Experimental constraints on the neutrino oscillations and a simple model of three-flavor mixing

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A simple model of neutrino mixing is considered which contains only one right-handed neutrino field coupled, via the mass term, to the three usual left-handed fields. This is the simplest model that allows for three-flavor neutrino oscillations. The existing experimental limits on the neutrino oscillations are used to obtain constraints on the two free-mixing parameters of the model. A specific sum rule relating the oscillation probabilities of different flavors is derived.

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In the past fifteen years, considerable effort has been made to detect the effect of neutrino oscillations [1]. A positive signal in the neutrino oscillation experiments would indicate nonzero neutrino masses and provide some information on the pattern of the lepton mixing. Constraints have been obtained on almost all possible types of neutrino oscillations. The results of the neutrino oscillation experiments are usually expressed in the form of limits on the mixing angle as a function of the difference in the squared neutrino masses, under the assumption that the oscillations transition occurs between only two weak-interaction eigenstates.

In this Brief Report, we consider a model of neutrino oscillations that goes beyond the two-flavor approximation, allowing for oscillations involving all three neutrino flavors, but which is much simpler than the most general case with a three-flavor mixing. This model contains only one right-handed neutrino field coupled, via the mass term, to the three usual left-handed fields. We show that the formulas for the neutrino oscillation probabilities in this model may be expressed in a compact form. We consider constraints on the neutrino mixing implied by the experimental limits on various oscillation probabilities. We show that these constraints have a simple geometric interpretation. We discuss, in some detail, the constraints from the presently available data from the accelerator neutrino oscillation experiments. Finally, we obtain in the considered model a sum rule relating the oscillation probabilities.

The effect of the neutrino oscillation in the vacuum is described by the formula for the probability $P_{ij}(L)$ of detecting the weak-interaction eigenstate v_j at the distance L from the region in which neutrinos in the weak interaction eigenstate v_i are produced [2]:

$$P_{ij} = \sum_k |U_{jk}|^2 |U_{ik}|^2$$

$$+ \text{Re} \sum_{k \neq l} U_{jk} U_{jl}^* U_{ik}^* U_{il} \exp\left[-i \frac{(m_k^2 - m_l^2)L}{2p} \right], \quad (1)$$

where U_{kl} is a unitary mixing matrix relating the elec-

troweak neutrino eigenstates v_i with the mass eigenstates N_k,

$$v_i = \sum_k U_{ik} N_k , \quad (2)$$

and p denotes the neutrino momentum. In (1), it is assumed that the neutrinos are ultrarelativistic.

If only two neutrino flavors are taken into account, the mixing matrix takes the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} , \quad (3)$$

and the formula (1) reduces to the well-known expression [3]

$$P_{12}(L) = \sin^2 2\theta_{12} \sin^2 \frac{\Delta m^2 L}{4p} , \quad (4)$$

where $\Delta m^2 = |m_1^2 - m_2^2|$. This formula is used in most analyses of the experimental data on the neutrino oscillations [1]. The experimentally determined limits on the transition probabilities P_{ij} are translated into constraints on the allowed values of $\sin^2 2\theta_{ij}$, depending on the value of Δm^2.

As an illustration of the present experimental situation, we show in Fig. 1(a) compilation of the best constraints obtained from the accelerator experiments on the $v_e\text{-}v_\mu, v_e\text{-}v_\tau, v_\mu\text{-}v_\tau$ oscillations and the neutrino disappearance reactions $v_e\text{-}v_x, v_\mu\text{-}v_x$, where v_x denotes a neutrino of arbitrary type. It should be noted, however, that it is impossible for nontrivial $v_e\text{-}v_\mu$, $v_e\text{-}v_\tau$, and $v_\mu\text{-}v_\tau$ oscillations to have, simultaneously, the two-state character. For example, if we assume that the $v_e\text{-}v_\mu$ oscillations have a two-state character, then at the same time we *a priori* exclude the possibility of any $v_e\text{-}v_\tau, v_\mu\text{-}v_\tau$ oscillations, while the probabilities for the $v_e\text{-}v_x$ and $v_\mu\text{-}v_x$ transitions are trivially reduced to the $v_e\text{-}v_\mu$ case. Therefore the set of constraints on $\sin^2 2\theta_{ij}$ given by the conventional analysis does not properly reflect the patterns of the neutrino mixing which are still allowed by the available experimental data.
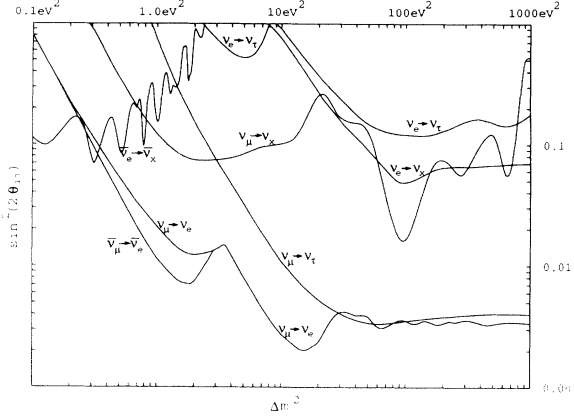
FIG. 1. Experimental limits on the neutrino oscillations obtained in the accelerator experiments, expressed in terms of constraints on the mixing angle (two-state analysis): v_e-v_μ [14], v_e-v_τ [15,16], v_μ-v_τ [15], v_e-v_x [17], v_μ-v_x [18]. All curves correspond to 90% C.L. limits. The limit on $\bar{v}_\mu$-$\bar{v}_e$ oscillations [19] has been indicated when it is stronger than the v_e-v_μ limit. Also the constraint on the $\bar{v}_e$-$\bar{v}_x$ transitions from the Goesgen reactor experiment [7] has been included for completeness.

On the other hand, a general analysis involving three-flavor neutrino mixing appears to be rather complicated. The three-flavor mixing matrix contains four free parameters which may be chosen, for example, in the form

$$U = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\delta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad (5)$$

φ, ϑ, and ψ denote mixing angles and δ denotes the CP-violating phase. (This parametrization is identical to the well-known Kobayashi-Maskawa parametrization [4], except for the permutation of axes and redefinition of δ.) The experimental constraints on these four parameters depend, in general, on two neutrino mass squared differences which are unknown. Another complication comes from the fact that experimentally measured probabilities are related to the theoretical probabilities, via a convolution, with phenomenological functions carrying the information on the energy spectrum of the neutrino beam or the geometry of the experimental apparatus. In a general three-flavor analysis such convolutions would have to be numerically reevaluated, taking into account three oscillating terms instead of one as in (4). A three-flavor analysis of the accelerator and reactor data was attempted in [5]. However, the results of [5] depend, in an essential way, on the data from the Bugey reactor experiment [6], which seemed to restrict one of the mass-squared differences to a narrow range, thus simplifying the whole analysis. Unfortunately, the Bugey reactor data were later found to be inconsistent with several other reactor experiments [7,8]. A three-flavor analysis may also be found in [9] in which the early oscillation experiments are discussed.

It may well be, however, that allowing for an arbitrary configuration of neutrino masses introduces an unnecessary complication. It seems reasonable to expect that one of the neutrino mass eigenstates would be much heavier than the remaining two. Such a hierarchical pattern is observed with other leptons and in the quark sector, and there is an argument involving the so-called seesaw mechanism [10] that neutrino masses should also have this structure. If this would be the case, then in the oscillation experiments sensitive to the heavier mass eigenstate, the two other states would appear as effectively massless [9]. This shows that it would be useful to consider a model of neutrino mixing and masses in which only one mass eigenstate has a mass different from zero [11,12].

The neutrino mass term of such a model may be written in the form

$$L_{v \text{ mass}} = -m \, (c_e \bar{v}_e^L + c_\mu \bar{v}_\mu^L + c_\tau \bar{v}_\tau^L) N_3^R + \text{H.c.} , \quad (6)$$

where

$$|c_e|^2 + |c_\mu|^2 + |c_\tau|^2 = 1 . \quad (7)$$

The parameters c_i in this mass term may always be redefined to be real (and positive), so that there is no CP violation. These parameters may be interpreted as directional cosines which fix the orientation of the massive eigenstate in the three-dimensional space spanned by the weak eigenstates

$$c_i = \cos\alpha_i . \quad (8)$$

It should be noted that the neutrino mass term (6) is the simplest extension of the standard model of electroweak interactions, which allows for nontrivial three-flavor neutrino oscillations, and therefore it may be of some interest in itself [13].

The expressions for the neutrino oscillation probabilities in this model may be derived by a simple argument. The form of the neutrino disappearance probabilities is immediately obtained when we note that the neutrino state vector that was initially a v_i weak eigenstate evolves in a two-dimensional space spanned by the massive component and this weak eigenstate. Therefore

$$P_{ix} = \sin^2 2\alpha_i \sin^2 \frac{m^2 L}{4p} . \quad (9)$$

The neutrino appearance probabilities may then be calculated as a solution of a system of equations

$$P_{e\mu} + P_{e\tau} = P_{ex} , \quad (10)$$

$$P_{\mu e} + P_{\mu\tau} = P_{\mu x} , \quad (11)$$

$$P_{\tau e} + P_{\tau\mu} = P_{\tau x} , \quad (12)$$

where we have $P_{ij} = P_{ji}$ because there is no CP violation in our model. In this way we find

$$P_{ij} = 4 \cos^2\alpha_i \cos^2\alpha_j \sin^2 \frac{m^2 L}{4p} . \quad (13)$$

We see that despite the fact that all types of neutrino oscillations may be consistently accommodated in our model, the formulas for the oscillation probabilities have a simple form.

The expressions (9) and (13) may also be obtained from the general formula (1) using the mixing matrix (5), provided that we identify the angles φ and ϑ as the two angles fixing the orientation of the massive state relative to the weak eigenstates, i.e.,

$$\cos\alpha_e = \cos\varphi\sin\vartheta, \quad \cos\alpha_\mu = \sin\varphi\sin\vartheta,$$

$$\cos\alpha_\tau = \cos\vartheta.$$

(14)

The value of the mixing angle ψ may be arbitrary because it corresponds to a redefinition of the massless states, which does not affect the oscillation probabilities.

Given the formulas for the oscillation probabilities, we may now use the available experimental data to exclude some values of the mixing angles φ and ϑ. The constraints obtained from the various experiments acquire a simple geometrical interpretation, when one represents different values of φ and ϑ as points on a unit sphere, corresponding to the locations of a "tip" of the massive eigenstate vector in the space spanned by the three weak eigenstates. It is enough to consider φ and ϑ in the range $0°-90°$. The boundaries of the relevant triangular region on the sphere ($\varphi=0°,90°$ and arbitrary $\vartheta,\vartheta=90°$ and arbitrary φ) correspond to the neutrino mixing which has a purely two-flavor character. From the formula (9), we see that for a given mass m the limits on the probabilities of ν_i-ν_x transitions exclude regions of the unit sphere bounded by circles of constant $\sin^2 2\alpha_i$. The limits on probabilities for ν_i-ν_j oscillations exclude regions bounded by the line on which the product $\cos^2\alpha_i\cos^2\alpha_j$ is constant, which are projections of hyperbolas on the sphere, and the side of the spherical triangle joining the ν_i and ν_j corners.

It is important that the actual numerical value of the constraints on $\sin^2 2\alpha_i$ may be obtained directly from the

limits on $\sin^2 2\theta_{ex}$ and $\sin^2 2\theta_{\mu x}$ extracted in the two-state analysis of the ν_e-ν_x and ν_μ-ν_x oscillation experiments, provided that the Δm^2 of the conventional analysis is now understood as the mass squared of the massive eigenstate. Similarly, the constraints on $4\cos^2\alpha_i\cos^2\alpha_j$ are obtained directly from the limits on $\sin^2 2\theta_{e\mu},\sin^2 2\theta_{e\tau}$ and $\sin^2 2\theta_{\mu\tau}$ considered in the conventional approach to ν_i-ν_j oscillations in the neutrino appearance experiments.

As an illustration, we show in Fig. 2 the pattern of constraints for $m^2=1$ eV^2 that corresponds to the 90% C.L. limits obtained in the two-state analysis (see Fig. 1). We find that the orientations of the massive eigenstate, allowed by all the available constraints at this mass, are confined to three rather small regions of rectangular shape lying near the corners of the spherical triangle. The angular dimensions of these rectangles are determined directly by the angles θ_{ij} obtained in the two-state analysis of the experimental data. For example, the size of the region near the ν_τ corner is determined by $\theta_{\mu x}$ and θ_{ex}. [For a general value of mass, the relevant bounds for this region are given by $\min(\theta_{\mu x},\theta_{\mu\tau})$ and $\min(\theta_{ex},\theta_{e\tau})$.] In Fig. 2, it may be seen that the limit on ν_e-ν_μ oscillations also restricts the strength of the eventual ν_e-ν_τ and ν_μ-ν_τ oscillations. This is a manifestation of a correlation
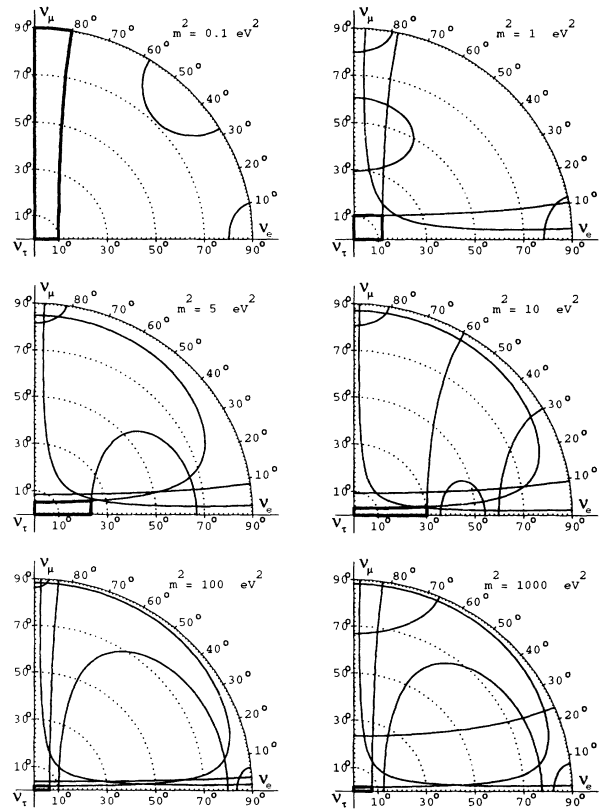
FIG. 3. Constraints on the mixing angles φ and ϑ for six values of the neutrino mass. The radial variable on these plots is ϑ, and the angular variable is φ. The indicated curves reflect the 90% C.L. limits on $\sin^2 2\theta_{ij}$ obtained in the two-state analysis of the accelerator data. Thick lines surround the regions of the allowed values of φ and ϑ for which the massive state is mostly the ν_τ weak eigenstate.
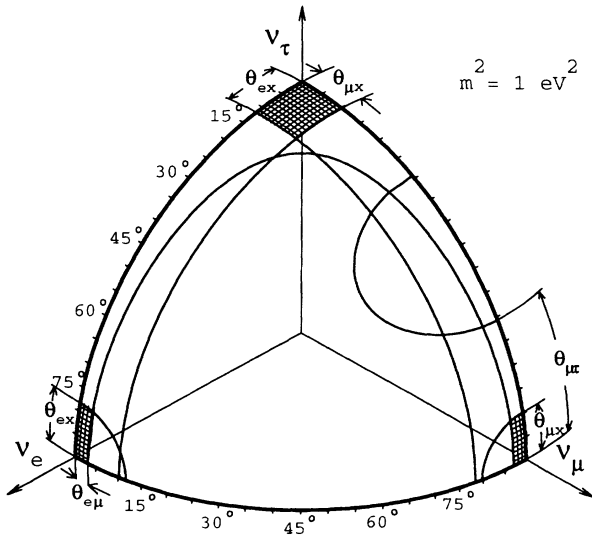

FIG. 2. Experimental constraints on the orientation of the massive neutrino state with $m^2=1$ eV^2. The indicated curves correspond to the 90% C.L. limits on $\sin^2 2\theta_{e\mu}$, $\sin^2 2\theta_{\mu\tau}$, $\sin^2 2\theta_{ex}$, and $\sin^2 2\theta_{\mu x}$ obtained in the two-state analysis. Dashed regions indicate orientations of the massive eigenstate consistent with all the indicated constraints.

between oscillations of different flavors which is present in the considered model.

In Fig. 2, we see that the various constraints represented on a surface of the sphere have a simple and highly symmetric form. However, for a detailed analysis of the experimental results, it is more convenient to use a two-dimensional plot in which ϑ plays the role of a radial variable and φ remains an angular variable. Such plots have been used in Fig. 3 to show how the available constraints evolve when m^2 is varied from 0.1 eV^2 to 1000 eV^2. The indicated curves reflect the 90% C.L. limits obtained in accelerator experiments [14–19]. (As exception is the plot for $m^2 = 0.1$ eV^2 in which the constraint from the Goesgen reactor experiment [7] on $\bar{\nu}_e$-$\bar{\nu}_x$ oscillations has been included.) The plot for $m^2 = 1000$ eV^2 represents the asymptotic form of the constraints for large neutrino mass. We see that for $m^2 \geq 1$ eV^2, the parameters consistent with all the constraints remain located in the approximately rectangular regions in the corners of the triangle. However, the sizes of these regions vary significantly. In Fig. 3, we clearly see changes in the character of the strongest constraints that determine the size of the allowed regions. For example, in the case of the ν_τ corner, which seems to be the most interesting from the phenomenological point of view, the dominant constraints at $m^2 = 1$ eV^2 come from the neutrino disappearance experiments for $m^2 = 5$ eV^2, from neutrino appearance experiments sensitive to ν_μ-ν_τ and ν_e-ν_τ oscillations, and for $m^2 \geq 10$ eV^2 from ν_e-ν_x and ν_μ-ν_τ experiments.

It should be noted that the allowed regions of the mixing parameters shown in Figs. 2 and 3 indicate parameters consistent with all constraints reflecting the 90% C.L. limits on $\sin^2\theta_{ij}$, provided that these constraints are treated independently. A more precise statistical analysis of the data within our model would require a consideration of joint probability distributions for φ and ϑ implied by all the experimental results. Such an analysis goes beyond the scope of the present paper.

It is interesting to note that within the considered model one may obtain a nontrivial sum rule relating the neutrino appearance probabilities, which reflects the fact that there are only two independent mixing parameters in this case. Indeed, let us denote

$$P_{ij} = R_{ij}\sin^2\frac{m^2L}{4p} \ . \tag{15}$$

Then we have

$$R_{e\mu}R_{e\tau}R_{\mu\tau}\left[\frac{1}{R_{e\mu}} + \frac{1}{R_{e\tau}} + \frac{1}{R_{\mu\tau}}\right]^2 = 4 \ . \tag{16}$$

If a positive signal for nontrivial ν_e-ν_μ, ν_e-ν_τ, and ν_μ-ν_τ oscillations is obtained, then this sum rule may be used as a test on the character of the neutrino mixing. If, for some m^2, the $R_{ij}^{\rm expt}$ factors, obtained from the experimentally-determined probabilities, would satisfy the sum rule (16) with a good accuracy, then this would be a strong argument in favor of the presence of a dominant massive neutrino eigenstate with this mass.

Summarizing, we may say that we have discussed the simplest model of neutrino mixing which allows for three-flavor neutrino oscillations. This model is of physical interest because of the expected hierarchical pattern of the neutrino masses. We have shown that, by using the experimental limits on the oscillation probabilities, it is easy to obtain constraints on the two parameters that characterize the neutrino mixing in this model. We have obtained the domain of the mixing parameters consistent with the available data on the accelerator neutrino oscillation experiments. We have found that this domain may be estimated directly from the properly reinterpreted results of the conventional two-flavor analysis. We have pointed out that there exists a sum rule relating the neutrino appearance probabilities in this model, which may be used as a test for the presence of a dominant massive neutrino state.

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