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# Bound on $m_\eta/m_{\eta'}$ for large $N_C$

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If the number of colors is large, the ratio $m_\eta/m_{\eta'}$ is bounded from above. The bound is not satisfied by the observed $\eta$ and $\eta'$ masses.

One of the classic statements of the U(1) problem in QCD was Weinberg's observation that a chiral U(1) broken only by quark masses (isospin symmetric and small enough to apply chiral Lagrangian arguments) implies the existence of a neutral meson state with a mass less than $\sqrt{3}m_\pi$ [1]. 't Hooft showed how nonperturbative effects could solve this problem by breaking the chiral U(1) [2]. If the chiral U(1) is broken strongly, it does not make sense to regard the $\eta'$ as a Goldstone boson. However, if the number of colors is large (and it is often speculated that 3 is large enough), the breaking of the chiral U(1) is suppressed and leading order chiral Lagrangian arguments can still be applied. In this Brief Report, I review the well-known[1] form for the pseudoscalar meson mass-squared matrix in this limit and note the existence of an upper bound on the ratio $m_\eta/m_{\eta'}$. The bound has the form

$$\frac{m_\eta^2}{m_{\eta'}^2} < \frac{3-\sqrt{3}}{3+\sqrt{3}} + \frac{3\sqrt{3}}{(3+\sqrt{3})^2}\left(\frac{m_u+m_d}{m_s}\right) + O\left(\left(\frac{m_u+m_d}{m_s}\right)^2\right) \tag{1}$$

with the $u$, $d$, and $s$ quark masses denoted by $m_u$, $m_d$, and $m_s$. What is perhaps slightly amusing about this bound is that it is *not satisfied* by the observed $\eta$ and $\eta'$ masses. This is a clear (if not very surprising) indication that higher order effects in the chiral Lagrangian are very important for the $\eta$-$\eta'$ system.

In leading nontrivial order in large $N$ and the momentum expansion, the chiral Lagrangian for the nonet of pseudoscalar mesons takes the form

$$\mathcal{L}(\pi) = f^2\Big\{\frac{1}{4}\mathrm{tr}\left(\partial^\mu U^\dagger\partial_\mu U\right) + \frac{1}{2}\mathrm{tr}\left(U^\dagger \mu M\right) + \frac{1}{2}\mathrm{tr}\left(U\mu M\right) + \frac{1}{2}m_0^2\left(\det U + \det U^\dagger\right)\Big\}, \tag{2}$$

where

$$U = \exp[2i\underset{\sim}{\Pi}/f]\,, \qquad \underset{\sim}{\Pi} = \sum_{a=0}^{8}\pi_a T_a\,, \tag{3}$$

where $f$ is a constant with dimensions of mass and $M$ is the quark mass matrix,

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{4}$$

The $U$ field transforms linearly under U(3)×U(3):

$$U \to U' = LUR^\dagger, \tag{5}$$

In the basis $(u\bar{u},\, d\bar{d},\, s\bar{s})$, (2) gives a mass-squared matrix of the flavor-neutral pseudo Goldstone bosons proportional to

$$\begin{pmatrix} x+m_u & x & x \\ x & x+m_d & x \\ x & x & x+m_s \end{pmatrix} \tag{6}$$

where the $x$'s arise from the $m_0^2$ term in (2) [3]. From (6), I will derive the bound for $m_u = m_d = 0$, where the algebra is simple. I will then indicate how to derive most easily the result to next order.

For $m_u = m_d = 0$, the mass-squared matrix (6) has one zero eigenvalue. The other eigenvalues are

$$\frac{3\,x}{2} + \frac{m_s}{2} \pm \frac{\sqrt{9\,x^2 - 2\,x m_s + m_s^2}}{2}. \tag{7}$$

The ratio is

$$r_0(x, m_s) \equiv \frac{3\,x + m_s - \sqrt{9\,x^2 - 2\,x m_s + m_s^2}}{3\,x + m_s + \sqrt{9\,x^2 - 2\,x m_s + m_s^2}}. \tag{8}$$

This is maximized for $x = m_s/3$, which gives the first term in the result (1).

The second term in (1) can be most easily obtained by setting $m_u = m_d$ and computing the ratio of the two largest eigenvalues in perturbation theory. It is easy to see that the general result to first order in $m_u/m_s$ and $m_d/m_s$ depends only on $m_u + m_d$, thus no information is lost by setting $m_u = m_d$. To first order, the ratio is

$$r_0(x, m_s) + \frac{m_u + m_d}{m_s} r_1(x, m_s)\,, \tag{9}$$

where

[1] See, for example, [3].

 

$$r_1(x,m_s) = \frac{6\,x^2 + 2\,m_s^2}{\left(3\,x + m_s + \sqrt{9\,x^2 - 2\,xm_s + m_s^2}\right)^2 \sqrt{9\,x^2 - 2\,xm_s + m_s^2}}\,. \tag{10}$$

Setting $x = m_s/3$ in (10) gives the second term in (1).

The first term in (1) gives a mass ratio bound of

$$\frac{m_\eta}{m_{\eta'}} < 0.518 \tag{11}$$

compared to the experimental value

$$\frac{m_\eta}{m_{\eta'}} \approx 0.572\,. \tag{12}$$

Including the effects of the nonzero $u$ and $d$ masses brings these closer, but not into agreement. Using generous values $m_d/m_s \approx 0.06$ and $m_u/m_d \approx 0.7$ [in both cases probably erring on the side of increasing $(m_u + m_d)/m_s$] gives

$$\frac{m_\eta}{m_{\eta'}} < 0.540\,. \tag{13}$$

Two brief comments are in order.

(1) Note the role of large $N$ in the difference between Weinberg's bound ( [1]) and (1). To obtain Weinberg's bound, you maximize the ratio of $m_\eta$ to $m_\pi$ under variations of the ratio of the decay constants of the octet and singlet pseudoscalars. In (1), the ratio of decay constants is fixed to 1 by large $N$, and what varies is the ratio of $m_s$ to the nonperturbative contribution to $m_{\eta'}$.

(2) It is not surprising that the large $N$, chiral perturbation theoretic analysis fails for the $\eta'$. The $\eta'$ mass in our world is sufficiently large that higher order terms in the chiral Lagrangian are probably important. Likewise, three colors is surely not enough to justify total neglect of nonleading terms in $1/N$. Nevertheless, I find it amusing that the failure happens the way it does. It is not that you can fit the masses and then the details like decay rates and branching ratios do not work. You cannot even get started.

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