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# Hydrodynamic detonation instability in electroweak and QCD phase transitions

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The hydrodynamic stability of deflagration and detonation bubbles for a first order electroweak and QCD phase transition has been discussed recently with the suggestion that detonations are stable. We examine here the case of a detonation more carefully. We find that in front of the bubble wall perturbations do not grow with time, but behind the wall modes exist which grow exponentially. We briefly discuss the possible meaning of this instability.

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A first order phase transition involves the nucleation of a bubble of one phase within a medium of the original phase. If the bubbles are large enough, they expand, collide, and coalesce until the original phase has been completely replaced by the new phase. Though many of the details of this process are not definitively known—the expansion rate of the bubbles, for instance—understanding the dynamics of such transitions may provide valuable insight into the conditions of the early Universe.

Of particular interest are the electroweak (EW) and QCD phase transitions and the possible ramifications towards the generation of a baryon asymmetry in the EW case and the concentration of baryons in the QCD case. If the EW phase transition is first order, the possibility of baryon asymmetry generation arises [1]. Furthermore, if the asymmetry is created by interaction with the bubble wall, it becomes important to understand in detail the shape and structure of the wall [2]. A first order QCD phase transition, on the other hand, may result in baryon concentrating effects [3]. Crucial to this is the concept of phase separation [4]. The generation of a baryon concentration may depend on whether, and how effectively, the two phases mix during the period of bubble expansion. The stability of the bubble wall and the possible existence of turbulence are thus important factors when considering the effects of these phase transitions.

Recent investigations [4–6] of the hydrodynamic stability of the bubble wall restrict their attention to the case of a deflagration front, namely a bubble wall which is propagating at a speed slower than sound relative to the old phase. Recent estimates support the assumption that the bubble wall will propagate subsonically [7]. Even if these estimates are correct, however, there may exist situations which allow for the supersonic propagation of the bubble, i.e., a detonation front. It was suggested by Kamionkowski and Freese [5] that the existence of an instability in the deflagration front could result in the acceleration of the front until it becomes a detonation, a phenomenon which is observed in laboratories studying combustion [8]. It is interesting to note that once a bubble becomes a detonation it does not necessarily become stable. Indeed, in the standard theory detonations are typically unstable [9]. The paper by Huet *et al.* [6] analyzed the dispersion relation obtained for a perfect relativistic fluid and concluded that there could be no perturbations which grow in time in the detonation case. They, however, only examine the region in front of the expanding bubble and not the region behind it. In this paper we employ the standard linear stability analysis used by others [4,6,10,11] to examine the possible existence of instabilities both in front of and behind the bubble wall in the detonation case.

Detonations and deflagrations consist of two different phases separated by a transition region referred to as the bubble wall. A detonation front propagates a velocity greater than the local speed of sound relative to the original phase. Behind the wall, the medium has a velocity equal to or less than the speed of sound in the second phase (see Steinhardt's article [12] for a discussion of relativistic detonation and shock waves). We choose to move into the frame of the moving wall and position it at $x=0$. To the left of the wall ($x<0$) is the original (e.g., quark) phase, while to the right ($x>0$) is the new (e.g., hadron) phase. In region 1 ($x<0$) the fluid has a positive velocity less than that of sound ($v_1>c_{s1}>0$). In region 2 ($x>0$) the fluid has a positive velocity less than that of sound ($0<v_2<c_{22}$) [12]. The situation is illustrated in Fig. 1. Because of the different velocities, the behavior of perturbations and their effect on the bubble wall will differ. Since perturbations cannot travel faster than the speed of sound, we expect that in region 1 they should be "swept away" as the fluid passes through the wall. This is in agreement with Refs. [5,6]. The situation behind the wall is considerably different, there being no *a priori* reason why one should expect perturbations to decay with time.

We first consider the behavior of a relativistic perfect fluid in the two separate regions. Using the metric $g=(+,-,-,-)$ the stress energy of a perfect fluid is $T^{\mu\nu}=wu^\mu u^\nu-pg^{\mu\nu}$ where we have taken $c=1$, with $e$ the energy density, $p$ the pressure, and $w=e+p$ the enthalpy density. The equations of motion are

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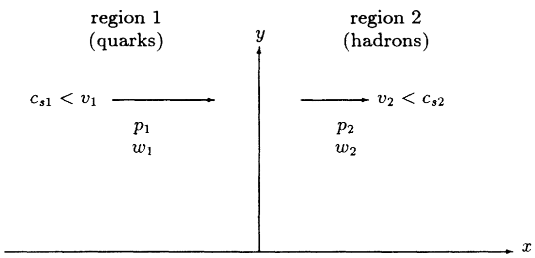


FIG. 1. Schematic of the bubble wall in the frame of the wall. The wall is located at $x=0$ with the old phase (e.g., quarks) in the half-plane $x<0$ and the new phase (e.g., hadrons) in the half-plane $x>0$.

$$u^{\mu}\partial_{\mu}p + c_s^2 w\partial_{\mu}u^{\mu}=0 ,$$

$$wu^{\mu}\partial_{\mu}u_{\nu} - \partial_{\nu}p + u_{\nu}u^{\mu}\partial_{\mu}p = 0 ,$$

where $u_{\mu}=(\gamma,\gamma\vec{v})$ and $\gamma=1/\sqrt{1-v^2}$. We take perturbations in the velocity and pressure of the fluid about a constant solution:

$$p=p_0+\delta p ,$$

$$\vec{v}=\vec{v}_0+\delta\vec{v} ,$$

$$\vec{v}_0=v_0\hat{\mathbf{x}} ,$$

$$\delta\vec{v}=\delta v_x\hat{\mathbf{x}}+\delta v_y\hat{\mathbf{y}} .$$

Keeping terms up to first order in $\delta p$ and $\delta\vec{v}$, we can write the equations of motion as

$$(1-c_s^2v_0^2)\frac{\partial}{\partial t}\delta p + v_0(1-c_s^2)\frac{\partial}{\partial x}\delta p + wc_s^2\left[\frac{\partial}{\partial x}\delta v_x + \frac{\partial}{\partial y}\delta v_y\right]=0 , \qquad (1)$$

$$\frac{v_0}{w\gamma^2}\frac{\partial}{\partial t}\delta p + \frac{\partial}{\partial t}\delta v_x + \frac{1}{w\gamma^2}\frac{\partial}{\partial x}\delta p + v_0\frac{\partial}{\partial x}\delta v_x = 0 , \qquad (2)$$

$$\frac{\partial}{\partial t}\delta v_y + v_0\frac{\partial}{\partial x}\delta v_y + \frac{1}{w\gamma^2}\frac{\partial}{\partial y}\delta p = 0 , \qquad (3)$$

where (1) comes from the conservation of energy and (2) and (3) are the relativistic Euler equations. It is more convenient to write this system as the matrix equation

$$\mathbf{A}_t\frac{\partial}{\partial t}\vec{W} + \mathbf{A}_x\frac{\partial}{\partial x}\vec{W} + \mathbf{A}_y\frac{\partial}{\partial y}\vec{W} = 0 , \qquad (4)$$

where

$$\vec{W}=\begin{bmatrix}\delta p\\ \delta v_x\\ \delta v_y\end{bmatrix} ,$$

$$\mathbf{A}_t=\begin{bmatrix}1-c_s^2v_0^2 & 0 & 0\\ \frac{v_0}{w\gamma^2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix} ,$$

$$\mathbf{A}_x=\begin{bmatrix}v_0(1-c_s^2) & wc_s^2 & 0\\ \frac{1}{w\gamma^2} & v_0 & 0\\ 0 & 0 & v_0\end{bmatrix} ,$$

$$\mathbf{A}_y=\begin{bmatrix}0 & 0 & wc_s^2\\ 0 & 0 & 0\\ \frac{1}{w\gamma^2} & 0 & 0\end{bmatrix} .$$

This system may be solved to obtain a solution of the form

$$\vec{W}(x,y,t)=\vec{F}(x)e^{-i(\omega t+ky)}$$

where

$$\vec{F}(x)=\sum_j a_j e^{-iq_jx}\vec{R}_j .$$

The $a_j$ are constants, and $R_j$ are eigenvectors. The $q_j$ are found by solving the characteristic equation for (4). Doing this, we obtain the dispersion relation

$$(qv_0+\omega)\left[\frac{1}{c_s^2}(qv_0+\omega)^2-(q+v_0\omega)^2-(1-v_0^2)k^2\right]=0 . \qquad (5)$$

The solutions for $q$ are

$$q_1=\frac{-\omega}{v_0} , \qquad (6)$$

$$q_{2,3}=\frac{1}{v_0^2-c_s^2}\left[(c_s^2-1)v_0\omega \pm c_s(1-v_0^2)\left[\omega^2+\frac{v_0^2-c_s^2}{1-v_0^2}k^2\right]^{1/2}\right] . \qquad (7)$$

These, (5)–(7), are the same as the equations obtained in Ref. [6], Eqs. (32), (34)–(36). It will be of further interest to note that in the case where $v_0=c_s$ we obtain two solutions, $q_1$ from above and

$$q_2=\frac{-(1+c_s^2)}{2c_s}\omega+\frac{c}{2}\frac{k^2}{\omega} . \qquad (8)$$

Thus, the solution for perturbations in a perfect relativistic fluid is

$$\vec{W}(x,y,t)=(a_1\vec{R}_1e^{-iq_1x}+a_2\vec{R}_2e^{-iq_2x} + a_3\vec{R}_3e^{-iq_3x})e^{-i(\omega t+ky)} \qquad (9)$$

where

$$\vec{R}_1=\begin{bmatrix}0\\ 1\\ \frac{q_1}{k}\end{bmatrix}$$

and

$$\vec{R}_j=\begin{bmatrix} 1 \\ \dfrac{-(q_j+v_0\omega)}{w\gamma^2(\omega+v_0q_j)} \\ \dfrac{-k}{\omega\gamma^2(\omega+v_0q_j)} \end{bmatrix} \quad \text{for } j=2,3 .$$

We now wish to examine whether there exist any unstable modes of $\vec{W}$ (i.e., modes where $\mathrm{Im}\omega>0$) which obey the boundary conditions $\vec{W}\to 0$ as $\to\pm\infty$. First, consider region 1 ($x<0$). In order to satisfy the boundary conditions as $x\to-\infty$, we must require either $\mathrm{Im}q_j>0$ or $a_j=0$. However, with some algebra, we can show from (7) that if $\mathrm{Im}\omega>0$ then $\mathrm{Im}q_j<0$ and therefore $a_j=0$ for $j=1,2,3$. That is, perturbations which grow in time cannot exist in front of the bubble wall. This is the conclusion reached in Ref. [6]. Let us next consider what occurs in region 2 ($x>0$). Here, the $x\to+\infty$ boundary condition requires either $\mathrm{Im}q_j<0$ or $a_j=0$. We find, if $\mathrm{Im}\omega>0$ then $\mathrm{Im}q_{1,2}<0$ and $\mathrm{Im}q_3>0$. Thus, behind the wall we need require only that $a_e=0$. That is, since we can satisfy the $x\to+\infty$ boundary condition without requiring $a_{1,2}=0$, the solution for the perturbations, Eq. (9), is not identically zero, i.e., perturbations behind the bubble wall that grown in time may exist.

The above work, however, is not enough to prove that instabilities do, in fact, occur. It is still necessary to impose the constraints of the boundary conditions across the bubble wall to determine whether the instabilities do not contradict the relevant conservation laws. Let us use notation where the subscripts of one or two indicate quantities in region 1 ($x<0$) or 2 ($x>0$), respectively. Also, we assume that there exists a perturbation of the bubble shape of the form

$$\Delta(y,t)=De^{-i(\omega t+ky)} .$$

We require, conservation of energy,

$$w_1\gamma_1^2v_1=w_2\gamma_2^2v_2 ,$$

conservation of momentum,

$$w_1\gamma_1^2v_1^2+p_1=w_2\gamma_2^2v_2^2+p_2 ,$$

and continuity of transverse velocity,

$$v_{1y}+v_1\frac{\partial\Delta}{\partial y}=v_{2y}+v_2\frac{\partial\Delta}{\partial y} .$$

When considering small perturbations, the surface term $-\sigma(\partial^2/\partial y^2-\partial^2/\partial t^2)\Delta$ must be added to the right-hand side of the momentum equation as a correction, where $\sigma$ is the surface tension. Recalling that $W=0$ in region 1, the above equations may then be linearized to obtain, in region 2,

$$\delta p=\frac{1}{\Gamma_-}\left[-2\gamma_2^2w_2v_2\frac{v_1-v_2}{v_1}(-i\omega)+\sigma(\omega^2-k^2)\right]\Delta , \tag{10}$$

$$\delta v_x=\frac{\Gamma_+}{\Gamma_-}\left[\frac{v_1-v_2}{v_1}\right](-i\omega)\Delta+\left[1-\frac{\Gamma_+}{\Gamma_-}\right]\frac{\sigma(\omega^2-k^2)}{2\gamma_2^2w_2v_2}\Delta , \tag{11}$$

$$\delta v_y=(v_1-v_2)(-ik)\Delta , \tag{12}$$

where $\Gamma_\pm=1\pm\gamma_2^2\theta_2v_2^2$ and $\theta_2=1+1/c_{s2}^2$. Then (10)–(12) give us the perturbations just interior to the bubble, i.e., as $x\to 0^+$. Since we require $a_3$ to be zero for $x>0$ we are left with $a_1$, $a_2$, and $D$ as undetermined constants. By matching the general solution for perturbations for $x>0$, Eq. (9), with Eqs. (10)–(12), we obtain three equations for these three unknowns. We will always have a solution for such a system provided that the determinant of the coefficient matrix vanishes. Let us write Eqs. (10)–(12) as

$$\vec{Y}\Delta\equiv\begin{bmatrix}\delta p\\ \delta v_x\\ \delta v_y\end{bmatrix} . \tag{13}$$

Matching solutions (9) with (13) we get

$$\vec{W}(0^+,y,t)=(a_1\vec{R}+a_2\vec{R}_2)e^{-i(\omega t+ky)}$$

$$=\vec{Y}De^{-i(\omega t+ky)} .$$

This equation may be rearranged into a single $3\times3$ matrix equation where the columns of the matrix are the vectors $\vec{R}_1$, $\vec{R}_2$, and $\vec{Y}$:

$$(\vec{R}_1|\vec{R}_2|\vec{Y})\begin{bmatrix}a_1\\ a_2\\ D\end{bmatrix}=0 .$$

By taking the determinant of the above $3\times3$ matrix we obtain an equation in $\omega$, $q_2$, $k$, $\sigma/w_2$, $v_1$, $v_2$, and $c_{s2}$:

$$\frac{1}{\Gamma_-v_2k(\omega+v_2q_2)}\left\{i\frac{v_1-v_2}{v_1}[(\Gamma_+-2v_2^2)\omega^3+v_2(\Gamma_+-2)q_2\omega^2+(2v_2-\Gamma_-v_1)v_2k^2\omega-\Gamma_-v_1v_2^2q_2k^2]\right.$$
$$\left.+\frac{\sigma(\omega^2-k^2)}{2\gamma_2^2w_2v_2}[\{\Gamma_+-\Gamma_-(1+2v_2^2)\}\omega^2+(\Gamma_+-3\Gamma_-)v_2q_2\omega+2\Gamma_-v_2^2k^2]\right\}=0 . \tag{14}$$

Combining Eqs. (7) and (14) and solving for $\omega$ we can eliminate $q_2$ and obtain an equation for $\omega$ as a function of $v_1$, $v_2$, $c_{s2}$, $\sigma/w_2$, and $k$. By studying this equation we can determine if there exist any cases where $\omega$ has a positive imaginary solution, thereby establishing that the detonation has a self-sustaining instability.

Of particular relevance is the case of a Chapman-Jouget detonation. This occurs when the velocity behind the wall is equal to the speed of sound ($v_2 = c_{s2}$). Steinhardt showed that any detonation with spherical symmetry must be of the Chapman-Jouget type [12]. It has also been postulated that any "naturally" occurring detonation will meet this condition [11]. In order to examine this case we use the value for $q_2$ given in Eq. (8). It is interesting to look at the case where the surface tension is zero ($\sigma = 0$); there are four solutions to $\omega$, two of which are positive imaginary. The solutions are

$$\omega = \pm i \left[ \frac{c_{s2} v_1}{2 - c_{s2}^2} \right]^{1/2} c_{s2} k \ , \quad \omega = \pm i \left[ \frac{1}{1 - c_{s2}} \right]^{1/2} c_{s2} k \ .$$

The full equation with surface tension cannot be solved analytically. Instead we look at the behavior of $\omega$ under different limits. In the long wavelength limit ($k \to 0$), keeping terms up to second order in $k$, Eq. (14) becomes

$$\frac{3\sigma(1-c_s^2)}{2w_2 c_s}(1-c_s^4)\omega^5 + i\frac{v_1 - c_s}{v_1}(c_s^4 - 3c_s^2 + 2)\omega^4 + \frac{\sigma(1-c_s^2)}{w_2 c_s}[-c_s^4 + \tfrac{3}{2}(c_s^2 - 1)]k^2\omega^3 + i(-c_s^3 v_1 - c_s^2 + c_s v_1 + 2)c_s^2 \frac{v_1 - c_s}{v_1} k^2 \omega^2 = 0 \ . \tag{15}$$

Notice that the quantity $\sigma / w_2$ provides a natural length scale. In Fig. 2, the behavior of $\mathrm{Im}\omega$ was plotted as a function of $k$ with values of $v_1 = 1.5 c_s$ and $c_s = 1/\sqrt{3}$. The particular choice for $v_1$ is arbitrary; the behavior of $\mathrm{Im}\omega$ is not altered significantly for different values. We have chosen a solution which matches with an unstable mode in the zero surface tension case, and we see that the rate of expansion increases slightly slower than linearly with the wave number, an effect of the surface tension. The discontinuity which appears as $k$ increases is an artifact of the small $k$ approximation; its location approaches the origin as the surface tension is increased. That is, with a larger surface tension we must go to a larger wavelength in order to insure that the $k \to 0$ approximation is valid, as we expect.

Next, consider the short wavelength limit. As $k$ goes to infinity only higher orders of $k$ contribute and we get

$$\frac{\sigma(1-c_s^2)}{w_2 c_s}[-c_s^4 + \tfrac{3}{2}(c_s^2 - 1)]k^2\omega^3 + i(-c_s^3 v_1 - c_s^2 + c_s v_1 + 2)c_s^2 \frac{v_1 - c_s}{v_1} k^2 \omega^2 - \frac{\sigma(1-c_s^2)^2}{2w_2} c_s k^4 \omega + i(v_1 - c_s)c_s^5 k^4 = 0 \ . \tag{16}$$

Figure 3 shows $\mathrm{Im}\omega$ as a function of $k$ with the same values as in Fig. 2. Here we notice that as $k$ gets larger $\mathrm{Im}\omega$ quickly approaches a constant value of approximately $0.144 w_2/\sigma$. Unlike in the deflagration case, where the surface tension results in there being a lower limit cutoff in the wavelength of instabilities, with a Chapman-Jouget detonation the growth rate does not drop to zero with shorter wavelengths, but rather reaches a maximum positive value.

Physically, we expect a cutoff wavelength because the surface energy associated with a perturbation of amplitude $A$ goes like $\sigma k^2 A^2$. The existence of a cutoff eliminates the problem of a divergent energy at extremely small wavelengths. In the present model, when the velocity behind the wall is less than that of sound ($v_2 < c_s$), we

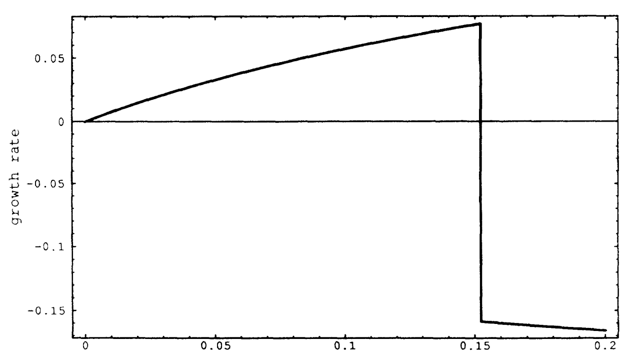


FIG. 2. Perturbation growth rate ($\mathrm{Im}\omega$) as a function of wave number ($k$) in the limit $k \to 0$, plotted with parameter values $c_s = 1/\sqrt{3}, v_1 = 1.5 c_s$ in units of $w_2/\sigma$.

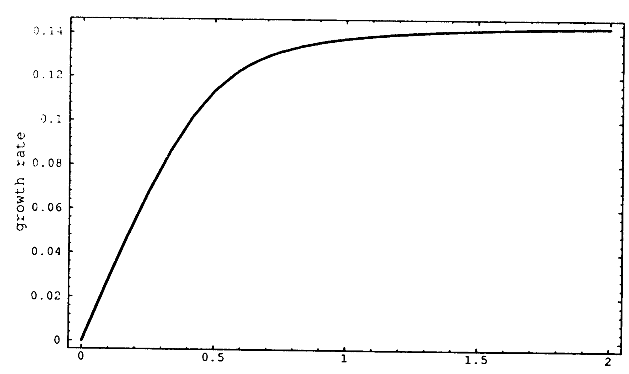


FIG. 3. The same as in Fig. 2, but in the limit $k \to \infty$.

obtain a critical wave length below which perturbations are stabilized. In fact, in the limit where $v_2 \ll c_s \ll 1$ the critical wavelength is the same as the one obtained by Link [4] for a deflagration. It is only in the case where $v_2 \rightarrow c_s$ that the cutoff wavelength drops to zero.

In order for these instabilities to be dynamically relevant the mode with the fastest growth time scale must be less than the time scale associated with the phase transition. The QCD transition lasts a time $t_H \sim 10^{-5}$ s and has a value of $\sigma/w_2 \sim 1$ fm [4]. We showed above that $\text{Im}\omega$ reached a maximum value of $\sim 0.14 w_2/\sigma$, corresponding to a time of $\sim 2.3 \times 10^{-23}$ s. Thus, there is ample time for the instabilities to mature.

The scales associated with the EW phase transition are much smaller. In this case, using the formulas and parameter values as given in Ref. [13], we have $\sigma \sim 0.09 T_c^3$ and $w_2 \sim 40 T_c^4$ where $T_c$ is the critical temperature. With a critical temperature of 150 GeV we get $\sigma/w_2 \sim 3 \times 10^{-6}$ fm. This leads to a maximum value of $\text{Im}\omega$ of $\sim 5 \times 10^4$ fm$^{-1}$ corresponding to a time of $\sim 6.7 \times 10^{-29}$ s. Since the phase transition lasts a time $\sim 0.005 t_H$ [13], where the Hubble time is $t_H \sim 10^{-11}$ s, we see that there is sufficient time for the instabilities to grow.

Our general picture of the instabilities, then, is the following. For very large wavelengths the instabilities grow only very slowly with time, vanishing as $\lambda$ approaches infinity. As the wavelength decreases $\text{Im}\omega$ rapidly increase with smaller $\lambda$ until a maximum rate is reached. Thus, unstable modes exist at wall wavelengths. Even though these calculations were done for a Chapman-Jouget type of detonation, it is unlikely that the science and time scale of the instabilities is so sensitive to the velocity behind the wall that this picture would be very much altered should $v_2 < c_s$. Additional work beyond the scope of this article would need to be done, however, to verify this assertion.

Note that these instabilities do not exist in front of the bubble wall, but rather behind it and at its surface. What, then, are the ramifications on the bubble wall and the fluid inside it? To answer this, let us examine similar types of instabilities which exist in the laboratory. D'yakov and others [10] have calculated the stability condition for shock waves in a classical nonrelativistic fluid. Though a shock wave and denotation are not identical, the fluid dynamics are quite similar. It is possible to established conditions in the laboratory which violate this stability requirement; that is, under certain circumstances there are unstable modes which exist on the surface of and behind the shock wave. Thompson *et al.* [14] have recently done experiments which violate this shock stability condition. In their report they show a series of photographs where we see shock waves transform from planar to a billowy cloudlike surface as the stability condition is violated. The surface of the wave is traveling faster than the speed of sound, and the instabilities do not propagate forward from the shock. The transformation of the shock surface from planar to irregular is described by them as "transition to turbulence" of the fluid behind the shock.

It is possible that a similar situation exists with the instabilities described above. The growth of perturbations behind the bubble wall may result in turbulence and a highly irregular surface. Such effects may entail an alternation in the picture of the phase transition proceeding through the growth of uniform spherical bubbles. This may very well be relevant when considering the possibility of baryon generation and concentration in the EW and QCD phase transitions, respectively.

Whether detonations actually do arise, though, is not yet known. A more highly first order transition would likely result in an increased possibility of detonations. Another mechanism could be the existence of instabilities in deflagrations as described in Ref. [5]. The above analysis, however, is limited to the linear regime and does not take into account any nonlinear effects that may arise. The result of such effects may be to stabilize the perturbations, as has been observed with deflagrations in the laboratory [8]. It remains to be seen whether non linearities are indeed important.

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