Graviton creation in an inflationary universe

Alfredo B. Henriques
Departamento de Física, Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal
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Graviton creation in an expanding universe is studied. Following the work of Parker, we calculate explicit expressions for the Bogoliubov coefficients α and β as functions of time, valid in the high-frequency limit and for an exponentially expanding phase of the universe. The power spectrum of the created gravitons is investigated.

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I. INTRODUCTION

As far as we know, Schrödinger was the first to realize that, in a universe expanding with acceleration, pairs of particles would be created, a phenomenon that he classified as alarming [1]. The fascinating thing is that, rather than pursuing the subject and exploring its consequences, Schrödinger decided instead to investigate those situations where this phenomenon would not occur. Later, in 1957, the problem of quantized particle creation in an expanding system was treated by Takahashi and Umezawa [2]. But the formalism we shall be using throughout this paper is the one developed in 1969 by Parker, in a pioneering work, for an expanding universe [3]. Important references are also the works of Zeldovich and Starobinskii [4] and of Birrell and Davies [5]. For a review on cosmological gravitational waves see Ref. [6].

In this paper we address the problem of the stochastic gravitational-wave background calculating, using directly the second-quantized formalism developed by Parker in Ref. [3], the Bogoliubov coefficients $\alpha_k(\tau)$ and $\beta_k(\tau)$ as a function of time. During an accelerated phase the creation of gravitons could be an important phenomenon [7–17]. We study the especially simple case of an exponentially accelerated phase and restrict ourselves to the high-frequency limit of the spectrum of the created gravitons. The problems related to the low-frequency end of the spectrum, for power-law expansion, were discussed in Refs. [20,21].

We shall closely follow Ref. [3], adapting it to the present situation, and use, with slight modifications, Allen's notation [16] for the exponential inflationary phase. For convenience we take a spatially flat universe and write the line element in the form

$$ds^2=a^2(\tau)(-d\tau^2+d\mathbf{x}^2)\ , \tag{1}$$

treating the gravitational metric as an unquantized external field. We have an inflationary phase between τ_2 and τ_1, undergoing at τ_1 a transition to a radiation-dominated universe. The scale factor in the conformal time of the metric (1) obeys

$$a(\tau)=a_2\frac{2\tau_1-\tau_2}{2\tau_1-\tau}\quad\text{for } \tau_2\le\tau\le\tau_1\ , \tag{2a}$$

with

$$a_2(2\tau_1-\tau_2)=(8\pi G\rho_{\text{vac}}/3)^{1/2}\ ,$$

and

$$a(\tau)=a_2\frac{2\tau_1-\tau_2}{\tau_1^2}\tau\quad\text{for } \tau>\tau_1\ . \tag{2b}$$

We shall write the gravitational perturbation $[g_{\mu\nu}=a^2(\tau)(\delta_{\mu u}+h_{\mu\nu})]$ as a Fourier expansion in terms of the creation and annihilation operators and of the two polarization states of the gravitational plane wave [21]:

$$h_\pm=\sum_{\mathbf{k}}[a_{\mathbf{k},\pm}(\tau)\phi(\tau)\exp(i\mathbf{k}\cdot\mathbf{x})+\text{H.c.}]\ . \tag{3a}$$

The physical frequency is $\omega=ck/a(\tau)$ (we take $c=1$ and $k=|\mathbf{k}|$) and $a^\dagger_{\mathbf{k},\pm}(\tau)$ and $a_{\mathbf{k},\pm}(\tau)$ are the corresponding creation and annihilation operators, which we take as time dependent. The gravitational perturbation thus written must satisfy the equation

$$h''_\pm(\tau)+2(a'/a)h'_\pm+k^2h_\pm=0\ , \tag{3b}$$

where primes denote $d/d\tau$. As Parker did, the function $\phi(\tau)$ will, in what follows, be written in the general form

$$\phi(\tau)=\frac{1}{a^{3/2}}\frac{1}{\sqrt{W(k,\tau)}}\exp\left[-i\int_{\tau_2}^{\tau}W(k,\tau')d\tau'\right]\ , \tag{3c}$$

with $W(k,\tau)$ an arbitrary function of k and τ, but on whose choice the form of the operators $a_{\mathbf{k},\pm}(\tau)$ will obviously depend. These time-dependent operators obey the canonical commutation relations

$$[a_{\mathbf{k},\pm}(\tau),a^\dagger_{\mathbf{k}'\pm}(\tau)]=\delta_{\mathbf{k},\mathbf{k}'}\quad\text{all the others being zero}\ , \tag{4a}$$

as can be proved by following a procedure exactly similar to the one developed by Ford and Parker in Ref. [21] [Sec. III; in particular, from Eqs. (3.30) up to (3.38)] and Parker [3], *provided* that a condition similar to Eq. (10) of Ref. [3] holds:

$$\frac{dh_\pm(\tau)}{d\tau}=\sum_{\mathbf{k}}\left[a_{\mathbf{k},\pm}(\tau)\frac{d}{d\tau}\left\{\frac{1}{a^{2/3}}\frac{1}{\sqrt{W(k,\tau)}}\exp\left[i\left[\mathbf{k}\cdot\mathbf{x}-\int_{\tau_2}^{\tau}W(k,\tau')d\tau'\right]\right]\right\}+\text{H.c.}\right]\ . \tag{4b}$$

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This condition in our case is also satisfied for all times. We now introduce an arbitrary time τ_3 such that $a_{\mathbf{k},\pm}$ is given at that time by

$$a_{\mathbf{k},\pm}(\tau_3)=A_{\mathbf{k},\pm}\ ; \tag{5a}$$

for later times we assume that $a_{\mathbf{k},\pm}(\tau)$ evolves as determined by the ansatz

$$a_{\mathbf{k},\pm}(\tau)=\alpha_k^*(\tau)A_{\mathbf{k},\pm}+\beta_k(\tau)A^\dagger_{-\mathbf{k}\pm} \tag{5b}$$

defining the Bogoliubov coefficients, which, later on, will be seen to obey the condition $|\alpha_k(\tau)|^2-|\beta_k(\tau)|^2=1$.

Exact solutions of (3b), corresponding to a choice of an infinite-order adiabatic vacuum state, where $\alpha_k=\text{const}=1$ and $\beta_k=0$ in (5b), are known for the cases (2a) and (2b) and correspond to the choices for $\phi(\tau)$ [16]:

$$\phi(\tau)=(a_2/a)\{1+i(a/a_2)[1/k(2\tau_1-\tau_2)]\}\times\exp[-iK(\tau-\tau_2)]\ , \tag{6a}$$

for the inflationary phase, and

$$\phi(\tau)=(a_2/a)\exp[-ik(\tau-\tau_1)]\ , \tag{6b}$$

for the radiation-dominated phase.

The inflationary phase can be preceded (for $\tau<\tau_2$) by another phase, but this is not important now.

The important point to realize is that, due to the expansion of the Universe, the state $|0\rangle$, which we define as the state that contains no particles at a certain instant of time τ_3 with respect to the infinite-order adiabatic vacuum state ($A_{\mathbf{k},\pm}|0\rangle=0$, for all $\mathbf{k}$), will evolve in such a way that, if we define $|0\rangle_\tau=U(\tau)|0\rangle$, $U(\tau)$ a well-defined unitary operator [3], as the state which contains no particles at a time $\tau>\tau_3$, then $|\langle 0|0\rangle_\tau|\neq 1$. The square of this quantity, which expresses the probability of finding at the time τ zero particles corresponding to the wave number k, can be expressed in terms of $\alpha_k(\tau)$ and $\beta_k(\tau)$. It is also shown in Parker that the average number of particles present at time τ, in the mode k is [3]

$$\langle N_k(\tau)\rangle=|\beta_k(\tau)|^2\ . \tag{7}$$

The interesting thing in Parker's formalism seems to be the possibility of choosing a no-particle initial condition at a finite time in the past. For such a no-particle condition defined with respect to the infinite-order adiabatic vacuum state, we need to guarantee that our state, defined by (3a) and (3c) is, at the time τ_3, sufficiently close to this infinite-order adiabatic vacuum state to warrant such a definition of the initial condition. We show in Sec. III that this indeed happens in the cases under study; at the moment, we just assume that $\tau_2\le\tau_3\le\tau_1$. In the actual practice τ_3 is such that, prior to τ_3, all the created gravitons will be so redshifted that their contribution to the power spectrum can be ignored.

II. CALCULATION OF THE BOGOLIUBOV COEFFICIENTS

Our purpose now is to find the functions $\alpha_k(\tau)$ and $\beta_k(\tau)$. To do that we first derive a pair of integral equations for these functions, which we then solve using an iterative method. The solution is given by Eqs. (25a) and (25b) below. To establish a connection with Parker's notation note that his function $h(\tau)$ is now given by

$$h(\tau)\equiv\frac{\bar h}{\sqrt{W(k,\tau)}}\sqrt{a(\tau)}\left[\alpha_k(\tau)\exp\left[-i\int_{\tau_2}^{\tau}W(k,\tau')d\tau'\right]+\beta_k(\tau)\exp\left[i\int_{\tau_2}^{\tau}W(k,\tau')d\tau'\right]\right]\ , \tag{8a}$$

where $\bar h$ is an arbitrary constant (left for normalization purposes). In what follows we shall suppress the explicit mention of the index k and use h as the function defined by this relation. Equation (16) of Ref. [3] then becomes, in conformal time,

$$h''-(a'/a)h'+[k^2+\tfrac{3}{4}(a'/a)^2-\tfrac{3}{2}(a''/a)]h=0 \tag{9a}$$

and, of course, a solution of this equation is $a^{2/3}\phi(\tau)$, with ϕ given by (6). From Eqs. (4), (29), and (30) of Parker [3] it is shown that his condition (10) is satisfied at all times and without any mathematical constraint on W, leaving W as an essentially arbitrary real function of k and time. Exactly the same happens here with our condition (4b). We shall make, in what follows, the choice $W=k$. This is the simplest choice that we can make and also allows us, later on, to establish in a simple way a connection with the solutions written by Allen representing the infinite-order adiabatic vacuum state [see Eqs. (26) and (27) below]. With this choice of W, we write the solution of Eq. (9a) in the form

$$h(\tau)=\frac{\bar h}{\sqrt{k}}\sqrt{a(\tau)}\left[\alpha(\tau)\exp\left[-i\int_{\tau_2}^{\tau}k\,d\tau'\right]+\beta(\tau)\exp\left[-\int_{\tau_2}^{\tau}k\,d\tau'\right]\right]\ . \tag{8b}$$

We now assume the initial conditions

$$\alpha(\tau_3)=1\quad\text{and}\quad\beta(\tau_3)=0\ , \tag{10}$$

for some convenient τ_3, with $\tau_2\le\tau_3\le\tau_1$. Later on τ_3 will be taken equal to τ_2.

Let us first note that the expression

$$h_0(\tau)=\frac{\bar h}{\sqrt{k}}\sqrt{a}\ \exp\left[-i\int_{\tau_2}^{\tau}k\,d\tau'\right] \tag{11}$$

satisfies the equation

$$h_0''-(a'/a)h_0'+[k^2+\tfrac{3}{4}(a'/a)^2-\tfrac{1}{2}(a''/a)]h_0=0\ , \tag{12}$$

which differs from Eq. (9a) only in the numerical coefficient multiplying a''/a. Equation (9a) can then be rewritten as

$$h''-(a'/a)h''+[k^2+\tfrac{3}{4}(a'/a)^2-\tfrac{1}{2}(a''/a)]h=2kSh\ , \tag{9b}$$

where the left-hand side (LHS) is the same as Eq. (12) and S is given by

$$2kS\equiv(a''/a)\ . \tag{13}$$

From this equation we see that $S=0$ in a radiation-dominated universe where $a\sim\tau$; we also know that no gravitons are produced in this case due to a conformal invariance in the equations [17]. $S=0$ and no gravitons are produced, also in the limiting case where a is constant.

Let us introduce the function

$$G(k,\tau,\tau')=-\frac{1}{2i}\frac{i}{k}\frac{\sqrt{a(\tau)}}{\sqrt{a(\tau')}}\left[\exp\left[-i\int_{\tau'}^{\tau}k\,d\tau''\right]-\exp\left[i\int_{\tau'}^{\tau}k\,d\tau''\right]\right]\ ; \tag{14}$$

this function satisfies Eq. (12) and the conditions $G(k,\tau,\tau)=0$ and $\partial G(k,\tau,\tau')/\partial\tau=1$, at $\tau=\tau'$. From a comparison of Eq. (9b) with Eq. (12), we are then led to write $h(\tau)$ in the form

$$h(\tau)=\frac{\bar{h}}{\sqrt{k}}\sqrt{a(\tau)}\,\exp\left[-i\int_{\tau_2}^{\tau}k\,d\tau'\right]+\int_{\tau_3}^{\tau}G(k,\tau,\tau')2kS(\tau')h(\tau')d\tau'\ , \tag{15}$$

which we can check to be a solution of Eqs. (9). If we use expression (8b) for $h(\tau)$ on both sides of (15), we finally find the system of integral equations for $\alpha(\tau)$ and $\beta(\tau)$:

$$\alpha(\tau)=1+i\int_{\tau_3}^{\tau}d\tau'S(\tau')\left[\alpha(\tau')+\beta(\tau')\exp\left[2i\int_{\tau_2}^{\tau'}k\,d\tau''\right]\right]\ , \tag{16a}$$

$$\beta(\tau)=-i\int_{\tau_3}^{\tau}d\tau'S(\tau')\left[\beta(\tau')+\alpha(\tau')\exp\left[-2i\int_{\tau_2}^{\tau'}k\,d\tau''\right]\right]\ . \tag{16b}$$

From Eqs. (16), their derivative, and conditions (10), we derive

$$|\alpha(\tau)|^2-|\beta(\tau)|^2=1\ , \tag{17}$$

a condition that α and β must satisfy at all times. As stated at the beginning of this section, our purpose is to solve the system of Eqs. (16). In order to simplify our task, following Ref. [18] we express α and β in terms of two new functions η and ζ:

$$\alpha(\tau)=\eta(\tau)\exp\left[i\int_{\tau_3}^{\tau}d\tau'S(\tau')\right]\ , \tag{18a}$$

$$\beta(\tau)=\zeta(\tau)\exp\left[-i\int_{\tau_3}^{\tau}d\tau'S(\tau')\right]\ , \tag{18b}$$

with $\eta(\tau_3)=1$ and $\zeta(\tau_3)=0$, as required from (10). From these equations and Eqs. (16) we get the system of equations

$$\eta'(\tau)=iS(\tau)\zeta(\tau)\exp[-i\theta(\tau)]\ , \tag{19a}$$

$$\zeta'(\tau)=-iS(\tau)\eta(\tau)\exp[i\theta(\tau)]\ , \tag{19b}$$

where we introduced the function

$$\theta(\tau)=2\int_{\tau_3}^{\tau}d\tau'S(\tau')-2k(\tau-\tau_2)\ . \tag{20}$$

It is Eqs. (19) that we are going to solve iteratively. Taking into account the conditions on η and ζ at τ_3, the result of this procedure is to express $\eta(\tau)$ and $\zeta(\tau)$ through the expansions

$$\eta(\tau)=1+\int_{\tau_3}^{\tau}d\tau'\int_{\tau_3}^{\tau'}d\tau''S(\tau')S(\tau'')e^{-i\theta(\tau')+i\theta(\tau'')}$$
$$+\int_{\tau_3}^{\tau}d\tau'\int_{\tau_3}^{\tau'}d\tau''\int_{\tau_3}^{\tau''}d\tau'''\int_{\tau_3}^{\tau'''}d\tau''''S(\tau')S(\tau'')S(\tau''')S(\tau'''')e^{-i\theta(\tau')+i\theta(\tau'')-i\theta(\tau''')+i\theta(\tau'''')}+\cdots\ , \tag{21a}$$

$$\zeta(\tau)=-i\int_{\tau_3}^{\tau}d\tau'S(\tau')e^{i\theta(\tau')}-i\int_{\tau_3}^{\tau}d\tau'\int_{\tau_3}^{\tau'}d\tau''\int_{\tau_3}^{\tau''}d\tau'''S(\tau')S(\tau'')S(\tau''')e^{i\theta(\tau')-i\theta(\tau'')+i\theta(\tau''')}-\cdots\ , \tag{21b}$$

where it is important to notice that $S(\tau)$ and $\theta(\tau)$ are known functions of τ; α and β can then be obtained with the help of Eqs. (18). In the case of the inflationary phase, with $a(\tau)$ given by (2a), we have

$$S(\tau)=\frac{1}{k(2\tau_1-\tau)^2}$$

and

$$\theta(\tau)=\frac{2}{k}\left[\frac{1}{2\tau_1-\tau}-\frac{1}{2\tau_1-\tau_2}\right]-2k(\tau-\tau_2)\ , \tag{22}$$

where, from now on, we take $\tau_3 \equiv \tau_2$. We calculate (21) in the high-frequency limit case, $k\tau > 1$, where the series (21) rapidly converges and only the first few orders in $1/k\tau$ will be necessary. To do this we rewrite the functions $\exp[i\theta(\tau)]$, appearing in Eqs. (21), in the form of a series in $1/k\tau$ as

$$e^{i\theta(\tau)} = e^{-2ik(\tau-\tau_2)} \left[1 + \frac{i}{k} \left[\frac{2}{2\tau_1 - \tau} - \frac{2}{2\tau_1 - \tau_2} \right] - \frac{1}{2k^2} \left[\frac{2}{(2\tau_1-\tau)^2} - \frac{2}{(2\tau_1-\tau_2)^2} \right]^2 - \cdots \right] . \tag{23}$$

The integrals in (21) are calculated by parts leading, in the high-frequency limit [19], to a rapidly convergent series in $1/k\tau$. As an example, consider the first term in $\zeta(\tau)$:

$$-i \int_{\tau_3 \equiv \tau_2}^{\tau} d\tau' S(\tau') e^{i\theta(\tau')} = -i \int_{\tau_2}^{\tau} d\tau' \frac{e^{-2ik(\tau'-\tau_2)}}{k(2\tau_1-\tau')^2} \left[1 + \frac{i}{k} \left[\frac{2}{2\tau_1-\tau'} - \frac{2}{2\tau_1-\tau_2} \right] - \cdots \right] . \tag{24}$$

Both the expansion of $\exp[i\theta(\tau)]$ and the integrations by parts must be made up to the required order in $1/k\tau$, remembering that, each time we integrate by parts the exponential term $\exp[-2ik(\tau'-\tau_2)]$, we get an extra factor $1/k$. Doing this in a systematic way, rearranging the terms and introducing the notation $x \equiv (2\tau_1 - \tau)$, $y \equiv (2\tau_1-\tau_2)$, $a \equiv \exp[-2ik(\tau-\tau_2)]$, and $b \equiv \exp[2ik(\tau-\tau_2)] \equiv a^*$, we find, for the first few terms of $\beta(\tau)$,

$$\beta(\tau) = -i \left[\frac{i}{2k^2} \left[\frac{a}{x^2} - \frac{1}{y^2} \right] + \frac{2}{2k^3} \left[\frac{a}{x^2 y} - \frac{1}{y^3} - \frac{1}{2}\frac{a}{x^3} + \frac{1}{2}\frac{1}{y^3} \right] \right.$$
$$\left. + \frac{i}{4k^4} \left[-\frac{a}{x^4} + \frac{1}{y^4} - \frac{4a}{x^2y^2} + \frac{4a}{x^3 y} \right] + O(1/(k\tau)^5) \right] \exp\left[-i \left[\frac{1}{k(2\tau_1-\tau)} - \frac{1}{k(2\tau_1-\tau_2} \right] \right] . \tag{25a}$$

For $\alpha(\tau)$ we get

$$\alpha(\tau) = \left[1 + \frac{i}{6k^3} \left[\frac{i}{x^3} - \frac{1}{y^3} \right] + \frac{1}{4k^4} \left[\frac{1}{2}\frac{1}{y^4} + \frac{1}{2}\frac{1}{x^4} - \frac{b}{x^2y^2} \right] + O(1/(k\tau)^5) \right]$$
$$\times \exp\left[i \left[\frac{1}{k(2\tau_1-\tau)} - \frac{1}{k(2\tau_1-\tau_2)} \right] \right] . \tag{25b}$$

We can check by direct substitution that $h(\tau)$, as defined by (8b) and (25), is indeed a solution of Eqs. (9) and that α and β also obey $|\alpha|^2 - |\beta|^2 = 1 + O(1/(k\tau)^5)$.

We now establish the connection between our solutions (8b) and (25) and the solutions found by Allen [16], which we can write in the form

$$h(\tau) = a_2 \sqrt{a(\tau)} \left[1 + i \frac{1}{k(2\tau_i - \tau)} \right] \exp[-ik(\tau-\tau_2)] \tag{26}$$

and its H.c.

Any other solution of Eqs. (9) can be expressed as a linear combination of the solutions (26) and, indeed, this is what happens in our case. Expanding the exponentials

$$\exp[\pm i(1/k(2\tau_1-\tau) - 1/k(2\tau_1-\tau_2))]$$

that appear in the expressions for α and β, and taking (10) into account, we find after some algebra that

$$h(\tau) \sim \sqrt{a} \left[\left[1 + \frac{i}{k(2\tau_1-\tau)} \right] \exp[-ik(\tau-\tau_2)] - \frac{i}{k(2\tau_1-\tau_2)} \left[1 + \frac{i}{k(2\tau_1-\tau)} \right] \exp[-ik(\tau-\tau_2)] \right.$$
$$- \frac{i}{2k^2(2\tau_1-\tau_2)^2} \left[1 + \frac{i}{k(2\tau_1-\tau)} \right] \exp[-ik(\tau-\tau_2)]$$
$$\left. - \frac{1}{2k^2(2\tau_1-\tau_2)^2} \left[1 - \frac{i}{k(2\tau_1-\tau)} \right] \exp[+ik(\tau-\tau_2)] + O(1/(k\tau)^5) \right] , \tag{27}$$

the terms in $1/(k\tau)^4$ canceling among themselves. Why and how does such a series appear? Where do the terms of higher order in $1/k\tau$ come from? We may try to understand it in the following way. Notice that the solution (26) does not reduce, at the finite instant of time $\tau = \tau_2 \equiv \tau_3$, to the form $h(\tau_2) \sim \sqrt{a(\tau_2)}$, as it should, given the form (8b) for $h(\tau)$ and the initial conditions defined in (10). In order to have $h(\tau_2) \sim \sqrt{a(\tau_2)}$, we see that we need to subtract from (26) a term $\sqrt{a(\tau)}\, i/k(2\tau_1-\tau_2)$ giving to (26) the form

$$h(\tau)\sim\sqrt{a}\left[1+\frac{i}{k(2\tau_1-\tau)}-\frac{i}{k(2\tau_1-\tau_2)}\right]\times\exp[-ik(\tau-\tau_2)]\ . \tag{28}$$

But now (28) is no longer a solution of Eq. (9); to regain a solution we have to add a further term, which will be of order $1/(k\tau)^2$.

$$h(\tau)\sim\sqrt{a}\left[1+\frac{i}{k(2\tau_1-\tau)}-\frac{i}{k(2\tau_1-\tau_2)}-\frac{i^2}{k^2(2\tau_1-\tau_2)(2\tau_1-\tau)}\right]\times\exp[-ik(\tau-\tau_2)]\ , \tag{29}$$

which, although again a solution of (9), does not reduce to $h(\tau_2)\sim\sqrt{a(\tau_2)}$; this in turn means that a new term, of still higher order in $1/k\tau$, must be added. In this way a series in $1/k\tau$ is constructed, which has its origin in the fact that we chose our no-particle initial state at a finite time $\tau_2\equiv\tau_3$ in the past. [Of course, this procedure of adding terms does not lead to a unique way to construct such a series for $h(\tau)$—for example, there are important phase factors that might easily be left out; moreover we would still need to separate the two series for $\alpha(\tau)$ and $\beta(\tau)$, a difficult job to perform without direct appeal to Eqs. (16) and (17).] However, as we show in the next section, for any reasonable inflationary model the most important contributions come from the first terms in Eqs. (25) and (27).

III. THE POWER SPECTRUM

We begin this section by showing that, for the usual parameters defining the inflationary epoch, the first terms in expressions (25a) and (25b) are indeed the most important ones. We begin with the equation

$$t=a_2\int_{\tau_i}^{\tau}(2\tau_1-\tau_2)/(2\tau_1-\tau')d\tau'$$
$$=a_2(2\tau_1-\tau_2)\ln\frac{2\tau_1-\tau_i}{2\tau_1-\tau} \tag{30}$$

relating the comoving time t to the conformal time τ, where

$$a_2(2\tau_1-\tau_2)=\left[\frac{8\pi G}{3}\rho_{\text{vac}}\right]^{-1/2};$$

the initial value τ_i can be made equal to τ_2, without loss of generality, and in chaotic models of inflation we can still take, if we wish, $\tau_2\simeq\tau_{\text{Pl}}$. To be definite, we assume a minimum model where inflation lasted for about 67 e-folds:

$$\frac{a(\tau=\tau_1)}{a(\tau=\tau_2)}=\frac{a_1}{a_2}\simeq e^{67}\simeq10^{29}\ , \tag{31}$$

and also that

$$\frac{a(\tau_0)}{a(\tau_1)}=\frac{a_0}{a_1}\simeq10^{32}\ , \tag{32}$$

where we take τ_0 as being the present time, for reasons to be explained below. From relations (30)–(32) and Eq. (2a), we can see that $\tau_2\simeq-10^{29}\tau_1$ and that any τ, separated from τ_1 by several e-folds, will be negative and will have an absolute value several orders of magnitude larger than τ_1. The result is that our solution $h(\tau)$ in (27) becomes practically equal to Allen's solution and our initial state is very close indeed to the infinite-order adiabatic vacuum state, as required by our considerations at the end of Sec I. Then we have that, at $\tau=\tau_1$, β is reduced to the expression

$$\beta(\tau=\tau_1)\simeq\frac{\exp[-2ik(\tau_1-\tau_2)]}{2k^2\tau_1^2}\ , \tag{33}$$

in agreement with Allen's result [16], apart from a phase factor, giving

$$|\beta(\tau_1)|^2\simeq\frac{1}{4k^4\tau_1^4}=\frac{(8\pi G\rho_{\text{vac}}/3)^2}{4\omega_0^4(a_0/a_1)^4}\ , \tag{34}$$

the index 0 again referring to the present time.

In the limit of a very short transition to the radiation period and with the choice made for W, giving $h(\tau)$ in the appropriate form for the radiation phase, we can take our calculation of β across τ_1, to $\tau_1+\epsilon$. Equation (34) may then be interpreted as giving the number of gravitons at the beginning of the radiation phase. Had the phase transition been a slow one, lasting a finite interval of time $\Delta\tau$, as investigated by Ford in Ref. [15], our calculations would have to include such an interval of time and β would have to be calculated at the end of the transition period. In fact, in this case, to call it a phase transition would be a slight misnomer, such a period being an integral part of the expansion itself. This is seen in Fig. 1 of Ref. [15], where the dilution in the number density of created particles (gravitons) is slightly compensated by the creation of new particles (gravitons), as shown by the small increase in na^3. An entirely different situation is the one where a new physical phenomena, such as a quantum tunneling effect, takes place; in this case $\beta(\tau_1)$ would have to be composed in the usual way with the β corresponding to the phase transition to get β_{final}.

Multiplying (34) by the density of states $\omega^2d\omega/2\pi^2c^3$ and summing over the two polarization states, we find, taking as a reference a frequency today of 10^3 rad/s,

$$P(\omega_0)d\omega_0\simeq0.83\times10^{-16}(\rho_{\text{vac}}/\rho_{\text{Pl}})^2(10^3/\omega_0)\times(10^{32}a_1/a_0)^4d\omega_0\ \text{erg/cm}^3\ . \tag{35}$$

During the radiation phase no further gravitons will be produced and we shall neglect the gravitons created during the dust-dominated phase, a reasonable approximation to make (with $S\sim a''/a\sim1/\tau^2$, late times give a small S and, thus a $\beta\approx0$). The next gravitons we have to take into account are those produced by the transition between the radiation- and the dust-dominated phases. Following Allen's considerations on the adiabatic theorem [16] (see also Ref. [15]), we do not expect effects from these gravitons on the power spectrum, for frequencies above 10^{-15} rad/s, these effects being then restricted to the frequency interval $10^{-17}\le\omega_0\le10^{-15}$ rad/s, where the lower-frequency cutoff comes from the present horizon of $\simeq 10^{28}$ cm. For frequencies today $\omega_0 \equiv \omega(\tau_0)$, such that

$$ck\tau_1 = \left[\frac{8\pi}{3} G\rho_{\rm vac}\right]^{-1/2} \omega_0 \frac{a_0}{a_1} > 1 ,$$

which corresponds to the high-frequency tail of Fig. 2 in the first part of Ref. [16], the power spectrum is well approximated by (35).

We see that $P(\omega_0)$ varies with the eighth power of the mass scale defining the inflationary era [$\rho_{\rm vac} \sim M^4$ and $\rho_{\rm vac}$ appears squared in (35)], giving 32 orders of magnitude difference between an inflationary model characterized by a mass scale of the order of the Planck scale $M_{\rm Pl} \sim 10^{19}$ GeV, like the chaotic models, and a model characterized by the grand unified theory (GUT) scale $M_X \sim 10^{15}$ GeV. This may be the difference separating the possibility of detecting or not detecting a cosmological gravitational-wave background. When it is detected, using expressions such as (35) we will be able to fix some of the parameters that characterize inflation. Constraints are also obtained from the cosmic microwave background (CMB) radiation and from the cosmological nucleosynthesis. To derive these consequences the low-frequency part ($k\tau < 1$) of the gravitational-wave background is necessary to calculate the integrated power spectrum. This is now under investigation and will be the subject of a future publication.

To conclude, we have derived the Bogoliubov coefficients α and β as functions of time, for the case of an inflationary expanding phase of the Universe [Eqs. (25a) and (25b)], using the formalism derived by Parker; this allowed us to calculate the gravitons produced and their power spectrum, in the high-frequency limit, Eq. (35). The connection between our solution and the known solutions was also established.

Note added. After this work was already completed, I came across a recent and excellent paper by Márcio Maia (Sussex) [22] also dealing with relic gravitons and their spectrum, albeit by an entirely different method than the one used in the present paper.

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