# Covariant path integral for Nambu-Goto string theory

Luiz C. L. Botelho

*Department of Physics, Universidade Federal do Pará, Campus Universitário do Guamá, 66.075-900, Belém-Pará-Brazil*

(Received 2 July 1993)

We clarify the covariant Nambu-Goto string path integral proposed on phenomenological grounds by Polchinski and Strominger.

PACS number(s): 11.25.Sq

The attempt to understand collective phenomena in field theories has become the central problem of quantum field theory [1,2]. One of the most promising frameworks to solve this fundamental problem in quantum field theories is to write the associated field theory path integral in loop space and, thus, search for string solutions for the loop space field equations of motion [3]. This effort, in turn, has recently led to intensive research into the problem of the correct meaning for the string path integral. Most of these studies were based on Polyakov's analysis of the conformal anomaly of two-dimensional massless fields interacting with induced DeWitt quantum gravity in two dimensions [1].

Unfortunately the Polyakov proposal of DeWitt two-dimensional quantum gravity as the correct meaning for the string path integral may be considered only as an effective action study for the full Nambu-Goto area functional, since it involves the full use of a mean field approximation [1].

It is the purpose of this report to clarify and justify, in the framework of covariant path integrals, the alternative Nambu-Goto string path integral recently proposed on phenomenological grounds by Polchinski and Strominger [4].

Let us start our analysis by considering the original Polyakov path integral for the Nambu-Goto string propagator in a form useful for non-Abelian gauge theories (Eq. (9.76) of Ref. [1]) and Ref. [3]:

$$G[C]=\sum_{\{g_{ab}\}}\sum_{\{X_\mu\}}\exp\left[-\frac{1}{2\pi x'}\int_D(\sqrt{g}\,)(\xi)d^2\xi\right]\times\delta^{(F)}_{\rm cov}[g_{ab}(\xi)-h_{ab}(X^\mu(\xi))]\ . \quad (1)$$

The continuous sum over the string world sheet vector position $X_\mu(\xi)$ and the intrinsic two-dimensional (2D) metric $g_{ab}(\xi)$ in Eq. (1) are defined by DeWitt functional metrics on hemispherical manifolds possessing as nontrivial boundaries the string configuration $\{C\}$:

$$\|\delta X^\mu\|^2=\int_D(\sqrt{g}\ \delta X^\mu\delta X^\mu)(\xi)d^2\xi\ , \quad (2a)$$

$$\|\delta g_{ab}\|=\int_D[\sqrt{g}\ \delta g_{ab}(g^{aa'}g^{bb'})\delta g_{a'b'}](\xi)d^2\xi\ . \quad (2b)$$

The $\delta$ functional inside Eq. (1) restricts the nonphysical variable (intrinsic metric) $g_{ab}(\xi)$ to be the world sheet induced metric

$$h_{ab}(X^\mu(\xi))=(\partial_a X^\mu)(\partial_b X^\mu)(\xi)\ .$$

Let us briefly recall Polyakov's covariant analysis. In his explicitly covariant scheme one writes the delta functional by means of a covariant Fourier path integral:

$$G[C]=\sum_{\{g_{ab}\}}\exp\left[-\frac{1}{2\pi x'}\int_D(\sqrt{g}\,)(\xi)d^2\xi\right]\left[\sum_{\{X^\mu\}}\sum_{\{\lambda_{ab}\}}\exp\left[i\int_D d^2\xi[\sqrt{g}\,\lambda_{ab}(\partial^a X^\mu\partial^b X^\mu-g^{ab})](\xi)\right]\right]\ . \quad (3)$$

By making the hypothesis of the exact validity of the covariant mean field average for the Lagrange multiplier (see Eq. (9.88a) of Ref. [1]),

$$\lambda_{ab}(\xi)=i\langle\lambda\rangle g_{ab}(\xi)\ , \quad (4)$$

one obtains Polyakov's result of 2D massless scalar fields interacting with DeWitt two-dimensional quantum gravity as a definition for the string path integral Eq. (1) after substituting Eq. (4) into Eq. (3) and defining an effective cosmological constant:

$$\mu_0=1/2\pi x'+\langle\lambda\rangle\ .$$

Unfortunately, in string theory the conditions for the full validity of Eq. (4) on the string energy phase space is still an open question. This, in turn, makes Polyakov's approach [1] a path integral effective theory for string quantization.

We, thus, make a departure from the above Polyakov approximate analysis and try to consider exactly the original expression Eq. (1) with the $\delta$ function without making any mean field approximation of the sort of Eq. (4).

The invariant measure associated with the DeWitt supermetric Eq. (2b) on the functional space of the fields $g_{ab}(\xi)$ in the path integral formalism was shown in Ref. [5] to be correctly defined by the DeWitt measure

$$\sum_{\{g_{ab}\}} = \int \prod_{(\xi,a,b)} \{[dg_{ab}(\xi)](\sqrt{g(\xi)})^{-6/4}\}\delta^{(F)}[M(g_{ab})] \left| \left| \prod_{(\xi,c)} d\epsilon_c(\xi)\sqrt{\det(\sqrt{g}g^{ab})} \right| \det[\delta M(g^{\epsilon}_{ab})/\delta\epsilon] \right| , \tag{5}$$

where $M(g_{ab})$ is a gauge-fixing functional and $[\epsilon_c(\xi)]$ denotes the infinitesimal vector field generators of a general coordinate transformation in $\mathbb{D}$. The powers of $\sqrt{g(\xi)}$ in the above written equation come from the root square of the DeWitt super metric determinant in the invariant measure Eq. (20) of Ref. [5] for $R^2$.

We point out that direct use of Eq. (5) for calculations is very subtle since it contains the usual Feynman product measure on the variables $dg_{ab}(\xi)$ and $d\epsilon_c(\xi)$ weighted with factors of the form $(\sqrt{g(\xi)})^m$ which, in turn, lead to the use of a new field reparametrization in the path integral in order to reduce the functional measure to the usual Feynman measure. For instance, if one wants to evaluate formally a path integral of the form

$$I = \sum_{\{g_{ab}\}} \exp\left[ -\int_{\mathbb{D}} d^2\xi\, L(g_{ab})(\xi) \right] , \tag{6}$$

where $L(g_{ab})$ denotes an invariant coordinate transformation action functional for the $g_{ab}(\xi)$ field, we must consider first the variable change

$$\frac{\partial\varphi_{ab}(\xi)}{\partial\xi_l} = \left[ \frac{\partial}{\partial\xi_l} g_{ab}(\xi) \right] (\sqrt{g(\xi)})^{-3/2} , \tag{7}$$

which will reduce the weighted measure Eq. (5) to the usual Feynman product measure

$$I = \int \left[ \prod_{(\xi,a,b)} d\varphi_{ab}(\xi) \right] \exp\left[ -\int_{\mathbb{D}} d^2\xi\, \tilde{L}(\varphi_{ab})(\xi) \right] , \tag{8}$$

where $\tilde{L}(\varphi_{ab})$ is the new expression of the action in terms of the new variable Eq. (7) added with the Faddeev-Popov ghost action. It is worth remarking that in the functional integral form Eq. (8), practical calculations are very cumbersome and not explicitly covariant under the action of the (diffeomorphism) group.

Fortunately, in two dimensions it is possible to obtain a closed expression for Eq. (6) in the conformal gauge $g_{ab}(\xi) = e^{\varphi(\xi)}\delta_{ab}$ as has been shown by Polyakov by directly using the DeWitt super metric Eq. (2b) to rewrite the covariant measure Eq. (5) in terms of the conformal factor [1,9]:

$$\begin{aligned} \sum_{\{g_{ab}=e^{\varphi(\xi)}\delta_{ab}\}} &= \int \prod_{\xi} [d(e^{\varphi(\xi)}\delta_{11}) d(e^{\varphi(\xi)}\delta_{22}) e^{-3\varphi(\xi)/2}] \exp\left[ -\frac{26}{48\pi} \int_{\mathbb{D}} d^2\xi \left[ \frac{1}{2}(\partial\varphi)^2 + \mu^2 e^{\varphi} \right](\xi) \right] \\ &= \prod_{\xi} d(e^{\varphi(\xi)/2}) \exp\left[ -\frac{26}{48\pi} \int_{\mathbb{D}} d^2\xi \left[ \frac{1}{2}(\partial\varphi)^2 + \mu^2 e^{\varphi} \right](\xi) \right] . \end{aligned} \tag{9a}$$

By making the choice $e^{\varphi(\xi)/2} = \gamma(\xi)$ as the correct dynamical degree of freedom, we get the final expression for the $g_{ab}$-invariant measure to be used in our study:

$$\sum_{g_{ab}} = \int \prod_{\xi} d[\gamma(\xi)] \exp\left[ -\frac{26}{48\pi} \int_{\mathbb{D}} d^2\xi \frac{1}{2} \left[ \frac{\partial_a(\gamma^2)}{\gamma^2} \right]^2 \right] \exp\left[ \lim_{\delta\to 0^+} \frac{1}{4\pi\delta} \int_{\mathbb{D}} d^2\xi \gamma^2(\xi) \right] . \tag{9b}$$

Next, we consider the $X_\mu(\xi)$ functional integral [6]. In order to reduce the covariant path integral over the world sheet string vector position to a Feynman functional measure as in Eq. (9b) we first consider the following covariant Gaussian functional integral which may be used to define the covariant sum in Eq. (1) [see Eq. (2a)]:

$$\hat{I}[g_{ab}] = \int \prod_{(\xi,\mu)} [dX^\mu(\xi)\, {}^4\sqrt{g(\xi)}] \exp\left[ -\frac{1}{2} \int_{\mathbb{D}} d^2\xi [\sqrt{g} X^\mu(-\Delta_g) X^\mu](\xi) \right] , \tag{10}$$

where $\Delta_g$ is the Laplace Beltrami operator associated with the metric $g_{ab}(\xi)$. Now we note that Eq. (10) is a Gaussian path integral:

$$\hat{I}[g_{ab}] = \det{}^{-D/2}(-\Delta_g) . \tag{11}$$

It is possible to write the above functional determinant as a local field action for the conformal factor $\gamma(\xi)$ [1]: namely,

$$\hat{I}[g_{ab} = \gamma^2\delta_{ab}] = \exp\left[ \frac{D}{48\pi} \int_{\mathbb{D}} d^2\xi \frac{l}{2} \left[ \frac{\partial_a(\gamma^2)}{\gamma^2} \right]^2 (\xi) \right] \exp\left[ + \lim_{\delta\to 0^+} \frac{D}{\delta} \int_{\mathbb{D}} d^2\xi\, \gamma^2(\xi) \right] . \tag{12}$$

Let us now consider a metric conformal scaling in Eq. (10) [6]:

$$g_{ab}(\xi) = e^{\lambda(\xi)} \hat{g}_{ab}(\xi) . \tag{13}$$

We, thus, write Eq. (10) as well as

$$\hat{I}[g_{ab}]=\int \prod_{(\xi,\mu)} [dX^\mu(\xi)\,[\hat{g}(\xi)]^{1/4}e^{\lambda(\xi)/2}]\exp\left[-\frac{1}{2}\int_{\mathbb{D}} d^2\xi[\sqrt{\hat{g}}X^\mu(-\Delta_g)X^\mu](\xi)\right] . \tag{14}$$

We remark that the classical action of massless scalar fields on a compact manifold without boundary (the domain $\mathbb{D}$) is conformally scale invariant, so it does not depend on the conformal factor. The effects of the conformal scaling are nontrivial only at the quantum level or, equivalently, at the level of the functional measures as may be seen from Eq. (14).

Now we note that change on the functional measure Eq. (13) is taken into account entirely by a Jacobian $J[\lambda(\xi)]$ which is a functional of the conformal scale factor (the well-known Fujikawa conformal anomaly factor [6,9]):

$$\left[\prod_{(\xi,\mu)} dX^\mu(\xi)e^{\lambda(\xi)/2}\,[\hat{g}(\xi)]^{1/4}\right] =J[\lambda(\xi)]\left[\prod_{(\xi,\mu)} dX^\mu(\xi)\,[\hat{g}(\xi)]^{1/4}\right] . \tag{15}$$

After substituting Eq. (15) into Eq. (14) and evaluating the resulting Gaussian covariant functional integral, we get the explicit expression for the above-mentioned Jacobian:

$$J[\lambda(\xi)]=\det{}^{-D/2}(-\Delta_{e^{\lambda}\hat{g}})/\det{}^{-D/2}(-\Delta_{\hat{g}}) . \tag{16}$$

Let us make use of Eqs. (14)–(16) for $\hat{g}_{ab}=\delta_{ab}$ and $\lambda(\xi)=2\ln\gamma(\xi)$, since we can always consider the conformal gauge in Eq. (10)

$$g_{ab}(\xi)=\gamma^2(\xi)\delta_{ab} .$$

As a result we obtain the following relation between the covariant measure and the Feynman product measure parametrization:

$$\left[\prod_{(\xi,\mu)} [dX^\mu(\xi)\gamma(\xi)]\right]=\exp\left\{\frac{D}{48\pi}\int_{\mathbb{D}} d^2\xi\left[\frac{1}{2}\left[\frac{\partial_a(\gamma^2)}{\gamma^2}\right]^2+\mu^2\gamma^2\right](\xi)\right\}\prod_{(\xi,\mu)}[dX^\mu(\xi)] . \tag{17}$$

At this point, we return to the original Eq. (3) and rewrite it in the conformal gauge by using the Feynman functional measure parametrization Eqs. (9) and (17):

$$\begin{aligned}G[C]=\int \left[\prod_{\xi} d\gamma(\xi)\right]\left[\prod_{(\xi,\mu)}[dX^\mu(\xi)]\right]\exp\left\{-\frac{26-D}{48\pi}\int_{\mathbb{D}} d^2\xi\left[\frac{1}{2}\left[\frac{\partial_a\gamma^2}{\gamma^2}\right]^2+\mu^2\gamma^2\right](\xi)\right\}\\ \times\exp\left[-\frac{1}{2\pi x'}\int_{\mathbb{D}} d^2\xi(\partial_a X^\mu)^2(\xi)\right]\delta^{(F)}_{\text{cov}}[\gamma^2(\xi)\delta_{ab}-h_{ab}(X^\mu(\xi))] .\end{aligned} \tag{18}$$

It is instructive to remark that we must rewrite the covariant delta functional inside Eq. (18) in a Feynman parametrization form. In order to implement this step of our study we consider the covariant Fourier path integral representation written directly in the conformal gauge $g_{ab}(\xi)=\gamma^2(\xi)\delta_{ab}$ [see Eq. (3)]:

$$\begin{aligned}\delta^{(F)}_{\text{cov}}[\gamma^2(\xi)\delta_{ab}-h_{ab}(X^\mu(\xi))]=\int \left[\prod_{\xi}[d\lambda_{11}(\xi)\gamma^{-1}(\xi)]\right]\left[\prod_{\xi}[d\lambda_{22}(\xi)\gamma^{-1}(\xi)]\right]\\ \times\exp\left\{i\int_{\mathbb{D}}\frac{\lambda_{11}(\xi)}{\gamma(\xi)}\frac{[\partial^1X^\mu\partial_1X^\mu)(\xi)-\gamma^2(\xi)]}{\gamma(\xi)}\right\}\\ \times\exp\left\{i\int_{\mathbb{D}}\frac{\lambda_{22}(\xi)}{\gamma(\xi)}\frac{[\partial^2X^\mu\partial^2X^\mu)(\xi)-\gamma^2(\xi)]}{\gamma(\xi)}\right\} .\end{aligned} \tag{19}$$

The covariant functional measure $\sum_{\{\lambda_{ab}\}}$ for the Fourier tensor field variable $\lambda_{ab}(\xi)$ in the conformal gauge $g_{ab}(\xi)=\gamma^2(\xi)\delta_{ab}$ used in Eq. (9) is still defined by us with the DeWitt covariant measure Eq. (2b) for two-dimensional tensors $\lambda_{ab}(\xi)$:

$$\|\delta\lambda_{ab}\|^2=\int_{\mathbb{D}} d^2\xi\left[\gamma^2(\xi)(\delta\lambda_{ab})(\xi)\frac{\delta^{aa'}}{\gamma^2(\xi)}\frac{\delta^{bb'}}{\gamma^2(\xi)}(\delta\lambda_{a'b'})(\xi)\right] . \tag{20}$$

Following the discussion after Eq. (6) about the correct meaning of a covariant path integral, we note that by making the variable change

$$\tilde{\lambda}_{11}(\xi)=\lambda_{11}(\xi)/\gamma(\xi)\ ,\quad \tilde{\lambda}_{22}(\xi)=\lambda_{22}(\xi)/\gamma(\xi)\ , \tag{21}$$

the covariant delta functional Eq. (19) in the conformal gauge has the same form of the delta functional defined from the usual Feynman product measure definition:

$$\begin{aligned}\delta^{(F)}_{\mathrm{cov}}[\gamma^2(\xi)\delta_{ab}-h_{ab}(X^\mu(\xi))]&=\int\left[\prod_{\xi}d\tilde{\lambda}_{11}(\xi)\right]\left[\prod_{\xi}d\tilde{\lambda}_{22}(\xi)\right]\exp\left[i\int_{\mathrm{D}}d^2\xi\left[\tilde{\lambda}_{11}\frac{(\partial^1X^\mu\partial^1X^\mu-\gamma^2)}{\gamma}\right](\xi)\right]\\&\quad\times\exp\left[i\int_{\mathrm{D}}d^2\xi\left[\tilde{\lambda}_{22}\frac{(\partial^2X^\mu\partial^2X^\mu-\gamma^2)}{\gamma}\right](\xi)\right]\\&=\delta^{(F)}[\gamma^2(\xi)\delta_{ab}-h_{ab}(X^\mu(\xi))]\ .\end{aligned} \tag{22}$$

Next, we can evaluate exactly the root-square conformal factor $\gamma(\xi)$ auxiliary functional integral due to the usual delta functional Eq. (21) which produces the result [4]

$$\begin{aligned}G[C]=\int\left[\prod_{(\xi,\mu)}dX^\mu(\xi)\right]&\exp\left[-\frac{1}{2\pi x'}\int_{\mathrm{D}}d\xi^+d\xi^-[(\partial_+X^\mu)(\partial_-X^\mu)](\xi^+,\xi^-)\right]\\&\times\exp\left[-\frac{26-D}{48\pi}\int_{\mathrm{D}}d\xi^+d\xi^-\left[\frac{(\partial_+^2X^\mu)(\partial_-X^\mu)(\partial_-^2X^\mu)(\partial_+X^\mu)}{[(\partial_+X^\mu)(\partial_-X^\mu)]^2}\right](\xi^+,\xi^-)\right]\ .\end{aligned} \tag{23}$$

Note that the use of the conformal gauge in Eq. (1) implicitly constrains the use of the orthonormal coordinates for the string world sheet vector position (Ref. [3], Appendix C):

$$(\partial_+X^\mu)(\partial_+X^\mu)=(\partial_-X^\mu)(\partial_-X^\mu)\equiv 0\ ,\quad (\partial_+X^\mu)^2=(\partial_-X^\mu)^2\ . \tag{24}$$

Equation (23) is, thus, the exact path integral meaning to the sum over surfaces Eq. (1) in the string world sheet orthonormal gauge as originally conjectured in Ref. [4].

At this point of our paper we remark that scalar scattering amplitudes as random surfaces which intercept point probabilities at the critical dimension $D=26$ [1] are given exactly by the usual nontachyonic dilaton scattering amplitudes which solve the problem of tachyonic excitation on this string theory.

If we now consider a further term, taking into account the surface rigidity extrinsic functional in Eq. (1), namely,

$$\exp\left[-\frac{k}{2}\int_{\mathrm{D}}d^2\xi[\sqrt{g}\,(-\Delta_gX^\mu)^2](\xi)\right]\ , \tag{25}$$

we obtain straightforwardly a well-defined path integral quantization of the extrinsic string on the conformal gauge, a result which was used in Ref. [7] without proof:

$$\begin{aligned}\bar{G}[C]=\int\left[\prod_{(\xi,\mu)}dX^\mu(\xi)\right]&\exp\left[-\frac{1}{2\pi x'}\int_{\mathrm{D}}d\xi^+d\xi^-[(\partial_+X^\mu)(\partial_-X^\mu)](\xi^+,\xi^-)\right]\\&\times\exp\left\{-k\int_{\mathrm{D}}d\xi^+d\xi^-\left[(\partial_+\partial_-X^\mu)(\partial_+\partial_-X^\mu)\frac{1}{(\partial_+X^\mu)(\partial_-X^\mu)}\right](\xi^+,\xi^-)\right\}\\&\times\exp\left[-\frac{26-D}{48\pi}\int_{\mathrm{D}}d\xi^+d\xi^-\left[\frac{(\partial_+^2X^\mu)(\partial^-X^\mu)(\partial_-^2X^\beta)(\partial_+X^\beta)}{(\partial_+X^\mu\partial_-X^\mu)^2}\right](\xi^+,\xi^-)\right]\ .\end{aligned} \tag{26}$$

Let us recall that it is a subtle problem if the Liouville terms Eqs. (23) and (26) do not disturb the ultraviolet theory renormalizability. In addition, by considering complex fermionic degrees of freedom belonging to the fundamental representation of an intrinsic group such as SU(22) we can cancel this nonpolynomial Liouville piece of the action [3].

Finally we call attention to the fact that if we had followed Polyakov [1] by using the complete conformal factor $\rho(\xi)=e^{\varphi(\xi)}$ instead of its square root $e^{\varphi(\xi)/2}$ as the scalar dynamical degree of freedom to be quantized in the $g_{ab}$-functional integral,

$$\sum_{\{g_{ab}\}}=\int\prod_{\xi}[d\rho(\xi)]\exp\left\{-\frac{26}{48\pi}\int_{\mathrm{D}}d^2\xi\left[\frac{1}{2}\left[\frac{\partial_a\rho}{\rho}\right]^2\right](\xi)\right\}\exp\left[\lim_{\delta\to 0^+}\frac{1}{4\pi\delta}\int_{\mathrm{D}}\rho(\xi)d^2\xi\right]\ , \tag{27}$$

we would have obtained the following delta functional for Eq. (22):

$$\delta^{(F)}_{\rm cov}[\rho(\xi)\delta_{ab}-h_{ab}(X^{\mu}(\xi))]$$

$$=\delta^{(F)}\left[\frac{\partial^1X^{\mu}\partial^1X^{\mu}-\rho}{\sqrt{\rho}}\right]\delta^{(F)}\left[\frac{\partial^2X^{\mu}\partial^2X^{\mu}-\rho}{\sqrt{\rho}}\right]$$

$$=\sqrt{(\partial^1X^{\mu}\partial^1X^{\mu})(\partial^2X^{\mu}\partial^2X^{\mu})}\,\delta^{(F)}(\partial^1X^{\mu}\partial^1X^{\mu}-\rho)$$

$$\times\delta^{(F)}(\partial^2X^{\mu}\partial^2X^{\mu}-\rho) \qquad (28)$$

as a simple result of the usual identity

$$\delta[(a-y)/\sqrt{y}\,]=\sqrt{a}\,\delta(y-a)$$

used in its functional integral version.

The result implied by Eq. (28) will lead us to consider a further weight of the form $\sqrt{h(X^{\mu}(\xi))}$ on the Feynman differentials $dX^{\mu}(\xi)$ in our final Eqs. (23) and (26) for a sum over surfaces in the orthonormal coordinates [see Eq. (24)]; and it is worth pointing out that a similar weighted path integral result was put forward some decades ago in Ref. [8].

I am thankful to the Alexander V. Humboldt Stiftung for partial financial support to complete this research in Berlin Freie Universität.

[1] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic, Chur, Switzerland, 1987).
[2] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. D **44**, 1740 (1991); A. L. Kholodenko and C. Quian, Phys. Rev. B **40**, 2477 (1989); H. Kleinert, Phys. Lett. A **114A**, 263 (1986).
[3] Luiz C. L. Botelho, J. Math. Phys. **30**, 2160 (1989).
[4] J. Polchinski and A. Strominger, Phys. Rev. Lett. **67**, 1681 (1991).
[5] Luiz C. L. Botelho, Phys. Rev. D **38**, 2464 (1988).
[6] Luiz C. L. Botelho, Phys. Rev. D **40**, 660 (1989).
[7] Z. Yang, Phys. Lett. B **279**, 47 (1992).
[8] J. L. Gervais and B. Sakita, Phys. Rev. Lett. **30**, 716 (1973).
[9] K. Fujikawa, Phys. Rev. D **25**, 2584 (1982).