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Optimal solution to the inverse problem for the gravitational wave signal of a coalescing compact binary

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Coalescing compact binaries are the most promising sources of the gravitational waves to be detected by planned long-arm laser interferometers. Estimation of the parameters of such a binary, such as the masses of its members, distance to the binary, and their distribution in the sky, will provide a wealth of astrophysical information. An important problem, called the *inverse problem*, is to determine the astrophysically interesting parameters of the *binary* (such as the distance to the binary and its position in the sky) from the parameters of the *detector's response function* to the gravitational wave signal of the binary. To solve the problem one needs the network of at least three detectors. We present the solution of the problem that gives the maximum likelihood estimators of the astrophysically interesting parameters with the least possible errors. This involves solving a complicated set of algebraic equations. We find that for the network of the three planned advanced LIGO and/or VIRGO detectors and for a binary consisting of two neutron stars of 1.4 solar masses each at a distance of 100 Mpc we can expect to determine its distance to an accuracy of the order of 10%, its mass parameter to an accuracy of the order of 10^{-6} solar masses, and its position in the sky to an accuracy of the order of 10^{-4} sr.

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I. INTRODUCTION

We can expect that at the turn of the century we shall have a network of three laser interferometric gravitational wave detectors as a result of the successful accomplishment of the Laser Interferometric Gravitational Wave Observatory (LIGO) [1] and the VIRGO [2] projects. Once the instruments have achieved their first objective of detecting the gravitational waves, then the whole network can serve as a powerful astronomical observatory providing information complementary to that obtained by means of electromagnetic wave observations [3,4]. The most promising source to be detected by the planned long-arm laser interferometers is the gravitational wave signal from a compact coalescing binary system [5]. It has been shown by Schutz [3] that to determine the distance to the binary and its position in the sky one needs a network of at least three detectors. An important problem called the *inverse problem* is to determine from the parameters of the response function obtained by means of linear filtering of the data in each detector the above astrophysically interesting parameters. The problem of detection of the gravitational wave signal from a coalescing binary and estimation of its parameters for a single detector has been studied by a number of authors [6–10]. In this paper we give two algorithms to find the maximum likelihood estimators of the parameters of the binary system for the case of the three detectors. We take into account the relations between the parameters of the gravitational wave signal estimated in each detector. This additional information about the parameters of the three signals improves the accuracy of the determination of the parameters of the binary. Our method is optimal within the maximum likelihood method. This does not exclude the possibility that by some other method one can find better estimators, for example, using *nonlinear filtering* proposed by Davis [11]. The near-optimal solution for the case of the general burst signals has already been found by Gürsel and Tinto [12].

The plan of the paper is as follows. In Sec. II we give a detailed formula for the response function of the laser interferometer to the gravitational wave signal from a coalescing binary as well as its Fourier transform in the stationary phase approximation. For the network of the three detectors we show the relation between the position of the source in the sky and the time delays in the arrival of the wave in the detectors. We also briefly review the linear filtering of the data and the parameter estimation theory. In Sec. III we present two methods of the optimal solution of the inverse problem and show that the two methods give the same errors in the estimators of the parameters of the binary. In Sec. IV we give a detailed numerical analysis of the accuracy of the estimation of the parameters of the binary for the planned LIGO and/or VIRGO network of detectors. Our conclusions are presented in Sec. V.

II. PRELIMINARIES

A. Single-detector response function

The single-detector response function has been studied by several authors [5,8,13–17]. Our treatment follows that of Schutz and Tinto [13]. Let the orthogonal Cartesian coordinates (X,Y,Z) be connected with a weak plane gravitational wave traveling in the $+Z$ direction. Then the metric perturbation $h_{\mu\nu}$ ($g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $|h_{\mu\nu}\ll 1$) may be written, in matrix form, as [assuming we work in the transverse traceless (TT) gauge]

$$(h_{\mu\nu})=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & H_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

where h_{+} and $h_{\times}$ are the two independent wave's polarizations. (Since we are working in the TT gauge, the metric perturbation $h_{\mu\nu}$ equals its trace reverse $\bar{h}_{\mu\nu}$.) Now, following Schutz and Tinto [13], let us introduce orthogonal Cartesian coordinates $(\tilde{x},\tilde{y},\tilde{z})$ in the detector's proper reference frame (we assume that the detector is nearly at rest in the TT coordinate system of the gravitational wave). The $(\tilde{x},\tilde{y})$ plane contains the detector, and it is tangent to the surface of the Earth; the $\tilde{x}$ axis bisects the right angle between the detector's arms. The direction of the $\tilde{y}$ axis is chosen in such a way that the $(\tilde{x},\tilde{y},\tilde{z})$ coordinate system is a right-handed one with the $\tilde{z}$ axis pointing outside the surface of the Earth. In the detector's reference frame, the three-dimensional matrix $\tilde{H}$ of the metric perturbation is given by

$$\tilde{H}=AHA^{T}, \tag{2}$$

where A is the three-dimensional orthogonal matrix transformation from (X,Y,Z) to $(\tilde{x},\tilde{y},\tilde{z})$ coordinates [the superscript T in Eq. (2) denotes the matrix transposition] and H stands for the three-dimensional matrix built up from the space-space components of the four-dimensional matrix $(h_{\mu\nu})$ from Eq. (1):

$$H=\begin{bmatrix} h_{+} & h_{\times} & 0 \\ h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

In the proper reference frame of a central mass of an interferometer, an incident gravitational wave produces tiny oscillations of end masses. If we denote by $\mathbf{r}_0$ the radius vector of any end mass with respect to the central mass (the subscript 0 denotes that there is no gravitational wave), then the oscillatory change $\delta\mathbf{r}$ of the $\mathbf{r}_0$ produced by the wave is

$$\delta\mathbf{r}=\tfrac{1}{2}\tilde{H}\mathbf{r}_0 . \tag{4}$$

Equation (4) can be derived from the equation of geodesic deviation [13] and is valid only if the size $|\mathbf{r}_0|$ of the detector is much smaller than the reduced wavelength $\lambda/2\pi$ of the gravitational wave.

The relative length change of one arm is defined as

$$\frac{\delta r}{r_0}=\frac{|\mathbf{r}_0+\delta\mathbf{r}|-|\mathbf{r}_0|}{|\mathbf{r}_0|} .$$

Using Eq. (4) and dropping terms of order higher than one in h_{+} and $h_{\times}$, the above equation can be written as

$$\frac{\delta r}{r_0}=\frac{1}{2}\mathbf{n}\cdot\tilde{H}\mathbf{n} ,$$

where $\mathbf{n}=\mathbf{r}_0/|\mathbf{r}_0|$ is a unit vector parallel to an arm and the centered dot stands for the standard scalar product in Cartesian space $\mathbf{R}^3$. The difference between the relative length changes of the two interferometer arms is thus equal to

$$\Delta\left[\frac{\delta r}{r_0}\right]=\frac{1}{2}\mathbf{n}_1\cdot\tilde{H}\mathbf{n}_1-\frac{1}{2}\mathbf{n}_2\cdot\tilde{H}\mathbf{n}_2 , \tag{5}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ denote the unit vectors parallel to arms Nos. 1 and 2, respectively. In the detector's coordinate system described at the beginning of this subsection, obviously

$$\mathbf{n}_1=\left[\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right], \quad \mathbf{n}_2=\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0\right], \tag{6}$$

where we assumed that arm No. 1 lies in the first quarter of the $(\tilde{x},\tilde{y})$ plane. Taking into account Eqs. (2), (3), and (6), from Eq. (5) we obtain

$$\Delta\left[\frac{\delta r}{r_0}\right]=(A_{11}A_{21}-A_{12}A_{22})h_{+} +(A_{11}A_{22}+A_{12}A_{21})h_{\times} , \tag{7}$$

where A_{ij} is the component of matrix A occupying its ith row and jth column. Let us introduce the functions F_{+} and $F_{\times}$ determined as

$$F_{+}=A_{11}A_{21}-A_{12}A_{22} ,$$
$$F_{\times}=A_{11}A_{22}+A_{12}A_{21} . \tag{8}$$

The dimensionless quantity $s(t)=\Delta(\delta r(t)/r_0)$ is called the *detector response function* and is thus a linear combination of the two independent wave's polarizations h_{+} and $h_{\times}$:

$$s(t)=F_{+}h_{+}(t)+F_{\times}h_{\times}(t) . \tag{9}$$

The orientation-dependent functions F_{+} and $F_{\times}$ are called the *beam-pattern functions.*

B. Beam-pattern functions

In the case of a network of three detectors, one needs a common orthogonal Cartesian coordinate system (x,y,z) with respect to which a position of a source in the sky will be determined. This coordinate system is taken from the paper by Gürsel and Tinto [12], and it is constructed as follows. The (x,y) plane of the system coincides with the plane defined by the positions of the detectors. The center of the system is chosen to coincide with the position of detector No. 1. The positive x semiaxis passes through the positions of detectors Nos. 1 and 2. The direction of the y axis is chosen in such a way that the y component of the vector connecting detector No. 1 with detector No. 3 is positive in this coordinate system, and the direction of the z axis is chosen to form a right-handed coordinate system. If we introduce the usual Euler angles θ, ϕ, and ψ (where θ and ϕ give the incoming direction of the wave and ψ is the angle between the node direction and the X axis of the wave coordinate system), then the orthogonal matrix transformation B from the wave's coordinate system (X,Y,Z) to the network coordinate system (x,y,z) is equal to [18]

$$B=\begin{bmatrix} \cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi & -(\sin\psi\cos\phi+\cos\theta\sin\phi\cos\psi) & \sin\theta\sin\phi \\ \cos\psi\sin\phi+\cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi+\cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} . \tag{10}$$

The position of a detector on the surface of the Earth will be determined in the (x_E,y_E,z_E) coordinate system the center of which coincides with the center of the Earth. The x_E axis lies in the direction of the line passing through the center of the Earth and the intersection of the meridian passing through Greenwich, England, and the equator. The z_E axis is chosen to lie in the direction of the line passing through the center of the Earth and the North Pole. The y_E axis is chosen to form a right-handed Cartesian coordinate system with the x_E and z_E axes. Let $\mathbf{r}_I$ $(I=1,2,3)$ denote the vector connecting the center of the Earth with the position of the Ith detector, and let β_I and γ_I be the latitude and the longitude, respectively, of the Ith detector. Then the unit vector in the direction of the vector $\mathbf{r}_1$ has, with respect to the (x_E,y_E,z_E) coordinate system, the coordinates

$$\mathbf{d}_I=(\cos\beta_I\cos\gamma_I,\cos\beta_I\sin\gamma_I,\sin\beta_I) . \tag{11}$$

Let us define the sequence of the unit vectors [19]:

$$\mathbf{f}=\frac{\mathbf{d}_2-\mathbf{d}_1}{|\mathbf{d}_2-\mathbf{d}_1|} , \quad \mathbf{g}=\frac{\mathbf{d}_3-\mathbf{d}_1}{|\mathbf{d}_3-\mathbf{d}_1|} ,$$

$$\mathbf{o}=\frac{\mathbf{f}\times\mathbf{g}}{|\mathbf{f}\times\mathbf{g}|} , \quad \mathbf{p}=\mathbf{o}\times\mathbf{f} .$$

The orthogonal matrix transformation C from the network coordinate system (x,y,z) to the Earth's coordinate system (x_E,y_E,z_E) can be written as

$$C=\begin{bmatrix} f_{x_E} & p_{x_E} & o_{x_E} \\ f_{y_E} & p_{y_E} & o_{y_E} \\ f_{z_E} & p_{z_E} & o_{z_E} \end{bmatrix} , \tag{12}$$

where, e.g., f_{x_E} denotes the xth component of the vector $\mathbf{f}$ with respect to (x_E,y_E,z_E) coordinates.

Finally, in each detector's reference frame, we introduce the orthogonal coordinate system $(\tilde{x}_I,\tilde{y}_I,\tilde{z}_I)$ $(I=1,2,3)$ as described in the previous subsection. To fix an orientation of a detector with respect to the local geographical directions, we must introduce a new angle α_I, which is the angle between the local east-west direction and the bisector of the arms of the Ith detector (i.e., the $\tilde{x}_I$ axis). The local east-west direction is oriented from the west to the east, and the orientation of the angle α_I is anticlockwise [20]. The orthogonal matrix transformation D_I from (x_E,y_E,z_E) to $(\tilde{x}_I,\tilde{y}_I,\tilde{z}_I)$ coordinates is of the form [13]

$$D_I=\begin{bmatrix} -(\cos\alpha_I\sin\gamma_I+\sin\alpha_I\sin\beta_I\cos\gamma_I) & \cos\alpha_I\cos\gamma_I-\sin\alpha_I\sin\beta_I\sin\gamma_I & \sin\alpha_I\cos\beta_I \\ \sin\alpha_I\sin\gamma_I-\cos\alpha_I\sin\beta_I\cos\gamma_I & -(\sin\alpha_I\cos\gamma_I+\cos\alpha_I\sin\beta_I\sin\gamma_I) & \cos\alpha_I\cos\gamma_I \\ \cos\beta_I\cos\gamma_I & \cos\beta_I\sin\gamma_I & \sin\beta_I \end{bmatrix} . \tag{13}$$

Now we are able to determine the three orthogonal matrix transformations A_I from (X,Y,Z) to $(\tilde{x}_I,\tilde{y}_I,\tilde{z}_I)$ coordinates as

$$A_I=D_ICB . \tag{14}$$

The beam-pattern functions F_{I+} and $F_{I\times}$ are defined, in accordance with Eqs. (8), as

$$\begin{aligned} F_{I+}&=(A_I)_{11}(A_I)_{21}-(A_I)_{12}(A_I)_{22} , \\ F_{I\times}&=(A_I)_{11}(A_I)_{22}+(A_I)_{12}(A_I)_{21} , \end{aligned} \tag{15}$$

and are the complicated functions of the three Euler angles θ, ϕ, and ψ and the nine angles α_I, β_I, and γ_I.

The functions F_{I+} and $F_{I\times}$ have some symmetric properties with respect to the angle ψ. Taking into account that the matrix A_I is the product of the Euler matrix B [Eq. (10)] and another matrix (equal to D_IC), one can show, using definitions (15), that they are a linear combination of $\sin 2\psi$ and $\cos 2\psi$:

$$\begin{aligned} F_{I+}&=a_I\cos 2\psi+b_I\sin 2\psi , \\ F_{I\times}&=b_I\cos 2\psi-a_I\sin 2\psi , \end{aligned} \tag{16}$$

where a_I and b_I do not depend on the angle ψ. Equations

(16) imply the symmetries

$$F_{I+}(\psi+\pi)=F_{I+}(\psi) ,\quad F_{I\times}(\psi+\pi)=F_{I\times}(\psi) , \tag{17}$$

$$F_{I+}\left[\psi+\frac{\pi}{2}\right]=-F_{I+}(\psi) ,\quad F_{I\times}\left[\psi+\frac{\pi}{2}\right]=-F_{I\times}(\psi) , \tag{18}$$

$$F_{I+}\left[\psi+\frac{\pi}{4}\right]=F_{I\times}(\psi) ,\quad F_{I\times}\left[\psi+\frac{\pi}{4}\right]=-F_{I+}(\psi) . \tag{19}$$

C. Relation between the source location and the time delays

The network of the three wideband detectors provides two independent relative time delays which in turn determine two possible source directions. In the network coordinate system (x,y,z), these directions are mirror images of each other with respect to the (x,y) plane, which can be demonstrated as follows. For two detectors the gravitational wave signal hitting the Earth gives the same value of time delay for many different incoming directions. All these directions lie on the surface of a cone with an axis which coincides with the line connecting the detectors. The value of the time delay determines the angle of the cone. For three detectors we have two such cones which, in general, intersect each other along two half lines. These half lines are mirror images of each other with respect to the plane in which the axes of both cones lie, that is, with respect to the (x,y) plane.

Let $\mathbf{n}$ be the unit vector pointing at the position of the gravitational wave source in the sky. Then, in the network coordinate system (x,y,z),

$$\mathbf{n}=(-\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ -\cos\theta) . \tag{20}$$

We denote by τ_{12} and τ_{13} the two independent time delays between detectors 1,2 and 1,3, respectively. Then we have

$$\tau_{12}=t_{a_1}-t_{a_2}=\mathbf{n}\cdot\mathbf{r}_{12}/c ,\quad \tau_{13}=t_{a_1}-t_{a_3}=\mathbf{n}\cdot\mathbf{r}_{13}/c , \tag{21}$$

where t_{a_1}, t_{a_2}, and t_{a_3} are the times of arrival of the signal in each of the detectors, $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$ are the vectors connecting detector 1 with detectors 2 and 3, respectively, and c stands for the speed of light. In network coordinate system we have

$$\mathbf{r}_{12}/c=(a,0,0) ,\quad \mathbf{r}_{13}/c=(b_1,b_2,0) , \tag{22}$$

where in accordance with the definition of the (x,y,z) coordinate system always $a>0$ and $b_2>0$ (cf. Sec. II B). Of course,

$$\mathbf{r}_{12}=\mathbf{r}_2-\mathbf{r}_1 ,\quad \mathbf{r}_{13}=\mathbf{r}_3-\mathbf{r}_1 , \tag{23}$$

where $\mathbf{r}_I$ is the vector connecting the center of the Earth with the Ith detector. If $\mathbf{d}_I$ denotes the unit vector in the direction of the vector $\mathbf{r}_I$, then

$$\mathbf{r}_I=R_E\mathbf{d}_I , \tag{24}$$

where R_E stands for the radius of the Earth. Combining Eqs. (23) and (24) together with Eqs. (22), we easily obtain formulas for the coefficients a, b_1, and b_2:

$$a=\frac{R_E}{c}|\mathbf{d}_2-\mathbf{d}_1| ,\quad b_1=\left[\frac{R_E}{c}\right]^2\frac{1}{a}(\mathbf{d}_2-\mathbf{d}_1)\cdot(\mathbf{d}_3-\mathbf{d}_1) ,\quad b_2=\left[\left[\frac{R_E}{c}\right]^2|\mathbf{d}_3-\mathbf{d}_1|^2-b_1^2\right]^{1/2} . \tag{25}$$

Substituting Eqs. (22) into Eqs. (21), we obtain equations from which we compute cosine and sine functions of the angles θ and ϕ:

$$\sin\theta=\frac{1}{ab_2}\sqrt{\Delta} ,\quad \cos\theta=\pm\frac{1}{ab_2}\sqrt{(ab_2)^2-\Delta} ,\quad \sin\phi=-\frac{b_2\tau_{12}}{\sqrt{\Delta}} ,\quad \cos\phi=\frac{a\tau_{13}-b_1\tau_{12}}{\sqrt{\Delta}} , \tag{26}$$

where

$$\Delta=(b_1\tau_{12}-a\tau_{13})^2+(b_2\tau_{12})^2 .$$

The above formulas determine the angle ϕ uniquely, while for the angle θ we have two possible values.

D. Response functions to the gravitational wave signal from a coalescing compact binary system and their Fourier transforms

We restrict ourselves to the Newtonian regime in which the gravitational waveform is calculated using the quadrupole formula. Moreover, we assume that when the gravitational wave signal enters the observational window of the laser interferometer (frequency above 10 Hz) the orbit of a binary is nearly circular due to radiation reaction forces.

The dimensionless response s_I $(I=1,2,3)$ of the Ith receiver is, as usual, the linear combination of the two wave's polarizations h_{I+} and $h_{I\times}$:

$$s_I(t)=F_{I+}h_{I+}(t)+F_{I\times}h_{I\times}(t) ,\quad t\geq t_{a_I} . \tag{27}$$

To compute the functions h_{I+} and $h_{I\times}$, one must first fix the orientation of the X and Y axes of the wave coordinate system (X,Y,Z) from Sec. II A with respect to the binary (the Z axis is along the direction of propagation of the gravitational wave). To do this let us define the binary coordinate system (x_S,y_S,z_S). The z_S axis is along the binary orbital angular momentum vector. The x_S axis is the projection of the Z axis onto the orbital plane (if the Z and z_S axes coincide, then there is no preferred direction for the x_S axis). The y_S axis is chosen to form a right-handed coordinate system. Now we can define the X and Y axes to be simply projections of the x_S and y_S axes, respectively, onto the plane perpendicular to the Z axis (i.e., onto the sky). Let us note that the y_S and Y axes of such defined systems coincide (cf. Sec. III A of [8]).

In the (X,Y,Z) coordinate system defined above, h_{I+} and $h_{I\times}$ are the following functions of time [5,8,9]:

$$h_{I+}(t)=\frac{4\pi^{2/3}G^{5/3}}{c^4}\frac{\mathcal{M}^{5/3}}{R}[f(t;t_{a_I},\mathcal{M})]^{2/3}\frac{1+\cos^2\epsilon}{2}\cos[g(t;t_{a_I},\mathcal{M})+\delta] ,$$
$$h_{I\times}(t)=\frac{4\pi^{2/3}G^{5/3}}{c^4}\frac{\mathcal{M}^{5/3}}{R}[f(t;t_{a_I},\mathcal{M})]^{2/3}\cos\epsilon\sin[g(t;t_{a_I},\mathcal{M})+\delta] , \tag{28}$$

where

$$g(t;t_{a_I},\mathcal{M})=2\pi\int_{t_{a_I}}^{t} f(t';t_{a_I},\mathcal{M})dt' . \tag{29}$$

R is the luminosity distance from the Earth to the binary, $\mathcal{M}$ is the *chirp mass* defined by $\mathcal{M}=\mu^{3/5}m^{2/5}$, where $m=m_1+m_2$ is the total mass of the binary and $\mu=m_1m_2/m$ is its reduced mass. The angle ϵ is the angle between the line of sight toward the earth and the binary orbital angular momentum vector; δ is an initial phase of the orbital motion. The function $f=f(t;t_{a_I},\mathcal{M})$ is the frequency of the gravitational wave (twice the orbital frequency) and is given by

$$f(t;t_{a_I},\mathcal{M})=\left[\frac{c^3}{G}\right]^{5/8}\times\frac{1}{\pi}\left[\frac{5}{256}\frac{1}{\mathcal{M}^{5/3}}\frac{1}{t_0(t_{a_I},\mathcal{M})-t}\right]^{3/8} , \tag{30}$$

where t_0 is the time at which coalescence occurs. The time t_{a_I} of arrival of the signal is defined in such a way that $f(t_{a_I};t_{a_I},\mathcal{M})=f_i$, where f_i is the initial frequency of the wave. Substituting t_{a_I} instead of t into Eq. (30), we easily obtain that the coalescence time t_0 is the following function of the parameters t_{a_I} and $\mathcal{M}$:

$$t_0(t_{a_I},\mathcal{M})=t_{a_I}+\frac{5}{256}\left[\frac{c^3}{G}\right]^{5/3}\frac{1}{(\pi f_i)^{8/3}}\frac{1}{\mathcal{M}^{5/3}} . \tag{31}$$

Taking into account the time delays τ_{12} and τ_{13} between the detectors 1,2 and 1,3, respectively [cf. Eq. (21)], the response functions s_I can also be written in the form

$$s_1(t)=F_{1+}h_{1+}(t)+F_{1\times}h_{1\times}(t) ,$$
$$s_2(t)=F_{2+}h_{1+}(t-\tau_{12})+F_{2\times}h_{1\times}(t-\tau_{12}) , \tag{32}$$
$$s_3(t)=F_{3+}h_{1+}(t-\tau_{13})+F_{3\times}h_{1\times}(t-\tau_{13}) .$$

The very form of the detectors' response functions [cf. Eqs. (27) and (28)] together with symmetries (17)–(19) of the beam-pattern functions F_{I+} and $F_{I\times}$ entails that the detectors' response functions do not change under the transformations ($\psi\to\psi+\pi$, $\delta\to\delta$) and $\psi\to\psi+\pi/2$, $\delta\to\delta+\pi$). Therefore, from gravitational wave observations, we are only able to determine the angle ψ up to $\pi/2$ and the angle δ up to π.

Let us rewrite the response function of the Ith detector in a slightly different form:

$$s_I(t)=\mathcal{A}_I[f(t;t_{a_I},\mathcal{M})]^{2/3}\sin[g(t;t_{a_I},\mathcal{M})+\xi_I] , \quad t\geq t_{a_I} , \tag{33}$$

where

$$\mathcal{A}_I=\frac{4\pi^{2/3}G^{5/3}}{c^4}\frac{\mathcal{M}^{5/3}}{R}\left[\left[F_{I+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{I\times}\cos\epsilon)^2\right]^{1/2} , \tag{34}$$

and where the initial phases ξ_I of the signals are given by the equations

$$\sin(\xi_I-\delta)=\frac{F_{I+}\frac{1+\cos^2\epsilon}{2}}{\left[\left[F_{I+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{I\times}\cos\epsilon)^2\right]^{1/2}} ,$$
$$\cos(\xi_I-\delta)=\frac{F_{I\times}\cos\epsilon}{\left[\left[F_{I+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{I\times}\cos\epsilon)^2\right]^{1/2}} . \tag{35}$$

The Fourier transform $\tilde{s}_I(f)$ of the function $s_I(t)$ can be calculated to a good accuracy by means of the stationary phase approximation [5,21,8,9]:

$$\tilde{s}_I(f)=k_If^{-7/6}\exp[i\Phi(f)] , \quad f\geq f_i , \tag{36}$$

where

$$k_I=\frac{1}{2\pi^{2/3}}\left[\frac{5}{6}\right]^{1/2}\frac{G^{5/6}}{c^{3/2}}\frac{\mathcal{M}^{5/6}}{R}\left[\left[F_{I+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{I\times}\cos\epsilon)^2\right]^{1/2} ,$$

$$\Phi(f)=-2\pi ft_{a_I}+\xi_I-\frac{\pi}{4}-\frac{1}{\mathcal{M}^{5/3}}\Psi(f) , \tag{37}$$

$$\Psi(f)=\frac{1}{128}\left[\frac{c^3}{G}\right]^{5/3}\left[\frac{5\pi f}{(\pi f_i)^{8/3}}+\frac{3}{(\pi f)^{5/3}}-\frac{8}{(\pi f_i)^{5/3}}\right] .$$

E. Linear filtering of the data and the parameter estimation theory

In general we know the form of the signal as a function of a number of parameters. For example, in the case of the signal from a coalescing binary the unknown parameters can be the time of arrival of the signal, the chirp mass, the amplitude, and the initial phase of the waveform. The classical estimation method proposed here is the maximum likelihood estimation [11,22–26]. In this subsection we summarize the maximum likelihood method for the case of the Gaussian noise. Our treatment follows Ref. [23]. A different treatment is given in Ref. [7].

Let $\boldsymbol{\theta}=(\theta_1,\theta_2,\dots,\theta_m)$ be the set of unknown parameters of the signal $s(t;\boldsymbol{\theta})$. As in the case of the completely known signal, we can consider two probability density functions $p_1[x(t);\boldsymbol{\theta}]$ and $p_0[x(t)]$ [where $x(t)$ is the stochastic process in question] depending on whether the signal is present or absent; i.e., when $x(t)=s(t;\boldsymbol{\theta})+n(t)$ [where $n(t)$ is the noise in the detector], we have p_1, and when $x(t)=n(t)$, we have p_0. Then we form the likelihood ratio $\Lambda[x(t);\boldsymbol{\theta}]=p_1[x(t);\boldsymbol{\theta}]/p_0[x(t)]$. Maximum likelihood estimators (MLE's) $\hat{\boldsymbol{\theta}}$ are those values of the parameters $\boldsymbol{\theta}$ that maximize the likelihood ratio $\Lambda[x(t);\boldsymbol{\theta}]$. Thus the MLE's can be found by the solution of the set of simultaneous equations

$$\frac{\partial}{\partial\theta_i}\Lambda[x(t);\boldsymbol{\theta}]=0 . \tag{38}$$

In the case of the Gaussian noise, the logarithm of the likelihood ratio is given by [27]

$$\ln\Lambda[x(t);\boldsymbol{\theta}]=\int_0^T x(t)q(t;\boldsymbol{\theta})dt-\frac{1}{2}\int_0^T s(t;\boldsymbol{\theta})q(t;\boldsymbol{\theta})dt , \tag{39}$$

where $q(t;\boldsymbol{\theta})$ is the solution of the integral equation [28]

$$s(t;\boldsymbol{\theta})=\int_0^T K_N(t,u)q(u;\boldsymbol{\theta})du \tag{40}$$

and where we assume that we make observations over a certain interval of time $[0,T]$. $K_N(t,u)$ is the autocorrelation function of the noise in the detector.

The maximum likelihood estimators $\hat{\boldsymbol{\theta}}$ are random variables since they are functionals of the random variable $x(t)$ determined by the set of equations (38). Let Γ be the matrix the components of which are given by

$$\Gamma_{ij}=-E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln\Lambda[x(t);\boldsymbol{\theta}]\right] , \tag{41}$$

where E means the expectation value. Γ is called the Fisher information matrix. We say that the estimate $\hat{\boldsymbol{\theta}}$ of the set of parameters $\boldsymbol{\theta}$ is unbiased if the expectation value of $\hat{\boldsymbol{\theta}}$ equals the true values of the parameters: i.e.,

$$E(\hat{\boldsymbol{\theta}})=\boldsymbol{\theta} . \tag{42}$$

There is a general result called the Cramer-Rao inequality [22] which provides bounds on the covariances of unbiased maximum likelihood estimators:

$$E[(\theta_i-\hat{\theta}_i)(\theta_j-\hat{\theta}_j)]\geq(\Gamma^{-1})_{ij} . \tag{43}$$

It can be shown that the right-hand side of the Cramer-Rao inequality is the better approximation of the covariance matrix of the estimators of parameters of the signal the higher the signal-to-noise ratio [23]. Following Helstrom, we shall call the inverse of the Fisher matrix the covariance matrix. However, one should remember that the inverse of the Fisher matrix provides lower bounds on the accuracy of the estimation of the parameters and in practice the errors will always be greater. The question arises how big should the signal-to-noise ratio be in order that the inverse of the Fisher matrix approximates well the covariance matrix. In a paper by one of the authors [29], where simulations of the detection of the gravitational wave signal from a coalescing binary and estimation of its parameters were performed, it was found that at a signal-to-noise ratio of 25 the agreement between the variances of the estimators obtained from numerical simulations and the theoretical covariance matrix is very good, whereas at a signal-to-noise ratio of 8 the variances of the estimators from the simulations are distinctly greater than that given by the inverse of the Fisher matrix.

In the Gaussian case, the Fisher information matrix is given by

$$\Gamma_{ij}=\left.\frac{\partial^2 H(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)}{\partial\theta_{1i}\partial\theta_{2j}}\right|_{\boldsymbol{\theta}_1=\boldsymbol{\theta}_2=\boldsymbol{\theta}} , \tag{44}$$

where

$$H(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)=\int_0^T q(t;\boldsymbol{\theta}_1)s(t;\boldsymbol{\theta}_2)dt \tag{45}$$

and $q(t;\boldsymbol{\theta})$ is the solution of Eq. (40). The integral H is called the ambiguity function or the correlation integral. With the filter given by Eq. (40), the correlation integral is given by

$$H(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)=4\int_0^\infty \mathrm{Re}\left[\frac{\tilde{s}(f;\boldsymbol{\theta}_1)\tilde{s}^*(f;\boldsymbol{\theta}_2)}{S_n(f)}\right]df , \tag{46}$$

where $\tilde{s}$ is the Fourier transform of the signal ($\tilde{s}^*$ is the complex conjugate of $\tilde{s}$) and S_n is the *one-sided* spectral density of noise.

If we have three detectors such that their noises are independent, then the likelihood ratio for the network of detectors is given as a product of the likelihood ratios of each detector, and consequently if the noises in each detector are Gaussian, the logarithm of the likelihood ratio for the network of detectors is given by

$$\ln\Lambda[x_I(t);\boldsymbol{\theta}]=\sum_{I=1}^{3}\int_0^T x_I(t)q_I(t;\boldsymbol{\theta})dt-\frac{1}{2}\sum_{I=1}^{3}\int_0^T s_I(t;\boldsymbol{\theta})q_I(t;\boldsymbol{\theta})dt , \tag{47}$$

where $\boldsymbol{\theta}$ denotes the set of unknown parameters and $q_I(t;\boldsymbol{\theta}_I)$ is the solution of the integral equation

$$s_I(t;\boldsymbol{\theta})=\int_0^T K_N(t,u)q_I(u;\boldsymbol{\theta})du . \tag{48}$$

We assume that the autocorrelation function of the noise in all three detectors is the same. Hence, in the case of Gaussian noise, the MLE's can be obtained by passing the data in each detector through a bank of linear filters for suitably spaced values of the parameters and each of the filters being determined by solution of the integral Eq. (48). The values of the parameters that maximize the sum of the outputs from linear filters from all three detectors are maximum likelihood estimators.

The correlation integral for the network is given by

$$H(\theta_1,\theta_2)=\sum_{I=1}^{3}\int_0^T q_I(t;\theta_1)s_I(t;\theta_2)dt \quad (49)$$

and is a sum of correlation integrals H_I in each detector. With the filters given by Eq. (48) each correlation integral is given by

$$H_I(\theta_1,\theta_2)=4\int_0^\infty \mathrm{Re}\left[\frac{\tilde{s}_I(f;\theta_1)\tilde{s}_I^*(f,\theta_2)}{S_n(f)}\right]df . \quad (50)$$

III. OPTIMAL SOLUTION TO THE INVERSE PROBLEM

A. Estimation of the parameters k_I, ξ_I, t_{a_I}, and $\mathcal{M}_I$ in each detector

In this subsection we generalize the solution of the maximum likelihood equations for one detector given in Ref. [10] to the case of three detectors. Let us first write the signal $s_I(t)$ in each detector as $s_I(t)=\mathcal{A}_I h_I(t)$. Then one can write the filter as $q_I(t)=\mathcal{A}_I r_I(t)$, where $r_I(t)$ is the solution of the integral equation [cf. Eq. (40)]

$$h_I(t)=\int_0^T K_N(t,u)r_I(u)du . \quad (51)$$

From Eq. (39) the logarithmic likelihood ratio $\ln\Lambda_I$ in the Ith detector is now given by

$$\ln\Lambda_I[x_I(t);\mathcal{A}_I,\xi_I,t_{a_I},\mathcal{M}_I] =\mathcal{A}_I\int_0^T r_I(t)x_I(t)dt-\frac{1}{2}\mathcal{A}_I^2\int_0^T r_I(t)h_I(t)dt . \quad (52)$$

Solving the equation $\partial\ln\Lambda_I/\partial\mathcal{A}_I=0$, we get an explicit formula for the maximum likelihood estimate of the amplitude $\mathcal{A}_I$:

$$\hat{\mathcal{A}}_I=\frac{\int_0^T r_I(t)x_I(t)dt}{\int_0^T r_I(t)h_I(t)dt} . \quad (53)$$

Let us substitute the estimate $\hat{\mathcal{A}}_I$ for $\mathcal{A}_I$ in the likelihood ratio (52). We get

$$\ln\Lambda_I[x_I(t);\hat{\mathcal{A}}_I,\xi_I,t_{a_I},\mathcal{M}_I]=\frac{1}{2}\frac{[\int_0^T r_I(t)x_I(t)dt]^2}{\int_0^T r_I(t)h_I(t)dt} . \quad (54)$$

Now the estimates of the remaining parameters of the chirp (phase, time of arrival, and chirp mass) are found by maximizing the above functional. We shall next show that an explicit formula can be found also for the estimate of the phase ξ_I. Let us write $h_I(t)$ as

$$h_I(t)=h_{Is}(t)\cos\xi_I+h_{Ic}(t)\sin\xi_I , \quad (55)$$

where

$$h_{Is}(t)=[f(t;t_{a_I},\mathcal{M}_I)]^{2/3}\sin g(t;t_{a_I},\mathcal{M}_I) ,$$
$$h_{Ic}(t)=[f(t;t_{a_I},\mathcal{M}_I)]^{2/3}\cos g(t;t_{a_I},\mathcal{M}_I) . \quad (56)$$

The function g is defined in Eq. (29). Let r_{Ic} and r_{Is} be solutions of the integral equations

$$h_{Ic}(t)=\int_0^T K_N(t,u)r_{Ic}(u)du ,$$
$$h_{Is}(t)=\int_0^T K_N(t,u)r_{Is}(u)du . \quad (57)$$

Consequently, $r_I(t)$ can be written as

$$r_I(t)=r_{Ic}(t)\sin\xi_I+r_{Is}(t)\cos\xi_I . \quad (58)$$

Let us now consider the denominator in Eq. (54). It can be written, by virtue of Eqs. (55) and (58), as

$$\int_0^T h_I r_I dt=\left[\int_0^T h_{Ic}r_{Ic}dt\right]\sin^2\xi_I+\left[\int_0^T h_{Is}r_{Is}dt\right]\cos^2\xi_I + \left[\int_0^T h_{Ic}r_{Is}dt+\int_0^T h_{Is}r_{Ic}dt\right]\sin\xi_I\cos\xi_I . \quad (59)$$

It can be shown (see Ref. [10]) that the following equations are satisfied:

$$\int_0^T h_{Ic}r_{Ic}dt=\int_0^T h_{Is}r_{Is}dt , \quad (60)$$

$$\int_0^T h_{Ic}r_{Is}dt=\int_0^T h_{Is}r_{Ic}dt=0 . \quad (61)$$

If all the signals $s_I(t)$ are entirely contained in the time interval $[0,T]$, then all three integrals of the type (60) are independent of the times of arrival and are equal to each other:

$$\mathcal{C}=\int_0^T h_{1c}r_{1c}dt=\int_0^T h_{2c}r_{2c}dt=\int_0^T h_{3c}r_{3c}dt . \quad (62)$$

Moreover, $\mathcal{C}$ is positive. Taking Eqs. (60) and (61) into account, we easily show that the denominator given by Eq. (59) is equal to $\mathcal{C}$ defined in Eq. (62). Consequently, the solution of the equation for the estimator of the phase ξ_I,

$$\frac{\partial}{\partial\xi_I}\ln\Lambda_I[x_I(t);\hat{\mathcal{A}},\xi_I,t_{a_I},\mathcal{M}_I]=0 , \quad (63)$$

is given by

$$\hat{\xi}_I=\arctan\left[\frac{\int_0^T r_{Ic}(t)x_I(t)dt}{\int_0^T r_{Is}(t)x_I(t)dt}\right] . \quad (64)$$

Substituting the estimate of the phase into the likelihood ratio (54) we get

$$\ln\Lambda_I[x_I(t);\hat{\mathcal{A}}_I,\hat{\xi}_I,t_{a_I},\mathcal{M}_I] =\frac{1}{2\mathcal{C}}\left[\left[\int_0^T r_{Ic}(t)x_I(t)dt\right]^2+\left[\int_0^T r_{Is}(t)x_I(t)dt\right]^2\right] . \quad (65)$$

The above procedure determines the optimal analysis of the data to find the maximum likelihood estimates of the parameters of the chirp in each detector. First, one passes the data in the Ith detector through *two* banks of linear filters: $r_{Ic}(t;t_{a_I},\mathcal{M}_I)$ and $r_{Is}(t;t_{a_I},\mathcal{M}_I)$ for suitably spaced values of the parameters t_{a_I} and $\mathcal{M}_I$. The maximum likelihood estimate of the time of arrival $\hat{t}_{a_I}$ and chirp mass $\hat{\mathcal{M}}_I$ are those values of t_{a_I} and $\mathcal{M}_I$ that maximize the functional

$$\mathcal{F}_I(t_{a_I},\mathcal{M}_I)=\frac{1}{2\mathcal{C}}\left[\left[\int_0^T r_{Ic}(t)x_I(t)dt\right]^2+\left[\int_0^T r_{Is}(t)x_I(t)dt\right]^2\right]. \quad (66)$$

Once maximum likelihood estimates of t_{a_I} and $\mathcal{M}_I$ are found by linear filtering, we calculate the maximum likelihood estimates of phases and amplitudes from formulas (53) and (64) with $t_{a_I}=\hat{t}_{a_I}$ and $\mathcal{M}_I=\hat{\mathcal{M}}_I$. In practice, one can perform the correlations in the above functional using the fast Fourier transforms. Then one shall need two banks of only *one* parameter filters parametrized by chirp mass $\mathcal{M}_I$. The position of the maximum of the functional $\mathcal{F}_I$ will determine the time of arrival. The fact that one needs only two filters to determine the phase of the coalescing binary signal has been discovered by Dhurandhar and Sathyaprakash [21]. The above analysis is a systematic derivation using the maximum likelihood equations.

Let us choose as the parameters of the chirp signal in the Ith detector the constant k_I given by Eq. (37), the phase ξ_I, the time of arrival t_{a_I}, and the chirp mass $\mathcal{M}_I$. Then, from Eq. (50) and the formula (36) for Fourier transform of the chirp, the correlation integral in each detector is given by

$$H_I(k_{I1},k_{I2},\xi_{I1},\xi_{I2},t_{a_{I1}},t_{a_{I2}},\mathcal{M}_{I1},\mathcal{M}_{I2})=4k_{I1}k_{I2}\int_{f_I}^{f_f}\frac{\cos[2\pi f(t_{a_{I2}}-t_{a_{I1}})+\Psi(f)(1/\mathcal{M}_2^{5/3}-1/\mathcal{M}_1^{5/3})-(\xi_{I2}-\xi_{I1})]}{f^{7/3}S_n(f)}df\ , \quad (67)$$

where the function $\Psi(f)$ is defined in Eq. (37) and $S_n(f)$ is the spectral density of noise in the interferometer referred to the dimensionless response s_I; f_i is the initial frequency of the chirp, whereas f_f is its final frequency. The components of the Fisher information matrix Γ_I for the Ith detector are computed from the ambiguity function given by Eq. (67) by means of Eq. (44). They are given by the formulas

$$\begin{aligned}
&(\Gamma_I)_{k_Ik_I}=4\int_{f_i}^{f_f}\frac{1}{f^{7/3}S_n(f)}df\ ,\quad (\Gamma_I)_{k_I\xi_I}=(\Gamma_I)_{k_It_{a_I}}=(\Gamma_I)k_{I\mathcal{M}_I}=0\ ,\\
&(\Gamma_I)_{t_{a_I}t_{a_I}}=16\pi^2k_I^2\int_{f_i}^{f_f}\frac{1}{f^{1/3}S_n(f)}df\ ,\quad (\Gamma_I)_{t_{a_I}\mathcal{M}_I}=\frac{40\pi k_I^2}{3\mathcal{M}_I^{8/3}}\int_{f_i}^{f_f}\frac{\Psi(f)}{f^{4/3}S_n(f)}df\ ,\\
&(\Gamma_I)_{t_{a_I}\xi_I}=-8\pi k_I^2\int_{f_i}^{f_f}\frac{1}{f^{4/3}S_n(f)}df\ ,\quad (\Gamma_I)_{\mathcal{M}_I\mathcal{M}_I}=\frac{100k_I^2}{9\mathcal{M}_I^{16/3}}\int_{f_i}^{f_f}\frac{\Psi(f)^2}{f^{7/3}S_n(f)}df\ ,\\
&(\Gamma_I)_{\mathcal{M}_I\xi_I}=\frac{20k_I^2}{3\mathcal{M}_I^{8/3}}\int_{f_i}^{f_f}\frac{\Psi(f)}{f^{7/3}S_n(f)}df\ ,\quad (\Gamma_I)_{\xi_I\xi_I}=4k_I^2\int_{f_i}^{f_f}\frac{1}{f^{7/3}S_N(f)}df\ .
\end{aligned} \quad (68)$$

The covariance matrix C_I for the parameters k_I, ξ_I, t_{a_I}, and $\mathcal{M}_I$ is the inverse of the Γ_I matrix.

The Γ matrix for the single detector in a different set of coordinates has been obtained by Finn and Chernoff [8]. They also give explicitly the components of the covariance matrix.

B. Estimation of the parameters of a coalescing binary: An algebraic method

The first optimal method of solving the inverse problem relies simply on compilation of the data obtained independently in individual detectors. The starting point here is the set of 12 parameters which we group into one vector $\boldsymbol{\eta}$ of 12 components:

$$\boldsymbol{\eta}=(k_1,\xi_1,t_{a_1},\mathcal{M}_1,k_2,\xi_2,t_{a_2},\mathcal{M}_2,k_3,\xi_3,t_{a_3},\mathcal{M}_3)\ . \quad (69)$$

Twelve-dimensional Fisher information matrix has the block structure [which immediately follows from Eqs. (44) and (49)]

$$\Gamma_\eta=\begin{vmatrix}\Gamma_1 & \mathcal{O} & \mathcal{O}\\ \mathcal{O} & \Gamma_2 & \mathcal{O}\\ \mathcal{O} & \mathcal{O} & \Gamma_3\end{vmatrix}\ , \quad (70)$$

where Γ_I ($I=1,2,3$) denotes a four-dimensional Fisher matrix computed for four parameters $k_I,\xi_I,t_{a_I},\mathcal{M}_I$ measured by the Ith detector and $\mathcal{O}$ stands for the 4×4 matrix of all components equal to zero. The problem is to estimate values of the astrophysically important parameters knowing the estimated values of the 12 parameters grouped into the vector $\boldsymbol{\eta}$.

The parameters that we are looking for are collected into the eight-dimensional vector $\boldsymbol{\mu}$:

$$\boldsymbol{\mu}=(R,t_a,\mathcal{M},\theta,\phi,\psi,\epsilon,\delta)\ . \quad (71)$$

t_a and $\mathcal{M}$ denote here the improved values of the time of arrival of the signal in the first detector and the chirp mass, respectively. Improvement relies on taking into account information from all three detectors. We also must estimate the covariance matrix C_μ for the parameters $\boldsymbol{\mu}$. To do this we assume that the covariance matrix C_η for the $\boldsymbol{\eta}$ parameters is given by the inverse of the Fisher matrix (70):

$$C\eta=\Gamma_\eta^{-1} . \tag{72}$$

The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ are related to each other by means of the following set of 12 equations:

$$k_I=K_I(R,\mathcal{M},\theta,\phi,\psi,\epsilon), \quad I=1,2,3, \tag{73}$$

$$\xi_I=\Xi_I(\theta,\phi,\psi,\epsilon,\delta), \quad I=1,2,3, \tag{74}$$

$$t_{a_1}=t_a , \tag{75}$$

$$t_{a_2}=t_a-\tau_{12}(\theta,\phi) , \tag{76}$$

$$t_{a_3}=t_a-\tau_{13}(\theta,\phi) , \tag{77}$$

$$\mathcal{M}_I=\mathcal{M}, \quad I=1,2,3 . \tag{78}$$

The functions K_I are given by Eqs. (37),

$$K_I(R,\mathcal{M},\theta,\phi,\psi,\epsilon)=\frac{1}{2\pi^{2/3}}\left[\frac{5}{6}\right]^{1/2}\frac{G^{5/6}}{c^{3/2}}\frac{\mathcal{M}^{5/6}}{R}\times\left[\left[F_{I+}(\theta,\phi,\psi)\frac{1+\cos^2\epsilon}{2}\right]^2+[F_{I\times}(\theta,\phi,\psi)\cos\epsilon]^2\right]^{1/2} , \tag{79}$$

whereas the functions Ξ_I are determined by Eqs. (35),

$$\Xi_I(\theta,\phi,\psi,\epsilon,\delta)=\delta+\arctan\left[\frac{F_{I+}(\theta,\phi,\psi)(1+\cos^2\epsilon)}{2F_{I\times}(\theta,\phi,\psi)\cos\epsilon}\right] . \tag{80}$$

The functions τ_{12} and τ_{13} can be easily obtained from Eqs. (20)–(22):

$$\begin{aligned}\tau_{12}(\theta,\phi)&=-a\sin\theta\sin\phi ,\\ \tau_{13}(\theta,\phi)&=\sin\theta(b_2\cos\phi-b_1\sin\phi),\end{aligned} \tag{81}$$

where the coefficients a, b_1, and b_2 are given in Eqs. (25).

There does not exist a unique way of solving the set of algebraic equations (73)–(78) with respect to the new parameters $\boldsymbol{\mu}$, because the set is overdetermined. For example, knowing the three times t_{a_I} we are able to compute the angle ϕ and the two possible values of the angle θ by means of Eqs. (76) and (77) (see Sec. II C); then, by combining the three equations (73), we can compute the angles ψ and ϵ, choose the true value of the angle θ out of the two possible values, and after that compute the distance R (in fact, in three different ways; see the next subsection). However, instead of Eqs. (73), we can use Eqs. (74) to obtain the angles ψ and ϵ (and δ). Each of these possibilities will give in general different values of the unknown parameters. To use optimally all the information, we must apply an iterative procedure well known in the context of the least-squares method (see, e.g., [30], Sec. 9.2). Let us shortly denote the set of Eqs. (73)–(78) as

$$\eta_k=f_k(\boldsymbol{\mu})=0, \quad k=1,\ldots,12 . \tag{82}$$

To perform the iteration we must know the first approximation of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$. As the first approximation of the vector $\boldsymbol{\eta}$, we take the values of these parameters measured by the individual detectors. To obtain the first approximation for the vector $\boldsymbol{\mu}$, we must algebraically solve the set of Eqs. (73)–(78) with respect to $\boldsymbol{\mu}$. This we shall do in the next subsection. The procedure described in detail, e.g., in [30], Sec. 9.2, gives also the covariance matrix C_μ for the parameters $\boldsymbol{\mu}$ and the improved covariance matrix $\tilde{C}_\eta$ for the parameters $\boldsymbol{\mu}$ after each step of iteration:

$$C_\mu=(J^T\Gamma_\eta J)^{-1} , \tag{83}$$

$$\tilde{C}_\eta=JC_\mu J^T , \tag{84}$$

where the Jacobi matrix J is given by

$$J_{mn}=\frac{\partial f_m}{\partial\mu_n}, \quad m=1,\ldots,12, \; n=1,\ldots,8 . \tag{85}$$

All the derivatives $\partial f_m/\partial\mu_n$ can be computed by means of Eqs. (73)–(78) together with Eqs. (79)–(81).

At the end of this subsection we mention that there are other methods of inverting an overdetermined system, e.g., the singular value decomposition [31].

C. Algebraic solution to the inverse problem

In the first step from the two independent time delays, we calculate the two possible positions of the source in the sky: (θ_1,ϕ) and (θ_2,ϕ), where $\theta_1+\theta_2=\pi$ (see Sec. II C). We must thus find a procedure which also chooses the true angle θ out of the angles θ_1 and θ_2. In the first place we derive equation for the angle ψ. Dividing k_1^2 by k_2^2 and k_1^2 by k_3^2 by virtue of Eqs. (73) we obtain

$$\left[\frac{k_1}{k_2}\right]^2=\frac{\left[F_{1+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{1\times}\cos\epsilon)^2}{\left[F_{2+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{2\times}\cos\epsilon)^2} , \tag{86}$$

$$\left[\frac{k_1}{k_3}\right]^2=\frac{\left[F_{1+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{1\times}\cos\epsilon)^2}{\left[F_{3+}\frac{1+\cos^2\epsilon}{2}\right]^2+(F_{3\times}\cos\epsilon)^2} , \tag{87}$$

After solving Eq. (86) with respect to $\cos^2\epsilon$ we obtain the only root such that $0\le\cos^2\epsilon\le1$:

$$\cos^2\epsilon=a-\sqrt{a^2-1} , \tag{88}$$

where

$$a=-2\frac{k_1^2F_{2\times}^2-k_2^2F_{1\times}^2}{k_1^2F_{2+}^2-k_2^2F_{1+}^2}-1 \ .$$

Substituting Eq. (88) into Eq. (87), after some algebraic manipulations, we obtain the equation for the angle ψ:

$$k_1^2(F_{2\times}^2F_{3+}^2-F_{3\times}^2F_{2+}^2+k_2^2(F_{3\times}^2F_{1+}^2-F_{1\times}^2F_{3+}^2) +k_3^2(F_{1\times}^2F_{2+}^2-F_{2\times}^2F_{1+}^2)=0 \ . \quad (89)$$

Equation (89) does not change under any permutation of the detectors' labels. Because by virtue of Eqs. (16) the functions F_{I+} and $F_{I\times}$ are linear combinations of $\sin 2\psi$ and $\cos 2\psi$, Eq. (89) is in fact a polynomial equation of second order in the two variables $\sin 4\psi$ and $\cos 4\psi$. Coefficients of this equation are very complicated functions of Euler angles θ,ϕ and the angles $\alpha_I,\beta_I,\gamma_I$ determining the positions and orientations of the three detectors. Using the software MATHEMATICA we have been able to simplify this equation considerably. In compact form it can be rewritten as

$$\tan 4\psi=-4\frac{m_{23}p_{230}k_1^2+m_{31}p_{310}k_2^2+m_{12}p_{120}k_3^2}{m_{23}p_{231}k_1^2+m_{31}p_{311}k_2^2+m_{12}p_{121}k_3^2} \ , \quad (90)$$

where m_{IJ} and p_{IJK} are determined by means of the equations

$$F_{2+}F_{3\times}-F_{3+}F_{2\times}=m_{23} \ ,$$

$$F_{3+}F_{1\times}-F_{1+}F_{3\times}=m_{31} \ ,$$

$$F_{1+}F_{2\times}-F_{2+}F_{1\times}=m_{12} \ ,$$

$$F_{2+}F_{3\times}+F_{3+}F_{2\times}=p_{230}\cos 4\psi+p_{231}\sin 4\psi \ ,$$

$$F_{3+}F_{1\times}+F_{1+}F_{3\times}=p_{310}\cos 4\psi+p_{311}\sin 4\psi \ ,$$

$$F_{1+}F_{2\times}+F_{2+}F_{1\times}=p_{120}\cos 4\psi+p_{121}\sin 4\psi \ .$$

Let us note that combinations of the type $F_{I+}F_{J\times}-F_{J+}F_{I\times}$ do not depend on the angle ψ. The coefficients m_{IJ} and p_{IJK} are still complicated functions of the angles θ,ϕ and $\alpha_I,\beta_I,\gamma_I$.

For each of the two values of the angle θ, we obtain from Eq. (90) the two values for the angle ψ from the interval $[0,\pi/2[$. Of course, the difference between these angles is equal to $\pi/4$. Then we substitute these values of the angle ψ into Eq. (88) to calculate $\cos^2\epsilon$. It turns out that for a fixed value of the angle θ only one out of the two possible values of the angle ψ gives $\cos^2\epsilon$ which satisfies the condition $0\le\cos^2\epsilon\le 1$. This can be demonstrated as follows. Let us denote by a_1 and a_2 the values of the quantity a from Eq. (88) for the angle ψ and $\psi+\pi/4$, respectively. By virtue of Eqs. (19), there is a relation between a_1 and a_2:

$$\frac{3-a_1}{1+a_1}=a_2 \ . \quad (91)$$

Equation (91) implies that if $a_1>1$, then always $a_2<1$. In the extreme case $a_1=1$, we have $a_2=1$.

In the current stage of our procedure, we are able to calculate for each of the two values of the angle θ the one angle ψ from the interval $[0,\pi/2[$ [by virtue of Eq. (90)] and the one value of $\cos^2\epsilon$ [by virtue of Eq. (88)]. For some true values of the angle θ, it may happen that the nontrue value $\pi-\theta$ yields [by virtue of Eq. (88)] the condition $\cos^2\epsilon<0$ (it means that then $a\le 0$). In such a case, evaluation of $\cos^2\epsilon$ is already a criterion of choosing the true value of the angle θ out of the two possible values. But this is not the general case. In general, after substituting the two possible values of the angle ψ and $\cos^2\epsilon$ into any of the three equations (73) we obtain the two possible values of the distance R to the binary. To choose the true values of the distance, the angle ψ, and $\cos\epsilon$, we must use information contained in the phases ξ_I of the signal.

Equations (74) can be solved with respect to the $\sin\delta$ and $\cos\delta$. For example, for $\cos\delta$ we get

$$\cos\delta=\frac{c_I\cos\xi_I+d_I\sin\xi_I}{\sqrt{c_I^2+d_I^2}} \ , \quad (92)$$

where

$$c_I=\pm\cos\epsilon F_{I\times} \ , \quad d_I=\frac{1+\cos^2\epsilon}{2}F_{I+} \ .$$

The $\pm$ sign in the equation for the coefficient c_I results from the fact that we do not directly calculate $\cos\epsilon$ but only $\cos^2\epsilon$ [by virtue of Eq. (88)]. The right-hand side of Eq. (92) can be calculated three times (i.e., for $I=1,2,3$) for the plus sign and three times for the minus sign. For the true values of the angle θ, the angle ψ, and $\cos\epsilon$, one obtains the same number, whereas for the nontrue values, three different numbers. This completes the procedure.

D. Estimation of the parameters of a coalescing binary: A linear filtering method

In the previous two subsections, we have presented the first solution of the inverse problem. We linearly filter the data in each detector and obtain the estimates of the amplitude, phase, time of arrival of the signal, and also the sharp mass. Then we perform the iterative procedure described in Sec. II B to obtain the parameters of the binary. The second method is the filtering of the data from all three detectors directly for the parameters of the binary taking into account the two constraints relating the angles θ and ϕ determining the position of the binary in the sky and the two independent time delays between the detectors. To find the optimal procedure, one has to solve Eqs. (38) directly for the parameters of the binary and take into account the two constraints. We shall find that the estimates of the distance R and the phase δ can be determined analytically. We also can find an equation for the angle ϵ; however, this equation is unmanageable. We still find that it is optimal to linearly filter the data in each detector. We take the constraints into account in the following way. The filters are parametrized by angles θ and ϕ. For each set of these angles, we calculate the time delays. We shift the outputs from the linear filters in the detectors according to the calculated time delays and then add the outputs (with suitable weights). We proceed until we find the maximum of the sum.

Let us write the response function s_I in the form

$$s_I(t)=\frac{\mathcal{B}_I}{R}[h_{Is}(t)\cos(\chi_I+\delta)+h_{Ic}(t)\sin(\chi_I+\delta)] , \tag{93}$$

where $\mathcal{A}_I=\mathcal{B}_I/R$, $\chi_I=\xi_I-\delta$, and the functions h_{Is} and h_{Ic} are defined by Eqs. (56). Consequently, the filter function can be written as

$$q_I(t)=\frac{\mathcal{B}_I}{R}[r_{Is}(t)\cos(\chi_I+\delta)+r_{Ic}(t)\sin(\chi_I+\delta)] , \tag{94}$$

where r_{Ic} and r_{Is} are solutions of the integral Eq. (57). Let us now substitute s_I and q_I given by Eqs. (93) and (94) into the equation for the logarithm of the likelihood ratio $\ln\Lambda$ [see Eq. (47)] and let us use Eqs. (60)–(62). Then we get

$$\ln\Lambda[x_I(t);\boldsymbol{\mu}]=-\frac{\mathcal{C}}{2R^2}\sum_{I=1}^{3}\mathcal{B}_I^2+\frac{1}{R}\sum_{I=1}^{3}\left[\mathcal{B}_I\left[\sin(\chi_I+\delta)\int_0^T x_I(t)r_{Ic}(t)dt+\cos(\chi_I+\delta)\int_0^T x_I(t)r_{Is}(t)dt\right]\right] , \tag{95}$$

where $\boldsymbol{\mu}$ denotes the eight-dimensional vector of the parameters of the binary, given by Eq. (71). We can easily analytically solve the equations for the maximum likelihood estimators of the distance R and the phase δ:

$$\frac{\partial\ln\Lambda}{\partial R}=0 , \quad \frac{\partial\ln\Lambda}{\partial\delta}=0 .$$

We get

$$\hat{R}=\frac{\mathcal{C}\sum_{I=1}^{3}\mathcal{B}_I^2}{\sum_{I=1}^{3}\{\mathcal{B}_I[\sin(\chi_I+\delta)\int_0^T x_I(t)r_{Ic}(t)dt+\cos(\chi_I+\delta)\int_0^T x_I(t)r_{Is}(t)dt]\}} , \tag{96}$$

$$\tan\hat{\delta}=\frac{\sum_{I=1}^{3}\{\mathcal{B}_I[\sin\chi_I\int_0^T x_I(t)r_{Ic}(t)dt-\sin\chi_I\int_0^T x_I(t)r_{Is}(t)dt]\}}{\sum_{I=1}^{3}\{\mathcal{B}_I[\sin\chi_I\int_0^T x_I(t)r_{Ic}(t)dt+\cos\chi_I\int_0^T x_I(t)r_{Is}(t)dt]\}} . \tag{97}$$

After substituting the above equations for the estimators of R and δ into Eq. (95) we obtain

$$\ln\Lambda[x_I(t);\hat{R},\hat{\delta},\boldsymbol{\nu}]=\frac{1}{2\mathcal{C}\sum_{I=1}^{3}\mathcal{B}_I^2}\left\{\left[\sum_{I=1}^{3}\mathcal{B}_I\left[\sin\chi_I\int_0^T x_I(t)r_{Ic}(t)dt+\cos\chi_I\int_0^T x_I(t)r_{Is}(t)dt\right]\right]\right.$$
$$\left.+\left[\sum_{I=1}^{3}\mathcal{B}_I\left[\cos\chi_I\int_0^T x_I(t)r_{Ic}(t)dt-\sin\chi_I\int_0^T x_I(t)r_{Is}(t)dt\right]\right]\right\} , \tag{98}$$

where $\boldsymbol{\nu}$ denotes the remaining six parameters of a binary. We attempted to find the analytic solutions for the estimators of the angles ϵ and ψ; however, we found the equations unmanageable. We only found the following equation of the fourth order for the cosine of the angle ϵ:

$$c_0+c_1\cos\epsilon+c_2\cos^2\epsilon+c_1\cos^3\epsilon+c_0\cos^4\epsilon=0 , \tag{99}$$

where c_0, c_1, and c_2 are very complicated functions of the Euler angles θ, ϕ, and ψ.

The optimal filtering procedure presented above would be very complicated since it would require a five-parameter bank of filters for the chirp mass $\mathcal{M}$, the three Euler angles, and the inclination angle ϵ.

The accuracy of the second method described here is given by the Fisher matrix Γ_μ for the parameters $\boldsymbol{\mu}$. There is a simple way of computing Γ_μ knowing Γ_η. By means of Eq. (44), we can write

$$(\Gamma_\mu)_{ij}=\left.\frac{\partial^2 H(\boldsymbol{\eta}_1(\boldsymbol{\mu}_1),\boldsymbol{\eta}_2(\boldsymbol{\mu}_2))}{\partial\mu_{1i}\partial\mu_{2j}}\right|_{\mu_1=\mu_2=\mu}$$
$$=\sum_{k,l=1}^{12}\left.\frac{\partial^2 H(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)}{\partial\eta_{1k}\partial\eta_{2l}}\right|_{\eta_1=\eta_2=\eta}\frac{\partial\eta_k}{\partial\mu_i}\frac{\partial\eta_l}{\partial\mu_j} ,$$

or, in matrix notation,

$$\Gamma_\mu=J^T\Gamma_\eta J , \tag{100}$$

where the Jacobi matrix J is given by $J_{ij}=\partial\eta_i/\partial\mu_j$. The covariance matrix C_μ is the inverse of Γ_μ. Comparing Eq. (100) with Eq. (83) we see that the two methods give the same errors of the parameters $\boldsymbol{\mu}$.

IV. NUMERICAL EXAMPLES

A. Positions and orientations of the detectors

In Table I we find the geographical latitudes β_I and longitudes γ_I of the three planned long-arm laser inter-

TABLE I. Positions and orientations of the detectors used in the calculations. β_I and γ_I are the latitude and longitude of the Ith detector; α_I is the angle between the bisector of the arms of the detector and the local east-west direction.

No. (I)	Site	Latitude β_I (deg)	Longitude γ_I (deg)	Orientation α_I (deg)
1	Louisiana	30.6	−90.8	243
2	Washington	46.5	−119.4	171
4	Pisa, Italy	43.6	10.25	117

ferometric detectors as well as their orientations α_I with respect to the local east-west direction (see Sec. II B for a detailed definition of the angle α_I). For this network of detectors, the coefficients a, b_1, and b_2 describing the spatial separations between the detectors [32] [given in Eqs. (25)] have the values

$$a=9.99 \text{ ms}, \quad b_1=2.76 \text{ ms}, \quad b_2=26.20 \text{ ms}.$$

B. Model of the noise, the value of the recycling knee frequency, and the signal-to-noise ratios

Following Finn and Chernoff [8] we take into account the five main sources of the noise in a laser interferometer; therefore, the total spectral density of the noise is the sum of the spectral densities related to the different noise sources:

$$S_n(f)=S_{\text{shot}}(f)+S_{\text{susp}}(f)+S_{\text{int}}(f)+S_{\text{seismic}}(f)+S_{\text{quantum}}(f). \quad (101)$$

S_{shot} is the photon shot noise:

$$S_{\text{shot}}(f)=\frac{\hbar\lambda}{\eta I_0}\frac{A^2}{L}f_k\left[1+\left(\frac{f}{f_k}\right)^2\right]. \quad (102)$$

S_{susp} and S_{int} denote the thermal noise in the suspensions and test masses, respectively:

$$S_{\text{susp}}(f)=\frac{2k_B T f_0}{\pi^3 m Q_0 L^2[(f^2-f_0^2)^2+(f f_0/Q_0)^2]}, \quad (103)$$

$$S_{\text{int}}(f)=\frac{2k_B T f_{\text{int}}}{\pi^3 m Q_{\text{int}} L^2[(f^2-f_{\text{int}}^2)^2+(f f_{\text{int}}/Q_{\text{int}})^2]}. \quad (104)$$

S_{seismic} stands for the seismic noise:

$$S_{\text{seismic}}(f)=\frac{S_0}{f^4(f^2-f_0^2)^{10}}. \quad (105)$$

The value of the proportionality constant S_0 we calculate, following Finn and Chernoff [8], from the relationship

$$S_{\text{seismic}}(10 \text{ Hz})=S_{\text{susp}}(10 \text{ Hz})+S_{\text{int}}(10 \text{ Hz}).$$

We get $S_0=1.54\times10^{-21}$ Hz. Last, S_{quantum} is the quantum noise:

$$S_{\text{quantum}}(f)=\frac{2\hbar}{\pi^2 m L^2 f^2}. \quad (106)$$

The meaning and the values of all quantities appearing in Eqs. (102)–(106) are given in the Table II (taken from Finn and Chernoff [8]).

The frequency f_k in Eq. (102) denotes the recycling knee frequency. We fix the value of f_k by maximizing the signal-to-noise ratio for a coalescing binary signal. The square of the signal-to-noise ratio is given by the formula [23]

$$\left(\frac{S}{N}\right)^2=4\int_{f_i}^{f_f}\frac{|\tilde{s}(f)|^2}{S_n(f)}df, \quad (107)$$

where f_i and f_f stand for the initial and final frequencies of the signal, respectively. By means of Eq. (36), the squared signal-to-noise ratio for a chirp signal s_I in the Ith detector is equal to

$$\left(\frac{S}{N}\right)_I^2=4k_I^2\int_{f_i}^{f_f}\frac{1}{f^{7/3}S_n(f)}df. \quad (108)$$

Assuming $f_i=10$ Hz and $f_f=1000$ Hz and taking into account the total spectral density of the noise given by Eq. (101), we maximize the integral on the right-hand side of Eq. (108) with respect to f_k. We get $f_k=109$ Hz. This optimum value of the knee frequency equal to 100 Hz has been obtained by Finn and Chernoff in Ref. [8] for $f_f=\infty$.

In Fig. 1 we find the contour plots of the signal-to-noise ratios in the three detectors as functions of the angles θ and ψ for a chirp signal coming from a sample binary system defined by the numbers

$$\begin{aligned} &m_1=m_2=1.4M_\odot, \\ &R=100 \text{ Mpc}, \\ &\mathcal{M}\approx1.22M_\odot, \\ &\psi=30^\circ, \\ &\epsilon=45^\circ, \quad \delta=0^\circ. \end{aligned} \quad (109)$$

We use 100 Mpc as a canonical distance for the neutron star binaries. Recent estimates [33,34] have shown that at such a distance one can expect only one binary inspiral per three years. Therefore, to get a reasonable number of events, one has to go to farther distances. The accuracies

TABLE II. Advanced Ligo interferometer characteristics. The spectral density of the noise in the interferometer depends on the details of the interferometer instrumentation. We consider in this paper the advanced LIGO instrumentation that is expected to be available much after the LIGO facilities first come on line [8].

Temperature (T)	300 K
Pendulum frequency (f_0)	1 Hz
Suspension quality (Q_0)	10^9
End mass (m)	1000 kg
End mass fundamental mode (f_{int})	5000 Hz
End mass quality (Q_{int})	10^5
Effective laser power ($I_0\eta$)	60 W
Laser wavelength (λ)	5139 Å
Mirror losses (A^2)	2×10^{-5}

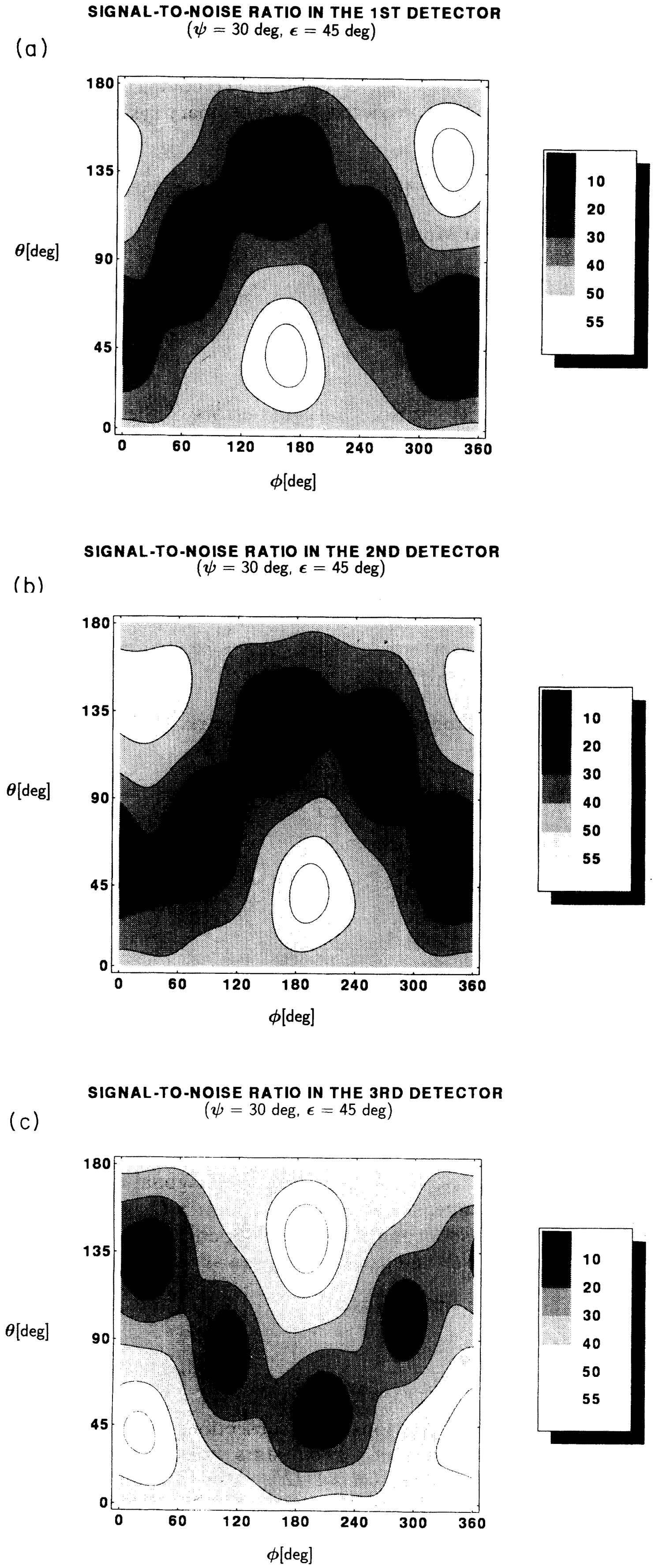

FIG. 1. Signal-to-noise ratios in the three detectors as functions of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22 M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

of the determination of the parameters scale in a simple way with distance as shown at the end of this section. The distance sensitivity of the LIGO and VIRGO detectors has been discussed in Refs. [8,35]. The contour plots reveal that for each detector there are four distinct regions in the sky, where the signal-to-noise ratio is very low and the two regions where the signal-to-noise ratio is very high (for the case presented in Fig. 1, the maximal value of the signal-to-noise ratio is about 60). It is clear from the plots that there is no simple transformation relating them.

C. Comparison of the one-detector accuracy with the network accuracy

The accuracy of the estimators of the parameters of the chirp signal in the Ith detector (i.e., k_I, ξ_I, t_{a_I}, and $\mathcal{M}_I$) is given by the four-dimensional covariance matrix C_I being the inverse of the four-dimensional Fisher matrix Γ_I of components given by Eqs. (68) [see discussion after Eq. (43)]. The accuracy of the estimators of the same parameters for the whole network describes the 12-dimensional covariance matrix $\tilde{C}_\eta$ of Eq. (84). To compare these two accuracies, we considered a sample binary system defined by the numbers

$$\begin{aligned} &m_1=m_2=1.4M_\odot\ ,\\ &R=100\ \text{Mpc}\ ,\\ &\mathcal{M}\approx 1.22M_\odot\ ,\\ &\theta=10^\circ\ ,\quad \phi=20^\circ\ ,\quad \psi=30^\circ\ ,\\ &\epsilon=45^\circ\ ,\quad \delta=0^\circ\ . \end{aligned} \tag{110}$$

The single-detector one-sigma errors of the estimators of the parameters of the chirp signal from the sample binary system are given by

$$\begin{aligned} &\sigma(k_1)=\sigma(k_2)=\sigma(k_3)=1.20\times 10^{-23}\ \text{s}^{-1/6}\ ,\\ &\sigma(\xi_1)=0.795\ \text{rad},\quad \sigma(\xi_2)=0.753\ \text{rad},\quad \sigma(\xi_3)=0.606\ \text{rad}\ ,\\ &\sigma(t_{a_1})=8.26\ \text{ms}\ ,\quad \sigma(t_{a_2})=7.82\ \text{ms}\ ,\quad \sigma(t_{a_3})=6.30\ \text{ms}\ ,\\ &\sigma(\mathcal{M}_1)=6.05\times 10^{-6}M_\odot\ ,\quad \sigma(\mathcal{M}_2)=5.73\times 10^{-6}M_\odot\ ,\quad \sigma(\mathcal{M}_3)=4.61\times 10^{-6}M_\odot\ , \end{aligned} \tag{111}$$

where $\sigma(\eta_i)$ is the square root of the diagonal element $(C_I)_{ii}$ of the four-dimensional covariance matrix C_I. For the network the errors are given by

$$\begin{aligned} &\sigma(k_1)=0.920\times 10^{-23}\ \text{s}^{-1/6}\ ,\quad \sigma(k_2)=0.916\times 10^{-23}\ \text{s}^{-1/6}\ ,\quad \sigma(k_3)=1.18\times 10^{-23}\ \text{s}^{-1/6}\ ,\\ &\sigma(\xi_1)=0.4064\ \text{rad}\ ,\quad \sigma(\xi_2)=0.4063\ \text{rad}\ ,\quad \sigma(\xi_3)=0.4065\ \text{rad}\ ,\\ &\sigma(t_{a_1})=4.2172\ \text{ms}\ ,\quad \sigma(t_{a_2})=4.2171\ \text{ms}\ ,\quad \sigma(t_{a_3})=4.2171\ \text{ms}\ ,\\ &\sigma(\mathcal{M}_1)=\sigma(\mathcal{M}_2)=\sigma(\mathcal{M}_3)=3.09\times 10^{-6}M_\odot\ , \end{aligned} \tag{112}$$

where $\sigma(\eta_i)$ is now the square root of the diagonal element $(\tilde{C}_\eta)_{ii}$ of the 12-dimensional covariance matrix $\tilde{C}_\eta$. We see that the errors of the phases ξ_I, the times of arrival t_{a_I}, and the chirp mass are about 1.5–2.0 times smaller, whereas the errors of the amplitudes are only 1.0–1.3 times smaller. The reason that the errors in the parameters are smaller for the network than for individual detectors is that in the calculation of the 12-dimensional covariance matrix the constraints, between the parameters are taken into account, i.e., the relations between the times of arrival [Eqs. (75)–(77)] and the fact that all detectors observe the same signal and consequently the chirp mass is the same [Eq. (78)]. The additional information contained in the constraints increases the accuracy. These numbers are also typical for other values of the angles θ, ϕ, ψ, ϵ, and δ.

D. $\sigma(R)/R$, $\sigma(\mathcal{M})/\mathcal{M}$, and $\Delta\Omega$ as functions of the angles θ, ϕ, ψ, and ϵ

In this subsection we compute the relative error $\sigma(R)/R$ of the luminosity distance R to the binary, the relative error $\sigma(\mathcal{M})/\mathcal{M}$ of the chirp mass $\mathcal{M}$, and the measure $\Delta\Omega$ of the solid angle inside which we are looking for the source in the sky. The standard deviations $\sigma(R)$ and $\sigma(\mathcal{M})$ are defined as

$$\sigma(R)=\sqrt{(C_\mu)_{RR}}\ ,\quad \sigma(\mathcal{M})=\sqrt{(C_\mu)_{\mathcal{M}\mathcal{M}}}\ ,$$

where $(C_\mu)_{RR}$ and $(C_\mu)_{\mathcal{M}\mathcal{M}}$ are the diagonal elements of the eight-dimensional covariance matrix C_μ given in Eq. (83). The solid angle $\Delta\Omega$ corresponds to the ellipse of semiaxes $\sigma(\theta)$ and $\sigma(\phi)$. It is given as

$$\Delta\Omega=\pi\sin\theta\sigma(\theta)\sigma(\phi)\ ,$$

where

$$\sigma(\theta)=\sqrt{(C_\mu)_{\theta\theta}}\ ,\quad \sigma(\phi)=\sqrt{(C_\mu)_{\phi\phi}}\ .$$

As in the previous two subsections, we consider a binary system of two stars of equal masses: $m_1=m_2=1.4M\odot$, ($\mathcal{M}\approx 1.22M\odot$), with luminosity distance $R=100$ Mpc; we also put $\delta=0^\circ$. Figure 2 shows the contour plot of $\sigma(R)/R$ as a function of the angles θ and ϕ. Figures 3 and 4 are the contour plots of $\sigma(\mathcal{M})/\mathcal{M}$

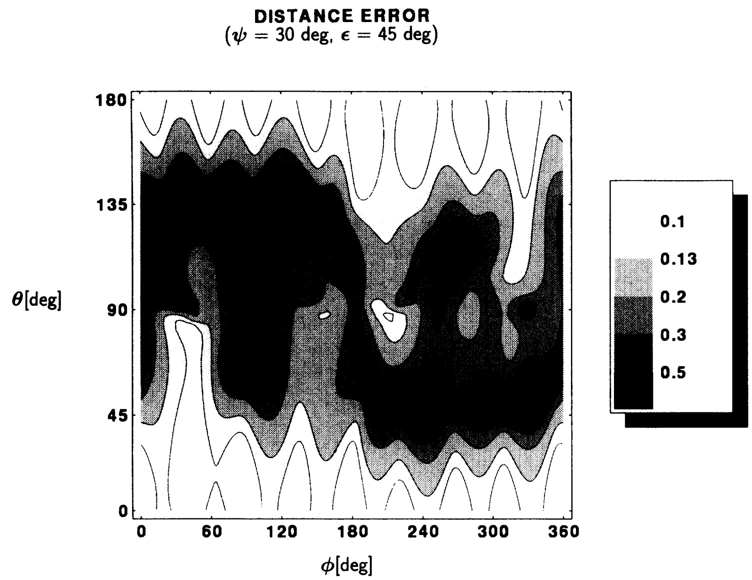

FIG. 2. Fractional standard deviation $\sigma(R)/R$ of the distance R to the binary as a function of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

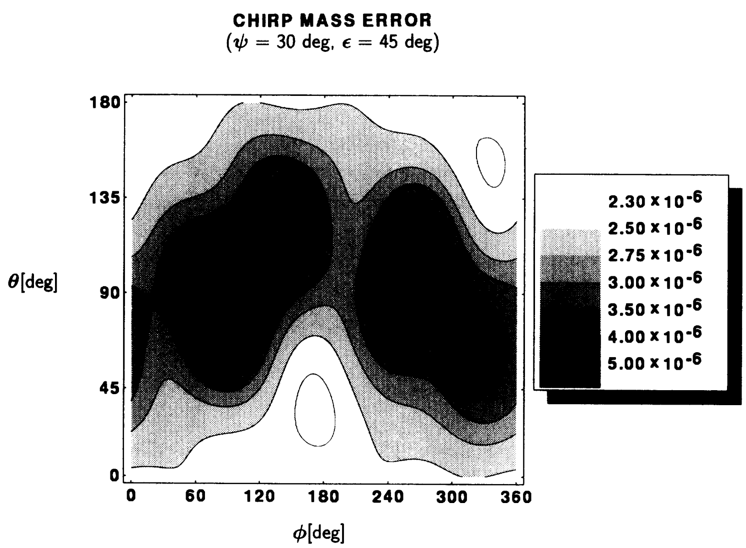

FIG. 3. Fractional standard deviation $\sigma(\mathcal{M})/\mathcal{M}$ of the chirp mass $\mathcal{M}$ as a function of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

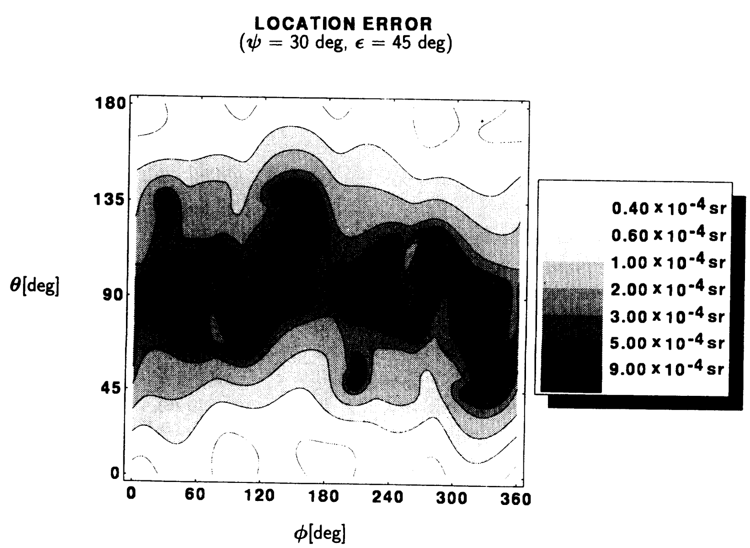

FIG. 4. Error $\Delta\Omega$ in the source location as a function of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

and $\Delta\Omega$, respectively, as functions of θ and ϕ. The value of the inclination angle ϵ for plots from Figs. 2–4 is equal to 45°, whereas the value of the polarization angle ψ is set to 30°. In Fig. 5 we find $\sigma(R)/R$, $\sigma(\mathcal{M})/\mathcal{M}$, and $\Delta\Omega$ plotted as functions of the inclination angle ϵ for the remaining angles fixed, whereas Fig. 6 shows these quantities as functions of the polarization angle ψ (for remaining angles fixed).

The analysis of the plots in Figs. 2–4 shows that for the fractional chirp mass error there are two distinct "bad" and two "good" regions in the sky, whereas for fractional distance and location errors the structure of "bad" and "good" regions is much more complicated. Detailed analysis of Fig. 5 shows that the plots of $\sigma(R)/R$ and $\Delta\Omega$ *are not* symmetric with respect to $\epsilon=90°$, whereas the plot of $\sigma(\mathcal{M})/\mathcal{M}$ *is* symmetric. Moreover, it is worth noting that $\sigma(\mathcal{M})/\mathcal{M}$ and $\Delta\Omega$ are smallest for $\epsilon=0°$ and $\epsilon=180°$, where the $\sigma(R)/R$ is the largest. $\sigma(R)/R$, $\sigma(\mathcal{M})/\mathcal{M}$, and $\Delta\Omega$ are all periodic with respect to the angle ψ with period equal to 90°. Compar-

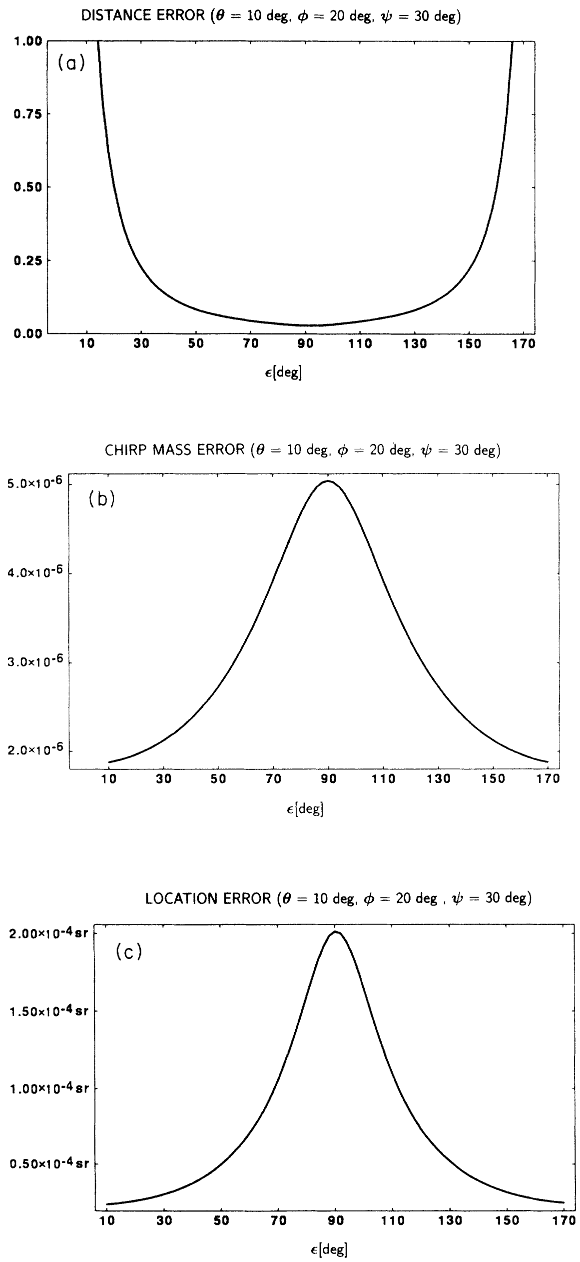

FIG. 5. Fractional standard deviation $\sigma(R)/R$ of the distance R to the binary, the fractional standard deviation $\sigma(\mathcal{M})/\mathcal{M}$ of the chirp mass $\mathcal{M}$, and the error $\Delta\Omega$ in the source location as functions of the angle ϵ for $\theta=10°$, $\phi=20°$, and $\psi=30°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22 M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

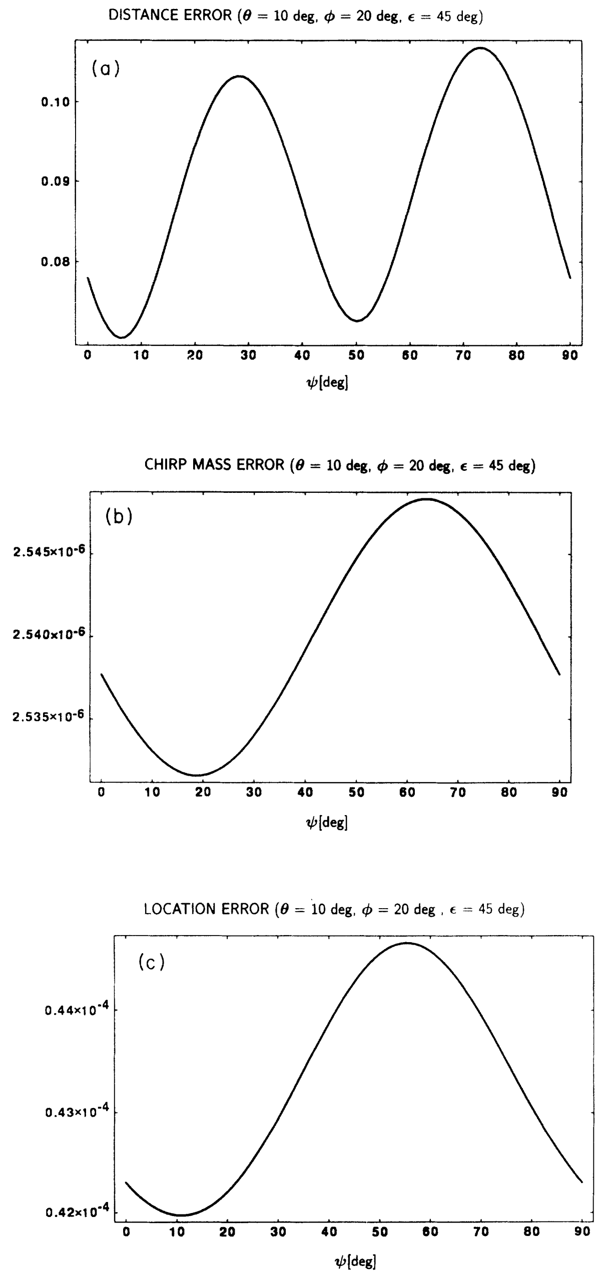

FIG. 6. Fractional standard deviation $\sigma(R)/R$ of the distance R to the binary, the fractional standard deviation $\sigma(\mathcal{M})/\mathcal{M}$ of the chirp mass $\mathcal{M}$, and the error $\Delta\Omega$ in the source location as functions of the angle ψ for $\theta=10°$, $\phi=20°$, and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22 M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

ison of plots in Figs. 5 and 6 shows that all the three errors depend on the angle ϵ more strongly than on the angle ψ.

$\sigma(R)/R$, $\sigma(\mathcal{M})/\mathcal{M}$, and $\Delta\Omega$ depend on the angles θ, ϕ, ψ, and ϵ in a rather complicated way. The dependence of these quantities on the luminosity distance R is simple:

$$\sigma(R)/R \sim R \ , \quad \sigma(\mathcal{M})/\mathcal{M} \sim R \ , \quad \Delta\Omega \sim R^2 \ .$$

V. CONCLUSIONS

From our numerical results, it follows that a typical compact binary at a distance of 100 Mpc can be located within its cluster of galaxies and its distance can be determined with 10% accuracy. Using an idea due to Schutz [3], one can expect to determine the Hubble constant to a good accuracy; however, a careful investigation is needed. How to measure cosmological parameters from observations by a single detector has already been investigated by Finn and Chernoff [36]. The chirp mass $\mathcal{M}$ is determined to a very impressive accuracy [38]. One reason for this is that the same chirp mass is estimated by the three independent detectors. However, a far more important cause for such an accuracy is the fact that the data analysis method presented here relies on correlating the signal for hundreds of cycles. Thus even a small difference in the chirp mass between the signal and filter results in a large phase difference over many cycles and consequently affects the value of the correlation significantly. These are spectacular examples of how a network of laser interferometers can serve as a powerful astronomical observatory.

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[19] Equations (A6) and (A7) from Appendix A of Ref. [12] contain errors: The order of subtracting vectors in these equations must be exchanged.
[20] In the paper by Schutz and Tinto [13], there is a certain inconsistency in using the angle α. Figure 2 in the paper suggests that the definitions of the angle α used by the authors and in the present paper are exactly the same, whereas in the beginning of Sec. III of [13] we find that the angle α is the angle between the bisector of the arms and the local *meridian*, not the local west-east direction.
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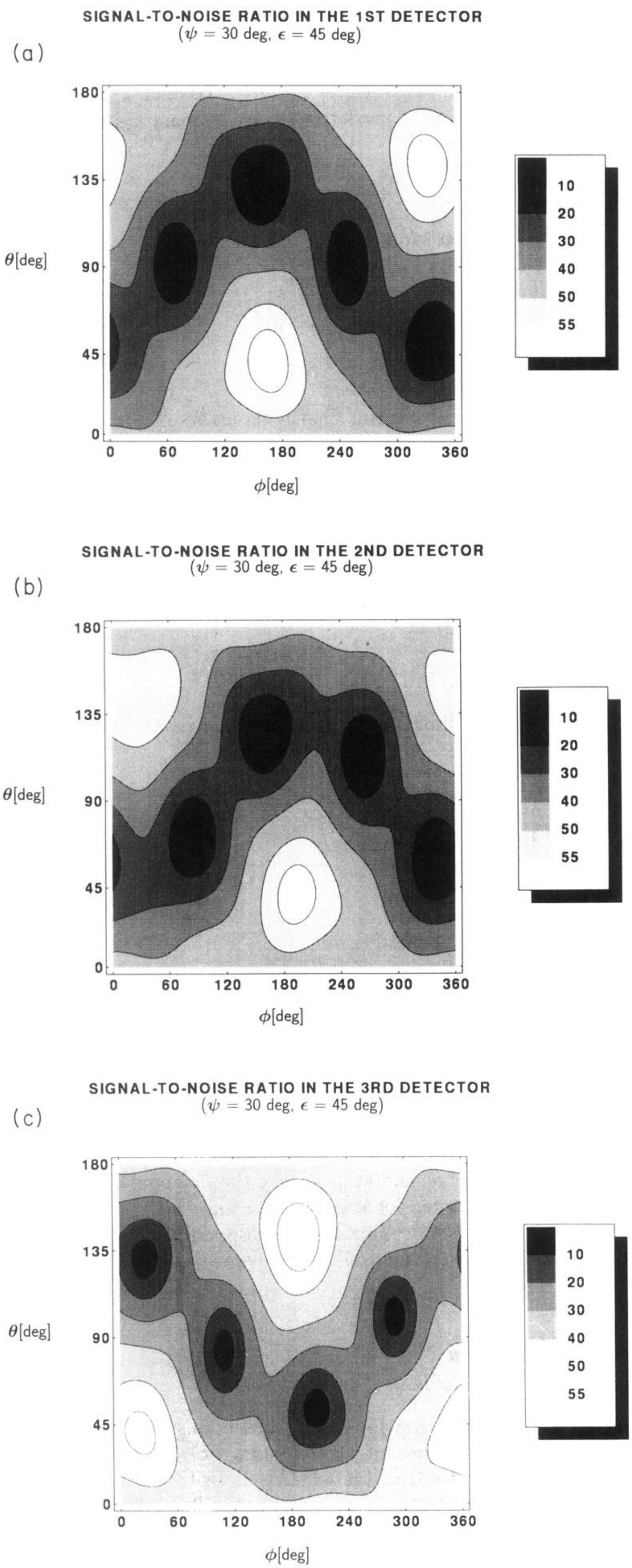

FIG. 1. Signal-to-noise ratios in the three detectors as functions of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22 M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

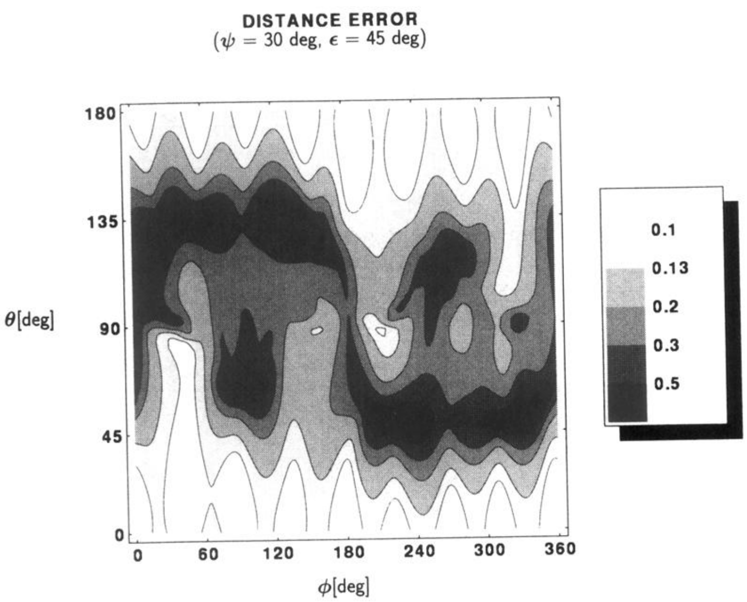

FIG. 2. Fractional standard deviation $\sigma(R)/R$ of the distance R to the binary as a function of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

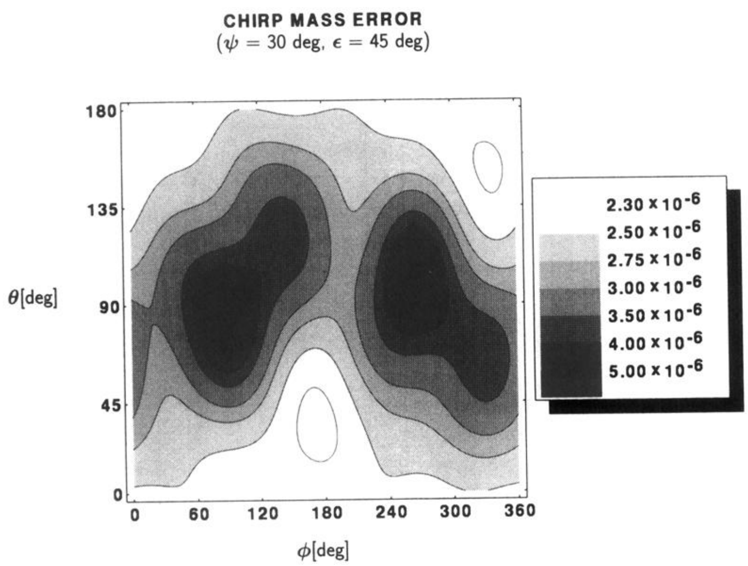

FIG. 3. Fractional standard deviation $\sigma(\mathcal{M})/\mathcal{M}$ of the chirp mass $\mathcal{M}$ as a function of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22 M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.

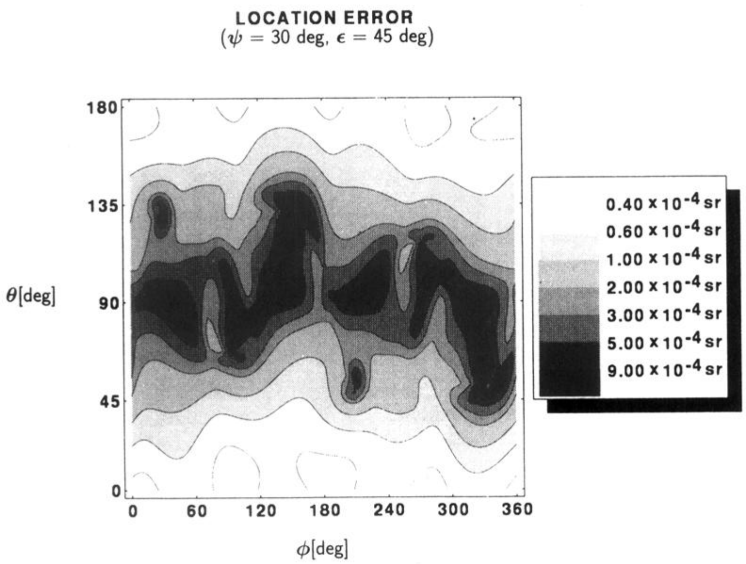

FIG. 4. Error $\Delta\Omega$ in the source location as a function of the angles θ and ϕ for $\psi=30°$ and $\epsilon=45°$. The signal comes from a binary system of $\mathcal{M}\approx 1.22 M_{\odot}$, $R=100$ Mpc, and $\delta=0°$.