PHYSICAL REVIEW D VOLUME 49, NUMBER 4 15 FEBRUARY 1994

Formation of topological defects in first order phase transitions

Mark Hindmarsh* and Anne-Christine Davis†
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

Robert Brandenberger
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 12 July 1993)

We analyze the evolution of scalar and gauge fields during first-order phase transitions and show how the Kibble mechanism for the formation of topological defects emerges from the underlying dynamics, paying particular attention to problems posed by gauge invariance when a local symmetry is spontaneously broken. We discuss also the application of the mechanism to semilocal defects and electroweak strings.

PACS number(s): 11.27.+d, 11.15.Ex, 98.80.Cq

I. INTRODUCTION

Topological defects [1] arising in theories with a spontaneously broken global or local symmetry play an important role in various branches of physics from cosmology [2,3] to condensed-matter physics [4–7]. In cosmology, topological defects are expected to form during phase transitions in the early Universe. It is important to know the number density of defects at the time of formation, as well as other statistics such as correlation functions. Depending on the initial number density, many physics models admitting local monopoles or domain walls are ruled out [8]. For defect-induced baryogenesis [9] it is important to know the initial separation of defects, and for structure formation scenarios using topological defects, such as cosmic strings [10], global monopoles [11], or global textures [12], it is also useful to understand the initial distribution of defects.

In 1976, Kibble [1] suggested a simple mechanism for the formation of defects in cosmological phase transitions, where the order parameter is a scalar quantum field ϕ (which maybe elementary or composite). It has two key ingredients: randomness of the phases of the scalar field ϕ on length scales larger than some correlation length ξ, and the geodesic rule for interpolating the values of ϕ on curves connecting points in different correlation volumes. The first assumption states that ϕ takes on random values in its vacuum manifold $\mathcal{M}$ at points separated by a distance greater than ξ; the second one states that along a curve in space connecting two such points the field ϕ traces out a geodesic path on the vacuum manifold.

For global defects, the Kibble mechanism has been verified in liquid crystal experiments [13] and in numerical simulations in field theory [14]. For local defects, convincing experiments are lacking (see, however, Refs. [15] for some numerical results for local vortex formation).

Recently, Rudaz and Srivastava [16] have argued that in local field theories, the geodesic rule is not justified and that the rate of defect formation might be much smaller than what would be obtained by naively applying the Kibble mechanism. The core of the objection is that the Kibble mechanism is formulated in a gauge-dependent way, and that a geodesic curve in the vacuum manifold of ϕ does not necessarily minimize the energy density and hence should not play a distinguished role. These objections highlight the need to reconsider the Kibble mechanism for defect formation in theories with local symmetry breaking.

In this paper we reconsider the Kibble mechanism for global and local defect formation at first-order phase transitions. We analyze the equations of motion for scalar and gauge fields, and demonstrate the validity of the geodesic rule for ϕ. To be specific, we consider vortex (in $2+1$ dimensions) or cosmic string (in $3+1$ dimensions) formation in a model in which a U(1) global or local symmetry is broken. However, our methods also work for more complicated models with $\pi_1(\mathcal{M})\neq 1$, for other types of topological defects, and for semilocal [17] and electroweak [18] strings. We do not consider here second-order transitions because the analysis is qualitatively different: the semiclassical methods we use ignore thermal fluctuations. We plan to discuss their role in a future publication.

In Sec. II we discuss theories with a first-order phase transition which proceeds via bubble nucleation. We discuss how far the analysis can be applied to semilocal defects and electroweak strings in Sec. III, and Sec. IV contains a brief summary of results. Throughout the paper, units in which $k_B=\hbar=c=1$ are employed. Greek indices run from 0 to 3 and our space-time signature is $(+,-,-,-)$.

*Present address: School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, United Kingdom.

†Also at King's College, Cambridge, England.

II. FIRST-ORDER PHASE TRANSITIONS

To be specific, we shall consider an Abelian Higgs model with a complex scalar field ϕ and a U(1) gauge connection A_μ. The Lagrangian of the system is

$$\mathcal{L}=(D_\mu\phi)^\dagger D^\mu\phi-V(\phi)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} , \tag{2.1}$$

where

$$D_\mu=\partial_\mu-ieA_\mu \tag{2.2}$$

is the covariant derivative (e is the gauge coupling constant),

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{2.3}$$

is the field strength, and $V(\phi)$ is a symmetry-breaking potential, whose zero-temperature form is $\lambda(|\phi|^2-\eta^2)^2/2$. We assume that $V(\phi)$ depends only on $|\phi|$. In this case, the Lagrangian (2.1) has a local U(1) symmetry. If $|\phi|=\eta(T)\neq 0$ is the absolute minimum of $V(\phi)$, then this symmetry is broken at low temperatures. We shall assume that $\phi=0$ is the minimum of the high-temperature effective potential $V_T(\phi)$. At a critical temperature T_c, $\phi=0$ ceases to be the absolute minimum of $V_T(\phi)$.

Considering the Abelian Higgs model in the context of an expanding Universe, we conclude that as the temperature drops below T_c, the system undergoes a symmetry-breaking phase transition. The order of the transition depends on the functional form of $V_T(\phi)$. It is generally thought that the transition is first order for $\beta\equiv\lambda/e^2\to 0$, and may be second order for $\beta\to\infty$ [19]. In the former case, the one-loop temperature-dependent effective potential has the form [20]

$$V_T(\phi)=\tfrac{1}{2}\lambda(T)(|\phi|^2-\eta^2)^2+\tfrac{1}{4}e^2T^2|\phi|^2-\frac{\sqrt{2}}{4\pi}e^3T|\phi|^2 . \tag{2.4}$$

In a first-order phase transition, $\phi=0$ remains a metastable fixed point below $T=T_c$. The transition to a state with $|\phi|=\eta$ proceeds by bubble nucleation [21]. There is a finite probability per unit volume per unit time $dP/dVdt$ that a bubble with $|\phi|=\eta$ will nucleate in a surrounding sea of "false vacuum" $\phi=0$. This probability is given by an expression of the form [21,22]

$$\frac{dP}{dVdt}=A\exp(-S_E[\bar{\phi},\bar{A}]) , \tag{2.5}$$

where $\bar{\phi}$ and $\bar{A}$ are extrema of the Euclidean action S_E. This field configuration represents a tunneling process. At high temperatures (high compared with the scalar and vector masses m_S and m_V) the action is effectively three dimensional, and the tunneling solution is spherically symmetric about a point [22]. If the energy difference between the minima at $\phi=0$ and $|\phi|=\eta(T)$ is small compared with the height of the barrier between them, then the solution is well approximated by a thin-walled spherical bubble of true vacuum inside the false one [21]. The width of the bubble wall is approximately m_S^{-1}, and the radius of the bubble is of order $V_b/m_S\Delta V$, where V_b is the barrier height and ΔV is the difference is free energy between the two phases. The bubble will then expand with a speed which depends on the interaction between the wall and the rest of the hot matter in the Universe. When the mean free path of the hot matter is much shorter than the wall thickness, the interactions are ineffective in slowing down the wall, and it reaches the speed of light [23]. The precise value of the wall velocity is not important to our analysis. The expansion of the bubble is fueled by the conversion of potential energy density $V(0)$ to wall kinetic and gradient energy. The phase transition is completed when neighboring bubbles collide and the fraction of space with $|\phi|=\eta$ approaches unity. The correlation length ξ for this transition is the mean separation of bubbles. Implicit in our analysis are some assumptions about time scales: we are taking the expansion rate to be much longer than the nucleation rate per Hubble volume, so that we are justified in taking the space-time to be Minkowski type. The expansion of the universe then serves to reduce the temperature as a known function of time.

According to the Kibble mechanism, there is a fixed probability p of the order 1 that a defect will form in any correlation volume ξ^3. The exact value of p depends on the type of defect, i.e., on the topology of the vacuum manifold $\mathcal{M}$ (see, e.g., Ref. [24] for a recent calculation of p for various models).

Let us illustrate the Kibble mechanism for our toy model, the Abelian Higgs model. The phase α of ϕ is assumed to take random values in different bubbles. After two bubbles meet, then if we follow a line in space connecting the centers of the two bubbles, α is assumed to interpolate more or less smoothly between its values in the two bubbles. The second part of Kibble's argument states that $\alpha(x)$ will follow a geodesic in $\mathcal{M}$ in order to minimize the potential energy.

A vortex can form when three bubbles collide, as illustrated in Fig. 1. If α_i, $i=1,2,3$, are the phases of ϕ in the three bubbles, then a vortex will form if the sum of the phase differences $\alpha_2-\alpha_1$ and $\alpha_3-\alpha_2$ exceeds π. In this case, $\alpha(x)$ will run from 0 to 2π as we go along the circle γ; i.e., the field configuration has winding number 1. There is then a topological obstruction to the scalar field reaching the vacuum manifold everywhere in the region bounded by the lines connecting the centers of the bubbles. The field must vanish somewhere, and this is where the vortex forms.

As discussed in Ref. [25], a bubble collision is a quite violent event. In order to justify the Kibble mechanism (both for global and local symmetry breaking), we must follow the evolution of amplitude and phase of ϕ and demonstrate that the geodesic rule is valid. In order to do this, we will write down the dynamical equations which follow from (2.1). We first consider a global theory and discuss the evolution of amplitude and phase of ϕ. Then, we make the transition to a local theory and show that the gauge fields do not strongly perturb the evolution of ϕ. This analysis is done in a particular gauge. However, the final result, the winding number, is gauge invariant.

In the Lorentz gauge, i.e., with the choice

$$\partial_\mu A^\mu=0 , \tag{2.6}$$

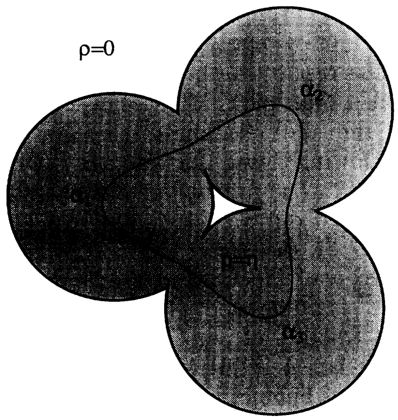

FIG. 1. Three bubbles of the broken symmetry phase ($\rho=\eta$) colliding. If the phase change of the scalar field around the loop γ is $\pm 2\pi$, a string (or antistring) is formed. If the phases α_i are ordered, then the requirement for a string is $\alpha_1+\pi<\alpha_3<\alpha_2+\pi$.

the variational equations which follow from (2.1) are

$$(\partial_\mu \partial^\mu - 2ieA_\mu \partial^\mu - e^2 A_\mu A^\mu)\phi + 2\frac{\partial V}{\partial |\phi|^2}\phi = 0 \qquad (2.7)$$

and

$$\partial_\mu \partial^\mu A_\nu - 2e^2 A_\nu |\phi|^2 = -ie\phi^* \overleftrightarrow{\partial}_\nu \phi \ . \qquad (2.8)$$

It is convenient to separate Eq. (2.7) into equations for the amplitude ρ and phase α of ϕ. Inserting

$$\phi = \rho e^{i\alpha} \qquad (2.9)$$

into (5) we obtain

$$\partial^2 \rho - (\partial \alpha - eA)^2 \rho - e^2 A^2 \rho + 2\frac{\partial V}{\partial \rho^2}\rho = 0 \qquad (2.10)$$

and

$$\partial^2 \alpha + 2(\partial^\mu \alpha - eA^\mu)\partial_\mu \rho \frac{1}{\rho} = 0 \ . \qquad (2.11)$$

The collision of two bubbles in the Abelian Higgs model was studied numerically in Ref. [25]. We now demonstrate that we can reproduce the essential features of the collision process using the above equations.

We first consider a theory with global U(1) symmetry and choose axes such that the centers of the bubbles lie on the x axis. We can additionally boost in the (y,z) plane to a frame in which the bubbles nucleate simultaneously [25], and translate in the x direction so that they collide at $x=0$. Provided the bubbles nucleate far from each other (far meaning much greater than the wall width so that the field is exponentially close to zero between the bubbles) the solution representing two bubbles nucleating at $\mathbf{x}_1$ and $\mathbf{x}_2$ can be approximated by a sum ansatz:

$$\bar{\phi}(\mathbf{x},0) = e^{i\alpha_1} f(\mathbf{x}-\mathbf{x}_1) + e^{i\alpha_2} f(\mathbf{x}-\mathbf{x}_2) \ , \qquad (2.12)$$

where f is the modulus of the single-bubble field. The action for this configuration is (exponentially) independent of α_1 and α_2: hence the phase of the two fields are well approximated by independent random variables. Without loss of generality we take the phase in one bubble to be $\alpha=0$ and in the other bubble $\alpha=\alpha_c$. Note that the phase α is not defined for points outside the bubbles.

When the bubbles meet (we take this to occur at time $t=0$), the phase $\alpha(x)$ is approximately a step function:

$$\alpha(x,t=0) = \alpha_c \theta(x) \ . \qquad (2.13)$$

In fact, the step will be smoothed on a scale m_S^{-1} by the finite thickness of the bubble walls. We shall work in the planar approximation $\partial_y \phi = \partial_z \phi = 0$. This is reasonable if the bubble radius is much greater than the wall thickness. In fact, the symmetry of the two-bubble collision reduces the problem to a two-dimensional one anyway [25], but for simplicity we choose not to exploit the coordinates in which this is manifest. Inserting (2.13) as initial condition into the phase equation [Eq. (2.11)], we see that phase wavefronts emerge which travel in $\pm x$ direction with the speed of light (see Fig. 2):

$$\alpha(x,t\geq 0) = \tfrac{1}{2}\alpha_c[\theta(x+t)+\theta(x-t)] \ . \qquad (2.14)$$

Equation (2.14) gives the solution of (2.11) because the phase waves are propagating inside the bubbles where $\partial_\mu \rho = 0$.

The phase waves which arise for $\alpha_c \neq 0$ carry away some of the kinetic energy of the walls, but not all. The rest of the energy goes into bubble walls. The region of false vacuum $\phi=0$ does not disappear at $t=0$. Rather, the walls which separate the region with $|\phi|=\eta$ from the false vacuum scatter and start to reexpand [25]. What is happening is that the modulus of the field has overshot $\rho=\eta$, and rolled back to $\rho=0$. We denote the wall positions by $\pm X(t)$. However, as $X(t)$ increases, the potential energy of the field configuration increases, thus creating a force which tends to restore X to 0. An approximate

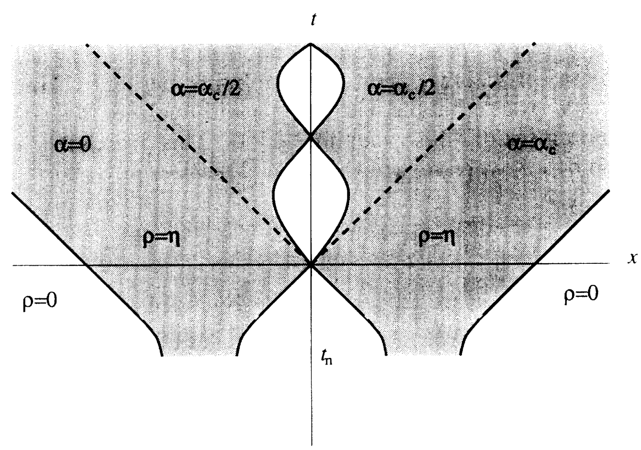

FIG. 2. Space-time diagram of two bubbles with different phases colliding. After nucleation at time t_n the bubble walls collide at $t=0$, when they are traveling at approximately c. Phase waves continue out from the collision site, spreading a region whose phase is halfway between the phases of the bubbles. The walls pass through each other, but eventually turn round and recollide. This may happen several times.

equation for $X(t)$ can be obtained from elementary physics considerations:

$$\sigma \ddot{X} = -\frac{\partial W}{\partial X} , \tag{2.15}$$

where σ is the mass per unit area of the wall, and $W(X)$ is the potential energy (per unit area) of the false vacuum region $x \in [-X,X]$. Obviously, $W = XV(0)$ and hence (2.13) becomes

$$\ddot{X} = -V(0)/\sigma . \tag{2.16}$$

Thus, there is a constant restoring force which causes the wall to recollapse on a time scale $\tau = 2\dot{X}(0)\sigma / V(0)$. There is a series of bounces, each losing energy through propagating oscillations in ρ, and thus leading to reduced $\dot{X}(0)$ and period τ (see Fig. 2). Note that the coupling of the scalar to other fields will result in particle production and consequent energy loss as the scalar field oscillates. In principle, these couplings could be strong enough to overdamp the oscillations, which would merely result in a reduction of the relaxation time.

An alternate way to derive Eq. (2.15) is to insert the ansatz

$$\rho(x,t) = \eta \theta(x - X(t)) \tag{2.17}$$

into the equation for the modulus ρ [Eq. (2.10)], and integrate the resulting distributional equation over x.

To summarize, we have verified that after a bubble collision, the phase $\alpha(x)$ along a path γ connecting the two bubble centers interpolates between its original values in the two bubbles. The interpolation happens in two jumps of width m_S^{-1} associated with excitations of the Nambu-Goldstone mode, spreading from the collision site at the speed of light. Thus, the second key ingredient of the Kibble mechanism, the "geodesic rule," has been established for global defects forming in a first-order phase transition. We stress that the geodesic rule follows from the equations of motion, not from minimizing the energy, as assumed in Ref. [16].

We now extend the analysis to theories with a local U(1) symmetry. We again study the collision of two bubbles and establish the applicability of the geodesic rule. In the bubble nucleation instanton, the magnetic field associated with A_μ vanishes. Thus at the time of nucleation t_n we have, even in the Lorentz gauge, some freedom left in specifying A_μ. In general,

$$\mathbf{A}(\mathbf{x},t_n) = \nabla \Lambda(\mathbf{x},t_n) , \quad A^0 = \dot{\Lambda}(\mathbf{x},t_n) , \tag{2.18}$$

an arbitrary function of position. However, it is clearly simplest to choose $\Lambda(\mathbf{x},t_n) = 0 = \dot{\Lambda}(\mathbf{x},t_n)$, which amounts to a complete specification of the gauge. During the bubble wall collision, a nonvanishing gauge connection is generated through the coupling to the phase difference in the scalar field across the bubble wall, which, in turn, feeds back into the evolution of the scalar field modulus ρ via (2.10). We will analyze first assuming $\lambda/e^2 \gg 1$. We assume that we may neglect eA in comparison to $\partial\alpha \sim \sqrt{\lambda}\eta$, so that Eq. (2.8) becomes

$$(\partial_t^2 - \partial_x^2) A_\nu = 2e|\phi|^2(\partial_\nu \alpha - e A_\nu) \simeq 2e\eta^2 \partial_\nu \alpha . \tag{2.19}$$

If we take for α Eq. (2.14), then we can easily solve for the gauge field. This is most easily expressed in light-cone coordinates $x^\pm = t \pm x$. After one integration we find $(t>0)$

$$\begin{aligned} \partial_- A_+ &= \tfrac{1}{4} e\eta^2 \alpha_c \theta(x^+) , \\ \partial_+ A_- &= \tfrac{1}{4} e\eta^2 \alpha_c \theta(x^-) , \end{aligned} \tag{2.20}$$

which corresponds to an electric field in the x direction of

$$\begin{aligned} F_{01} &= 2(\partial_- A_+ + \partial_+ A_-) \\ &= \tfrac{1}{2} e\eta^2 \alpha_c [\theta(x+t) - \theta(x-t)] . \end{aligned} \tag{2.21}$$

This field exists only between the phase waves, which are carrying off equal and opposite charges (per unit area) $\alpha_c/2$. Behind the phase wave the Higgs vacuum screens the charges, and the gauge field is exponentially damped beyond light-cone distances $\Delta x^\pm \sim (e\eta)^{-1}$. Thus, we conclude that the gauge field is bounded in modulus by $\sim \eta$. Our approximation is self-consistent, for $e\eta \ll \sqrt{\lambda}\eta$. At the phase wave itself the gauge field vanishes, so it should be a good approximation to ignore its effect on the propagation of the wave (this feature was originally found by Hawking *et al.* in their numerical simulations [25]).

This is perhaps not the correct limit to use if we are assuming a first-order phase transition, where λ/e^2 is supposed to be small. In this limit we cannot approximate the phase wave by a step function. Instead we take $(t>0)$

$$\alpha(x) = \tfrac{1}{2}\alpha_c [1 + \tfrac{1}{2} W(m_S x^+) + \tfrac{1}{2} W(m_S x^-)] , \tag{2.22}$$

where W is the phase wave profile, interpolating between -1 and 1 as its argument changes from $-\infty$ to $+\infty$. Then we find that Eq. (2.8) becomes

$$(4\partial_+\partial_- + m_V^2) A_\pm = \pm \tfrac{1}{2}\alpha_c e\eta^2 m_S W'(m_S x^\pm) . \tag{2.23}$$

In a gradient expansion in powers of m_S/m_V, the first term is $A_\pm \simeq \partial_\pm \alpha / e = \pm \alpha_c m_S W'(m_S x^\pm)/4$. Thus, the current vanishes to this order, and the equations of motion for α and ρ are affected only at higher order in m_S/m_V.

Physically, what is happening is that in the limit $m_S \gg m_V$, the wall collision contains enough high-frequency modes to create longitudinal gauge bosons. In the opposite limit, the gauge bosons are too massive to appear from scalar bosons of frequency $\sim m_S$: instead, the gauge field tracks the phase to ensure that the current vanishes.

In either case, the key point is that the interpolation of the phase between the bubbles is not affected by the presence of A_ν. Hence, by choosing a particular gauge, we establish the geodesic rule for local theories. In a three-bubble collision, the geodesic rule can be applied between each pair of bubbles provided the collision happens before the advancing wall of the other bubble reaches the collision point. We can, in fact, boost along the normal to the plane containing the centers of the three bubbles so that they nucleate simultaneously, so this is a constraint on the positions of the centers: they must form an acute triangle. The winding number around a closed curve connecting these points, along which the scalar field vanishes nowhere, is a gauge-invariant quantity. This is given by

$$n_\gamma = -\frac{i}{4\pi}\oint \frac{\phi^* \overleftrightarrow{\partial}_\mu \phi}{|\phi|^2} ds^\mu \ . \tag{2.24}$$

We can compute it in our chosen gauge and be confident that the result is gauge invariant.

These results imply that in first-order phase transitions, there is a finite probability that a field configuration of nontrivial winding (and hence a vortex) emerges during the collision of three bubbles. This is true for both local and global theories. If α_1, α_2, and α_3 are the phases of ϕ in the three bubbles (without loss of generality $\alpha_1<\alpha_2<\alpha_3$) and if $\alpha_1+\pi<\alpha_3<\alpha_2+\pi$, then by the geodesic rule, after bubble collision, $\alpha(x)$ will smoothly go from 0 to 2π along γ, yielding a configuration with winding number 1. As we have seen, the phases in the three bubbles are random, since the nucleation probability is independent of their values and so we get winding number 1 (with probability $\frac{1}{4}$ [24]).

III. EXTENSIONS

Our analysis has been based on dynamical rather than on topological considerations. Hence, the arguments may carry over to the case of defects such as semilocal strings [17] or electroweak strings [18], whose stability or otherwise is dynamical in origin.

The key to our analysis in the previous sections was to establish the validity of the geodesic rule, i.e., of the statement that after completion of the phase transition, the phases of the Higgs fields along a line in space connecting the centers of two initial bubbles will interpolate smoothly between their initial values in the bubbles, thus forming a section of geodesics on the vacuum manifold $\mathcal{M}$. The question of determining the probability of defect formation then reduces to the statistics problem of finding the probability that a closed geodesic will have a nonvanishing winding number.

We now generalize this approach to semilocal and electroweak strings. We first verify the geodesic rule, thus reducing the problem to a probability problem. However, the probability calculation, will, in general, be much harder than for topological defects.

Semilocal strings [17] arise in models with a large global symmetry group G of which only subgroup G_l is gauged. If this gauge group and its unbroken subgroup H_l obey the usual topological condition $\pi_1(G_l/H_l)\neq 0$, then stable string solutions exist only if the scalar mass is small enough relative to the vector mass [27,28]. The simplest example is the model of Ref. [17]: a U(2) symmetry acts on a complex scalar double Φ, but only the U(1) generated by the identity matrix is gauged. When Φ gains an expectation value the remaining symmetry is a global U(1).

The vacuum manifold of the theory is S^3, defined by $|\Phi|^2=\eta^2$, and it is fibered by the action of the local U(1) into a bundle which is locally $S^2\times S^1$— the Hopf bundle [27,28]. In the string solution, the scalar field wraps around one of these fibers outside the string, and vanishes at the origin. In fact, it is simply an embedding of the Nielsen-Olesen [26] vortex in the full theory. It is only stable if the scalar mass is less than the vector mass.

The equations of motion of this theory are essentially the same as those of Sec. II. Both components ϕ_1 and ϕ_2 of Φ satisfy Eq. (2.5),

$$\partial^2\phi_i - 2ieA_\mu\partial^\mu\phi_i - e^2A^2\phi_i + 2\frac{\partial V}{\partial|\phi|^2}\phi_i = 0 \ , \qquad i=1,2 \ , \tag{3.1}$$

and the gauge field satisfies the analogue of Eq. (2.8):

$$\partial^2 A_\nu - 2e^2A_\nu(\phi_1^2+\phi_2^2) = -ie(\phi_1^*\overleftrightarrow{\partial}_\nu\phi_1 + \phi_2^*\overleftrightarrow{\partial}_\nu\phi_2) \ . \tag{3.2}$$

In particular, the gauge connection A_μ does not mix ϕ_1 and ϕ_2. The same arguments as in Sec. II imply that the geodesic rule applies in our Lorentz gauge with $A_\mu(\mathbf{x},t_n)=0$.

However, this is a geodesic rule in the full vacuum manifold S^3, in which there is no topological obstruction to the scalar field reaching $|\Phi|=\eta$ everywhere in the triangular region formed by a three-bubble collision. This is related to the fact that the analogue of (2.24), or

$$m_\gamma = -\frac{i}{4\pi}\oint \frac{\Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi}{|\Phi|^2} dS^\mu \ , \tag{3.3}$$

is not necessarily an integer. The probability of forming a semilocal string is, in fact, a complicated dynamical question. It seems likely, however, that the "closer" the field in the three bubbles lies to the same U(1) orbit, the more likely is the formation of a string. We can, in fact, give this closeness a precise geometrical meaning.

Let us choose coordinates (σ,ψ,χ) on S^3 such that, in its vacuum manifold,

$$\Phi = \eta \begin{bmatrix} (\cos\chi/2)e^{i(\sigma-\psi)/2} \\ \sin(\chi/2)e^{i(\sigma+\psi)/2} \end{bmatrix} \ . \tag{3.4}$$

Then σ parametrizes the S^1 fibers, and (χ,ψ) are polar coordinates on the base space S^2. This can be seen by projecting onto a unit three vector:

$$\hat{\varphi}^a = \Phi^\dagger\sigma^a\Phi/|\Phi|^2 = (\sin\chi\cos\psi, \sin\chi\sin\psi, \cos\chi) \ . \tag{3.5}$$

The semilocal string has (χ,ψ) constant around it, with σ changing by 4π. Thus, the measure of closeness of the field values in the three bubbles (σ_i,χ_i,ψ_i) to the string configuration is the area of the spherical triangle defined by (χ_i,ψ_i). The smaller this area, the closer the field is to a pure phase change around the curve joining the centers of the bubbles (that is, the closer m_γ is to 1), and the more likely a string is to form. Unfortunately, without dynamical simulations, we cannot say how the string formation probability depends on the area A. The only piece of information we can extract geometrically is the probability $P(A)$ that a random spherical triangle has area less than or equal to A, which is [29]

$$P(A)=\left[\frac{A}{2}+\frac{1}{2}\sin\frac{A}{2}\cos\frac{A}{2}-\pi\sin^2\frac{A}{2}\right.$$
$$\left.+\frac{1}{8}(\pi-A)(3\pi-A)\tan\frac{A}{2}\right]\Big/\pi\cos^2\frac{A}{2}\ . \quad (3.6)$$

For small A

$$P(A)\simeq\frac{3A}{4\pi}\left[1+\frac{\pi^2}{4}\right]. \quad (3.7)$$

For fields Φ with d components we would expect the probability to go as V, where V is a $(2d-1)$-dimensional volume, although the coefficient is a more difficult exercise in geometric probability.

For electroweak strings the geometrical considerations are identical: our two-component semilocal model is just the bosonic sector of the electroweak theory in the limit that the weak mixing angle approaches $\pi/2$. However, we can expect the fields to evolve toward a string solution only if there is a locally stable solution towards which to evolve. In the standard model the string is unstable [30], so we do not expect to see them formed. However, metastable strings exist in theories with a more complex Higgs sector, [31] and in those we can expect some string formation in first-order phase transitions, although we are unable to estimate the probability.

IV. CONCLUSIONS

We have studied the formation of topological defects in first-order phase transitions. We showed that both for local and for global defects, the assumptions on which the Kibble mechanism [1] is based can be established by using the equations of motion.

The result of the analysis is that there is a probability p that a defect will form per correlation volume of the field. For topological defects, p is of the order 1 [24], for nontopological defects such as semilocal strings and electroweak strings, p depends on as yet ill-understood dynamics, but we guess it to be typically much smaller than 1. However, in no circumstance is the formation probability Boltzmann suppressed.

Our technique is based on first studying the dynamics of the scalar fields in the absence of gauge fields (a gauge-dependent analysis), establishing the existence of winding number in the final state (a gauge-independent conclusion), and then bounding the effects of the gauge fields, for convenience working in Lorentz gauge with the additional choice $A_\mu=0$ at the time of bubble nucleation.

We have not included thermal fluctuations into our consideration. We are therefore implicitly assuming that the root-mean-square amplitude of the thermal fluctuations on the length scales we discuss is much less than the magnitude of the scalar field $\eta(T)$, which is, in fact, not unreasonable [32].

Note added in Proof. Defect formation in a first-order phase transition (at the end of inflation) has also been considered in Ref. [33].

ACKNOWLEDGMENTS

For useful discussions and correspondence we are grateful to T.W.B. Kibble and A. Srivastava. We are also indebted to G.W. Gibbons for pointing out Ref. [29]. This work was supported in part by DOE Contract No. DE-FG02-91ER40688, Task A, and by NSF-SERC Collaborative Research Award NSF INT-9022895, and by SERC GR/G37149.

[1] T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).
[2] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[3] N. Turok, in *Particles, Strings and Supernovae,* Proceedings of the Theoretical Advanced Study Institute, Providence, Rhode Island, 1988, edited by A. Jevicki and C.-I. Tan (World Scientific, Singapore, 1989).
[4] N. Mermin, Rev. Mod. Phys. **51**, 591 (1979).
[5] M. Salomaa and G. Volovik, Rev. Mod. Phys. **59**, 533 (1987).
[6] P. de Gennes, *The Physics of Liquid Crystals* (Clarendon, Oxford, England, 1974).
[7] A. Abrikosov, Zh. Eksp. Teor. Fiz. **32**, 1442 (1957) [Sov. Phys. JETP **5**, 1174 (1957)].
[8] Ya. B. Zel'dovich and M. Khlopov, Phys. Lett. **79B**, 239 (1978); J. Preskill, Phys. Rev. Lett. **43**, 1365 (1979).
[9] R. Brandenberger, A.-C. Davis, and M. Hindmarsh, Phys. Lett. B **263**, 239 (1991); R. Brandenberger and A.-C. Davis, *ibid.* **308**, 79 (1993).
[10] Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. **192**, 663 (1980); A. Vilenkin, Phys. Rev. Lett. **46**, 1169 (1981); R. Brandenberger, Phys. Scr. **T36**, 114 (1991).
[11] D. Bennett and S. Rhie, Phys. Rev. Lett. **65**, 1709 (1990).
[12] N. Turok, Phys. Rev. Lett. **63**, 2625 (1989); Phys. Scr. **T36**, 135 (1991).
[13] I. Chuang, R. Durrer, N. Turok, and B. Yurke, Science **251**, 1336 (1991); M. Bowick, L. Chandar, E. Schiff, and A. Srivastava, Syracuse University Report No. SU-HEP-4241-512, 1992 (unpublished).
[14] J. Ye and R. Brandenberger, Mod. Phys. Lett. A **5**, 157 (1990); A. Srivastava, Phys. Rev. D **45**, 3304 (1992).
[15] J. Ye and R. Brandenberger, Nucl. Phys. **B346**, 149 (1990).
[16] S. Rudaz and A. Srivastava, Mod. Phys. Lett. A **8**, 1443 (1993).
[17] T. Vachaspati and A. Achucarro, Phys. Rev. D **44**, 3067 (1991).
[18] T. Vachaspati, Phys. Rev. Lett. **68**, 1977 (1992).
[19] J. March-Russell, Phys. Lett. B **296**, 364 (1992).
[20] D. A. Kirzhnits and A. D. Linde, Zh. Eksp. Teor. Fiz. **67**, 1263 (1974) [Sov. Phys. JETP **40**, 628 (1974)]; S. Weinberg, Phys. Rev. D **9**, 3320 (1974); L. Dolan and R. Jackiw, *ibid.* **9**, 3320 (1974); D. A. Kirzhnits and A. D. Linde, Ann. Phys. (N.Y.) **101**, 195 (1976).
[21] M. B. Voloshin, I. Yu. Kobzarev, and L. B. Okun, Yad. Fiz. **20**, 1229 (1974) [Sov. J. Nucl. Phys. **20**, 644 (1975)]; S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callan and S. Coleman, *ibid.* **16**, 1762 (1977).
[22] A. D. Linde, Nucl. Phys. **B216**, 421 (1983).
[23] M. Dine, R. G. Leigh, P. Huet, A. Linde, and D. Linde, Phys. Rev. D **46**, 550 (1992); N. Turok, Phys. Rev. Lett. **68**, 1803 (1992).
[24] T. Prokopec, Phys. Lett. B **262**, 215 (1991); R. Leese and T. Prokopec, Phys. Rev. D **44**, 3749 (1991).

[25] S. Hawking, I. Moss, and J. Stewart, Phys. Rev. D **26**, 2681 (1982).
[26] H. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[27] M. Hindmarsh, Phys. Rev. Lett. **68**, 1263 (1992).
[28] M. Hindmarsh, Nucl. Phys. **B392**, 461 (1993).
[29] *Mathematical Questions with Their Solutions,* edited by W. J. Miller (Hodgson, London, 1868), Vol. VIII, p. 21.
[30] M. James, L. Perivolaropoulos, and T. Vachaspati, Phys. Rev. D **46**, R5232 (1992).
[31] R. Holman, S. Hsu, T. Vachaspati, and R. Watkins, Phys. Rev. D **47**, 5352 (1992).
[32] M. Gleiser, Phys. Rev. D **42**, 3350 (1990).
[33] E. Copeland, E. Kolb, and A. Liddle, Phys. Rev. D **42**, 2911 (1990).

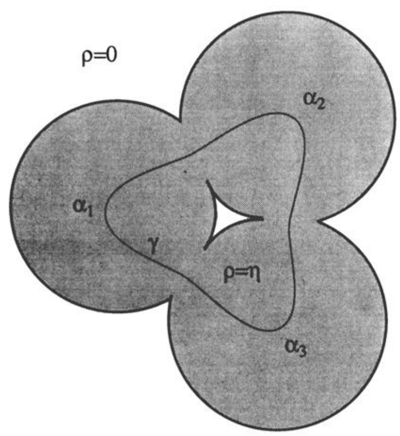

FIG. 1. Three bubbles of the broken symmetry phase ($\rho=\eta$) colliding. If the phase change of the scalar field around the loop γ is $\pm 2\pi$, a string (or antistring) is formed. If the phases α_i are ordered, then the requirement for a string is $\alpha_1+\pi<\alpha_3<\alpha_2+\pi$.

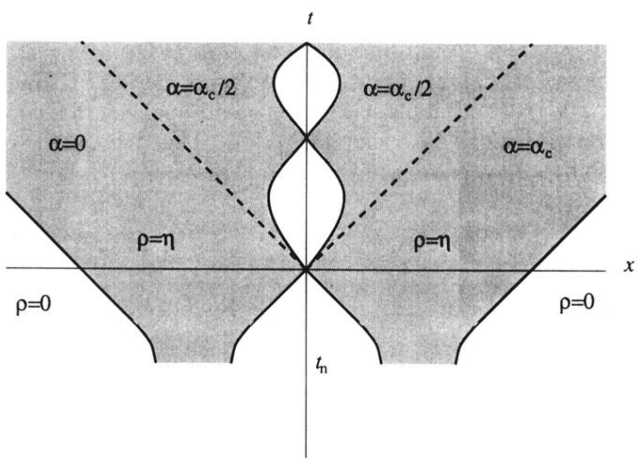

FIG. 2. Space-time diagram of two bubbles with different phases colliding. After nucleation at time t_n the bubble walls collide at $t=0$, when they are traveling at approximately c. Phase waves continue out from the collision site, spreading a region whose phase is halfway between the phases of the bubbles. The walls pass through each other, but eventually turn round and recollide. This may happen several times.