s)$.] Note that when we substitute ϕ_2 back in (10) we will in general get two solutions for ϕ_1 depending on the sign $\pm \lambda$. This is explicitly shown in (14). Note that at present the range of k_2 goes from $-\infty$ to ∞. However, we have not yet obtained all the eigenstates of the theory. There exists one bound state for ϕ_1 which can be obtained by solving (13). This cannot be obtained from a solution of ϕ_2 because for

this eigenstate $\phi_2 = 0$. This bound state is given by

$$\Phi_{\text{chiral}} = \begin{bmatrix} \text{sech} m_0 s \\ 0 \end{bmatrix} . \tag{15}$$

Along with this we have the complete set of states of the theory though they are not yet orthonormal. We can take linear combinations and form an orthonormal basis for the theory, which we find to be

$$\Psi_{\mathbf{k},\lambda\pm,\text{odd}} = \left[\frac{1}{8\pi^3} \frac{1}{\omega_k(\omega_k \pm k_1)} \right]^{1/2}$$

$$\times \begin{bmatrix} k_2 \cos k_2 s - m(s)\sin k_2 s \\ -(\pm\omega_k + k_1)\sin k_2 s \end{bmatrix} e^{-ik_0 t - ik_1 x} ,$$

$$\Psi_{\mathbf{k},\lambda\pm,\text{even}} = \left[\frac{1}{8\pi^3} \frac{1}{\omega_k(\omega_k \pm k_1)} \right]^{1/2}$$

$$\times \begin{bmatrix} k_2 \sin k_2 s + m(s)\cos k_2 s \\ (\pm\omega_k + k_1)\cos k_2 s \end{bmatrix} e^{-ik_0 t - ik_1 x} ,$$

$$\tag{16}$$

$$\Psi_{\mathbf{k},\lambda_0,\text{chiral}} = \left[\frac{1}{4\pi^2} \frac{m_0}{2} \right]^{1/2} \begin{bmatrix} \text{sech} m_0 s \\ 0 \end{bmatrix} e^{-ik_0 t - ik_1 x} ,$$

with eigenvalues $\lambda_\pm, \lambda_\pm, \lambda_0 = k_0 - k_1$, respectively.

The subscript $\mathbf{k}, \lambda, \alpha$ characterizes the different eigenstates. λ refers to the eigenvalue, α refers to odd, even or chiral, and $\mathbf{k}$ refers to (k_0, k_1, k_2). The allowed range of

$\mathbf{k}$ is given by

$$-\infty < k_0, k_1 < +\infty \quad \text{and} \quad 0 \leq k_2 < +\infty . \tag{17}$$

The asymmetry in the range for k_2 arises because we have made linear combinations of $+k_2$ and $-k_2$ to form odd and even states.

The orthonormality can be tested by showing

$$\int \Psi^\dagger_{\mathbf{k},\lambda,\alpha} \Psi_{\mathbf{k}',\lambda',\alpha'} = \delta_{\alpha,\alpha'} \delta_{\lambda,\lambda'} \delta^3(\mathbf{k} - \mathbf{k}')$$

when α and α' are not chiral. If both α and α' are chiral, then the three-dimensional δ function is replaced by a two-dimensional δ function in k_0 and k_1. The chiral eigenstate is orthogonal to both the odd and the even eigenstates. Thus we have an explicit calculation for the eigenstates of the theory. We can now go ahead and construct the propagator as described by (8).

III. PROPAGATOR

The construction of the propagator is straightforward though the derivation of the explicit expression in closed form will be complicated. We use (8) to obtain an integral representation in "momentum" space for the propagator. After substituting and rearranging the terms and also extending the limits of the k_2 integral from $-\infty$ to $+\infty$, we get

$$S(z,z') = S_{\text{chiral}}(z,z') + S_{\text{massive}}(z,z') , \tag{18}$$

where the chiral part is due to the chiral mode in the summation in (8) and the massive part is due to the rest. The two terms are given by

$$S_{\text{chiral}} = \frac{1}{2}(1 + i\gamma^2)\frac{m_0}{2}\text{sech}(m_0 s)\text{sech}(m_0 s') \int \frac{d^2 k}{(2\pi)^2} \frac{\gamma^a k_a}{k_0^2 - k_1^2 + i\epsilon} e^{-ik_0(t-t')} e^{-ik_1(x-x')} , \tag{19}$$

$$S_{\text{massive}} = \int \frac{d^3 k}{(2\pi)^3} \frac{\gamma^\mu k_\mu + M}{k^2 - m_0^2 + i\epsilon} e^{-ik_0(t-t')} e^{-ik_1(x-x')} e^{-ik_2(s-s')} , \tag{20}$$

where M is a matrix in spinor space given by

$$M = \begin{bmatrix} -m(s) & \dfrac{k_0 + k_1}{k_2^2 + m_0^2}\{m(s)m(s') + ik_2[m(s') - m(s)] - m_0^2\} \\ 0 & -m(s') \end{bmatrix} . \tag{21}$$

To make the notation clear, we will use Greek letters to run from 0 to 2 and Latin letters run from 0 to 1, so that $\gamma^a k_a$ means $\gamma^0 k_0 + \gamma^1 k_1$ unlike $\gamma^\mu k_\mu$, which means $\gamma^0 k_0 + \gamma^1 k_1 + \gamma^2 k_2$. The above form of the propagator has a structure similar to the usual fermion propagator in three dimensions except for the massless chiral term, which reflects the chiral modes on the wall and the unusual mass matrix M. The integral in the mass matrix M is the only thing that seems quite difficult to explicitly evaluate. Note that in three dimensions the other integrations can be easily done.

At this stage we will analytically continue to the *Euclidean space* so that we can explicitly evaluate the propagator and study the gauge transformation properties of the one loop effective action. This can be most easily done in Euclidean space where things are well defined. The continuation to Euclidean space means

$$t = -i\tau, \quad k_0 \to ik_0, \quad \Gamma^0 = \gamma^0, \quad \Gamma^1 = -i\gamma^1, \quad \Gamma^2 = -i\gamma^2 , \tag{22}$$

where the Γ's are the gamma matrices in Euclidean space and τ is Euclidean time. After the explicit calculations are done, we find the final result for the *Euclidean* propagator is

$$S_E(z,z') = -\frac{\Gamma^\mu}{4\pi} \frac{r_\mu}{r^3}(1+m_0 r)e^{-m_0 r} + \frac{1}{4\pi}\frac{e^{-m_0 r}}{r}\begin{bmatrix} m(s) & 0 \\ 0 & m(s') \end{bmatrix}$$

$$-\frac{1}{8\pi}(1-\Gamma^2)\frac{\Gamma^a \varepsilon_a}{\varepsilon^2}\frac{m_0 e^{-m_0 r}}{r}\left[r\left\{\frac{m(s)}{m_0}\frac{m(s')}{m_0}-1\right\}+(s-s')\left\{\frac{m(s')}{m_0}-\frac{m(s)}{m_0}\right\}\right]$$

$$-\frac{1}{8\pi}(1-\Gamma^2)\frac{\Gamma^a \varepsilon_a}{\varepsilon^2}m_0 \mathrm{sech}(m_0 s)\mathrm{sech}(m_0 s') , \tag{23}$$

where the various symbols are defined as

$$r^\mu = (\tau - \tau', x - x', s - s'), \quad \varepsilon^a = (\tau - \tau', x - x') ,$$
$$\varepsilon = \sqrt{(\tau - \tau')^2 + (x - x')^2}, \quad r = \sqrt{\varepsilon^2 + (s - s')^2} . \tag{24}$$

The last term of the propagator can be easily identified with the *chiral* propagator except for the $\mathrm{sech}(m_0 s)\mathrm{sech}(m_0 s')$ term, which says that these modes are bound to the wall. The first two terms combined become essentially the three-dimensional massive propagator with the mass term modified into a matrix. The third term seems to be new. It has the character of the chiral term (the last term), but at the same time is massive. To make this explicit, we rewrite (23) in a slightly different form by combining the third and fourth terms. After substituting $m(s) = m_0 \tanh m_0 s$, we get

$$S_E(z,z') = -\frac{\Gamma^\mu}{4\pi}\frac{r_\mu}{r^3}(1+m_0 r)e^{-m_0 r} + \frac{1}{4\pi}\frac{e^{-m_0 r}}{r}\begin{bmatrix} m_0 \tanh m_0 s & 0 \\ 0 & m_0 \tanh m_0 s' \end{bmatrix}$$

$$-\frac{1}{8\pi}(1-\Gamma^2)\frac{\Gamma^a \varepsilon_a}{\varepsilon^2}[1-f(\varepsilon, s-s')]m_0 \mathrm{sech}(m_0 s)\mathrm{sech}(m_0 s') , \tag{25}$$

where the function $f(\varepsilon, s-s')$ is given by

$$f(\varepsilon, s-s') = \frac{e^{-m_0 r}}{r}[r \cosh m_0(s-s') + (s-s')\sinh m_0(s-s')] . \tag{26}$$

We have written the third term in (23) in terms of the function f so as to unite it with the chiral term. Now the effects of this term are clearer. It modifies the singularity structure of the massless chiral term. This suggests that the massive modes "regulate" the chiral modes. This also suggests that, when the theory is coupled to a gauge field, a cancellation of the anomaly between the massive modes and the massless chiral modes might be more involved. Hence we investigate the cancellation of the anomaly in the next section.

IV. ANOMALY CANCELLATION AT ONE LOOP

Having found the free propagator explicitly, we can treat the gauge coupling perturbatively and construct the one loop effective gauge action induced by integrating out the fermion fields. Note that the effective action being calculated here is the one loop gauge action with no fermion fields. It is still three dimensional, though parts of it might look two dimensional due to the presence of the chiral pieces that are nonzero only on the wall. This is important to remember as there are many other kinds of effective action that can be considered, for example, by integrating out only the massive fermion fields and keeping the effective action dependent on the chiral fields. This would be natural when one wants to study the effective chiral fermion theory in the low energy limit. However, we are not doing this. We want to study the gauge invariance of the full theory and that can be done by integrating out all the fermion fields, and studying the

full three-dimensional effective gauge theory. However, we will treat the fluctuations in the gauge fields as small compared to the mass of the massive modes of the fermion fields in order to motivate a low energy effective gauge theory. We will then show that this low energy effective theory is gauge invariant. The actual problem at issue here is the gauge invariance of this low energy effective gauge theory, as there are nontrivial low energy terms that are induced by both the massless chiral fermion fields and also the massive fermion fields.

Having explained our motive, let us look at the one loop effective gauge action, which is given by

$$S_{\mathrm{eff}}[A] = \frac{1}{2}\int d^3z\, d^2z'\, A_\mu(z)V^{\mu\nu}(z,z')A_\nu(z') , \tag{27}$$

where

$$V^{\mu\nu} = \mathrm{tr}[\Gamma^\mu S_E(z,z')\Gamma^\nu S_E(z',z)] . \tag{28}$$

The trace is over the spinor space.

The expression for $V^{\mu\nu}$ clearly would be quite a long and complicated expression when $S_E(z,z')$ is substituted; but, as we will be interested finally in the $m_0 \to \infty$ limit, it makes sense to just look at the expression for the limit. Also we will focus attention on the part of $V^{\mu\nu}$ which has either a two-dimensional ϵ^{ab} or a three-dimensional antisymmetric tensor $\epsilon^{\mu\nu\rho}$. This is because the potential anomaly occurs in this part. The appropriate expression for $V_a^{\mu\nu}$ (where the label a denotes this potentially anomalous part), keeping only the nontrivial terms that survive in the limit $m_0 \to \infty$, is found to be

$$V_a^{\mu\nu} = \frac{1}{8\pi^2}\left[-i\epsilon^{\mu\nu\rho}\frac{r_\rho}{r^4}(1+m_0 r)e^{-2m_0 r}[m(s)+m(s')] - \frac{m_0^2}{2}\frac{\epsilon^\mu\epsilon^{*\nu}+\epsilon^\nu\epsilon^{*\mu}}{\epsilon^4}[1-f(\epsilon,s-s')]^2\text{sech}^2(m_0 s)\text{sech}^2(m_0 s')\right],$$

(29)

where $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor with $\epsilon^{012}=1$ and $\epsilon^{*a}=i\epsilon^{ba}\epsilon_b$ is the dual of ϵ^a. Note that ϵ^a was defined in (24). Here ϵ^{ab} is the antisymmetric tensor in two dimensions. Note also that whenever the two-vector ϵ^μ is encountered, the component ϵ^2 is assumed to be zero.

As stated above, it is assumed that $m_0 \to \infty$ will be eventually taken and only the relevant terms in this limit are given. The important point to note is that though the function $f(\epsilon, s-s')$ comes from the massive modes we cannot throw it away because in the limit $\epsilon \to 0$ this function goes to 1 and hence contributes to the anomaly in a nontrivial way. If the above expression for $V_a^{\mu\nu}$ is substituted in (28) and some simplification is done, we get

$$\mathscr{S}_{\text{eff}}^a = \mathscr{S}_{\text{eff}}^{\text{CS}} + \mathscr{S}_{\text{eff}}^{\text{chiral}}$$

$$= \frac{-i}{8\pi}\int d^3z \, \text{sgn}(s)\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu$$

$$- \frac{1}{16\pi^2}\int d^3z \, d^3z' \, A_a(z)\frac{1}{2}\frac{\epsilon^a\epsilon^{*b}+\epsilon^b\epsilon^{*a}}{\epsilon^4}A_b(z')m_0^2[1-f(\epsilon,s-s')]^2\text{sech}^2(m_0 s)\text{sech}^2(m_0 s') .$$

(30)

The first term is the Chern-Simons term which is mentioned in [1]. The limit of $m_0 \to \infty$ turns $\tanh m_0 s$ to $\text{sgn}(s)$, which is the origin of the $\text{sgn}(s)$ in (30). We have also used the limit

$$\lim_{m_0 \to \infty} m_0\frac{e^{-2m_0 r}}{r^2} = 2\pi\delta(\mathbf{r}) .$$

(31)

Note also that we are analyzing the term in the effective action which has in it the antisymmetric tensor and hence the superscript a in the action. We must now show that $\delta\mathscr{S}_{\text{eff}}^a$, the gauge variation, is zero.

First let us consider the Chern-Simons term denoted by $\mathscr{S}_{\text{eff}}^{\text{CS}}$. We then have

$$\delta\mathscr{S}_{\text{eff}}^{\text{CS}} = \frac{-i}{8\pi}\int d^3z \, \text{sgn}(s)\epsilon^{\mu\rho\nu}(\delta A_\mu\partial_\rho A_\nu + A_\mu\partial_\rho\delta A_\nu) ,$$

(32)

where $\delta A_\mu = \partial_\mu\Theta$. Substituting this and also using the antisymmetry of $\epsilon^{\mu\rho\nu}$, we get

$$\delta\mathscr{S}_{\text{eff}}^{\text{CS}} = \frac{-i}{8\pi}\int d^3z \, \text{sgn}(s)\epsilon^{\mu\rho\nu}\partial_\mu\Theta\partial_\rho A_\nu = \frac{i}{8\pi}\int d^3z \, 2\delta(s)\epsilon^{ab}\Theta\partial_a A_b = \frac{i}{4\pi}\int d^2z \, \Theta(z)[\epsilon^{ab}\partial_a A_b] .$$

(33)

Now let us consider the second term; we shall call this the *chiral* term. Note that there are factors of m_0 present in this term because we cannot take the limit $m_0 \to \infty$ before the integration. Making a gauge variation as above, we get

$$\delta\mathscr{S}_{\text{eff}}^{\text{chiral}} = -\frac{1}{16\pi^2}\int d^3z \, d^3z' \, m_0^2\text{sech}^2(m_0 s)\text{sech}^2(m_0 s')[\partial_a\Theta(z)A_b(z') + A_a(z)\partial_b\Theta(z')]\frac{1}{2}\frac{\epsilon^a\epsilon^{*b}+\epsilon^b\epsilon^{*a}}{\epsilon^4}[1-f(\epsilon,s-s')]^2$$

$$= \frac{1}{8\pi^2}\int d^3z \, d^3z' \, m_0^2\text{sech}^2(m_0 s)\text{sech}^2(m_0 s')\Theta(z)A_b(z')\partial_a\left[\frac{1}{2}\frac{\epsilon^a\epsilon^{*b}+\epsilon^b\epsilon^{*a}}{\epsilon^4}[1-f(\epsilon,s-s')]^2\right] .$$

(34)

Using the fact that the singularity as $\epsilon \to 0$ is not severe due to the factor $[1-f(\epsilon,s-s')]^2$, we can compute the partial derivative. We get the result

$$\partial_a\left[\frac{\epsilon^a\epsilon^{*b}+\epsilon^b\epsilon^{*a}}{\epsilon^4}[1-f(\epsilon,s-s')]^2\right] = \frac{\epsilon^a\epsilon^{*b}+\epsilon^b\epsilon^{*a}}{\epsilon^4}\frac{\partial}{\partial\epsilon}\{[1-f(\epsilon,s-s')]^2\}\frac{\epsilon_a}{\epsilon}$$

$$= i\epsilon^{cb}\frac{\epsilon^c}{\epsilon^3}\frac{\partial}{\partial\epsilon}\{[1-f(\epsilon,s-s')]^2\} ,$$

(35)

where we have used the fact that $\epsilon^{*a}\epsilon_a = 0$ and the definition of the dual $\epsilon^{*b} = i\epsilon^{cb}\epsilon^c$. Using the above result, we have

$$\delta\mathscr{S}_{\text{eff}}^{\text{chiral}} = \frac{i}{16\pi^2}\int d^3z \, d^3z' \, m_0^2\text{sech}^2(m_0 s)\text{sech}^2(m_0 s')\Theta(z)\epsilon^{cb}A_b(z')\frac{\epsilon^c}{\epsilon^3}\frac{\partial}{\partial\epsilon}\{[1-f(\epsilon,s-s')]^2\} .$$

(36)

Now we can use the property that m_0 is large and that contributions to the integral come only from the region where $z^\mu - z'^\mu = r^\mu$ tends to zero. So we can expand $A_b(z')$ about $z'=z$ and keep only the terms that do not vanish in the

$m_0 \to \infty$; we get

$$\delta \mathcal{S}_{\text{eff}}^{\text{chiral}} = \frac{i}{16\pi^2} \int d^3z \, d^3r \, m_0^2 \text{sech}^2(m_0 s)\text{sech}^2(m_0 s')\Theta(z)\epsilon^{cb}\partial_a A_b(z)(-\epsilon^a)\frac{\epsilon^c}{\epsilon^3}\frac{\partial}{\partial \epsilon}\{[1-f(\epsilon,s-s')]^2\}$$

$$= -\frac{i}{32\pi^2} \int d^3z \, d^{s'} d^2\epsilon \, m_0^2 \text{sech}^2(m_0 s)\text{sech}^2(m_0 s')\Theta(z)\epsilon^{ab}\partial_a A_b(z)\frac{1}{\epsilon}\frac{\partial}{\partial \epsilon}\{[1-f(\epsilon,s-s')]^2\} \ .$$

Using $d^2\epsilon = 2\pi\epsilon \, d\epsilon$, we get

$$\delta \mathcal{S}_{\text{eff}}^{\text{chiral}} = -\frac{i}{16\pi} \int d^3z \, ds' \, m_0^2 \text{sech}^2(m_0 s)\text{sech}^2(m_0 s')\Theta(z)\epsilon^{ab}\partial_a A_b(z)\int_0^\infty d\epsilon \frac{\partial}{\partial \epsilon}\{[1-f(\epsilon,s-s')]^2\} \ . \tag{37}$$

The ϵ integration can be trivially done. At the upper limit $\epsilon = \infty$, $[1-f(\epsilon,s-s')]^2 = 1$, and at the lower limit $\epsilon = 0$, $[1-f(\epsilon,s-s')]^2 = 0$, which can be easily verified using the definition of $f(\epsilon,s-s')$ in (26). Also, the remaining s and s' integrations can be trivially done since in the limit $m_0 \to \infty$ we get

$$m_0^2 \text{sech}^2(m_0 s)\text{sech}^2(m_0 s') = 4\delta(s)\delta(s') \ . \tag{38}$$

Using these results, we get

$$\delta \mathcal{S}_{\text{eff}}^{\text{chiral}} = -\frac{i}{4\pi} \int d^2z \, \Theta(z)[\epsilon^{ab}\partial_a A_b(z)] \ . \tag{39}$$

Hence, using (33) and (39), we finally get

$$\delta \mathcal{S}_{\text{eff}}^a = \delta \mathcal{S}_{\text{eff}}^{\text{CS}} + \delta \mathcal{S}_{\text{eff}}^{\text{chiral}} = 0 \ . \tag{40}$$

V. GAUGE INVARIANCE IN TERMS OF CURRENTS

Having shown the cancellation of the gauge variation of the one-loop effective action, we can try to look at the same phenomena in terms of the currents. If we define the current as $J^\mu = \delta \mathcal{S}_{\text{eff}}/\delta A_\mu$, we see from (30), after some simplification, that the current also can be thought of as consisting of two components, Chern-Simons and chiral parts:

$$J = J^{\text{CS}} + J^{\text{chiral}} \ , \tag{41}$$

where we get

$$J_\mu^{\text{CS}} = -\frac{i}{4\pi}\text{sgn}(s)\epsilon_{\mu\rho\nu}\partial^\rho A^\nu + \frac{i}{4\pi}\delta_{\mu a}\epsilon_{ab} A^b \delta(s) \tag{42}$$

and

$$\partial^\mu J_\mu^{\text{chiral}} = \frac{i}{4\pi}\epsilon_{ab}\partial^a A^b \delta(s) \ , \tag{43}$$

which is the *consistent* anomaly in two dimensions as discussed in [5] and [4]. In the second term in (42) and (43), the values of a and b go over only 0,1. Now consider the second term in (42). This term is nonzero only on the wall, which suggests that it be considered with J^{chiral}, which is nonzero only on the wall. Then we obtain the following splitting of the currents:

$$J = J^{\text{GW}} + J^{\text{cov}} \ , \tag{44}$$

where

$$J_\mu^{\text{GW}} = -\frac{i}{4\pi}\text{sgn}(s)\epsilon_{\mu\rho\nu}\partial^\rho A^\nu \ ,$$

$$J_\mu^{\text{cov}} = J_\mu^{\text{chiral}} + \frac{i}{4\pi}\delta_{\mu a}\epsilon_{ab} A^b \delta(s) \ . \tag{45}$$

Clearly, J^{cov} gives the covariant anomaly in two dimensions, which is twice the consistent anomaly as it should be. As seen in (45), in addition to the usual chiral current, an extra term is needed to make the anomaly covariant. This extra piece comes from the Chern-Simons piece as noted in [4]. J^{GW}, on the other hand, can be calculated using the methods of Goldstone and Wilczek [2] by considering points far from the wall where the approximation required for such an analysis is valid.

VI. CONCLUSION

We have shown explicitly that the potentially anomalous contributions in the gauge variation of the effective action cancel between the Chern-Simons term and the chiral term. This was suggested in [1] and [4], but here we show that their analysis can be made exact even close to the wall. We find the Chern-Simons term with the space-dependent coefficient as suggested before valid exactly even close to the wall. Further, we find that the effect of the massive modes is not only to produce the Chern-Simons term, but also to produce a chiral term in the propagator which plays a critical role in the cancellation of the gauge variation. A closer look indicates that the effect of the massive modes is to make the ultraviolet behavior softer for the chiral modes, acting as a regulator. Further investigations of additional effects of massive modes on the wall might be interesting in the context of [6], where the effects of the massive modes on the two-dimensional domain wall is crucial. Also, the claim that the anomaly on the two-dimensional wall is a covariant anomaly can be better understood from this explicit calculation. The expectation of Naculich [4] appears to have been borne out by our calculation.

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