Mass and the $E1$ decay rate of the singlet P state of charmonium

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We calculate the hyperfine splitting of the P states of charmonium using the perturbative QCD hyperfine interaction to order α_s^2, in an improved quasistatic approximation whereby the quark-antiquark scattering amplitude is expanded in powers of p^2/p_0^2 instead of p^2/m^2 and terms of up to first order in p^2/p_0^2 are kept. We evaluated the hyperfine splitting using the wave functions obtained from the unperturbed Hamiltonian of Gupta, Radford, and Suchyta. We find the splitting $\Delta M_P = M_{\text{c.o.g.}}^{^3P_J} - M^{^1P_1}$ to be -0.63 MeV. Our result is very similar to the result of Halzen, Olson, Olsson, and Stong who find $\Delta M_P = -0.7 \pm 0.2$ MeV, using various potential models. It also confirms the recent published experimental result on ΔM_P. We also note that if we had used the improved quasistatic approximation of Gupta to extract the hyperfine interaction from the $q\bar{q}$ scattering amplitude to order α_s^2 in QCD, we would have obtained entirely different results for the P-wave hyperfine splitting in charmonium. We also calculate the electric dipole decay rate of the process $1\,^1P_1 \to 1\,^1S_0 + \gamma$ and find it to be about 630 keV for charmonium.

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Recently there has been much interest [1,2] in the calculation of the mass of the singlet P state in charmonium. This interest is motivated by the ongoing experimental efforts of the E760 group [3] at Fermilab to detect this state in the $\bar{p}p$ collisions and to measure its mass. They recently detected the $^1P_1$ state of charmonium [3] and determined its mass to be 3526.2 MeV. Meanwhile Halzen, Olson, Olsson, and Stong have shown that to first order in α_s, the hyperfine splitting of the P states is zero and to second order in α_s the expression for the splitting is explicitly independent of the renormalization scheme and scale. They concluded that the singlet P state is 0.7 ± 0.2 MeV above the triplet center of gravity, consistent with the measurement of the E760 group [3]. The conclusions of Ref. [1] are based on a hyperfine interaction potential which was evaluated from the one gluon exchange diagram and all its one loop corrections in QCD for the quark-antiquark scattering, in the approximation of keeping only terms up to first order in $\mathbf{p}^2/m^2$ in the expression for the scattering amplitude, where $\mathbf{p}$ is the quark momentum in the center-of-mass frame. In this Brief Report we investigate the hyperfine splitting of the P states in an improved quasistatic approximation [4] whereby the scattering operator is expanded in powers of $\mathbf{p}^2/p_0^2$ and only terms of first order in $\mathbf{p}^2/p_0^2$ are kept. Here

$$p_0^2 = m^2 + \mathbf{p}^2 \,. \tag{1}$$

This should be a better approximation since the quantity $\mathbf{p}^2/p_0^2$ is always less than $\mathbf{p}^2/m^2$ and is better behaved in the high momentum limit. To first order in $\mathbf{p}^2/p_0^2$ the quark-antiquark scattering kernel in momentum space (or essentially the Fourier transform of the QCD potential in coordinate space) pertaining to the hyperfine splitting can be written as

$$V_s(\mathbf{p},\mathbf{p}') = \frac{32}{9}\pi\alpha_s \frac{\mathbf{s}_1\cdot\mathbf{s}_2}{p_0^2} - \frac{8}{27}\alpha_s^2 \frac{\mathbf{s}_1\cdot\mathbf{s}_2}{p_0^2}\left\{(33-2n_f)\ln\frac{k^2}{\mu^2}+18\right\} - \frac{8}{3}\frac{\alpha_s^2}{p_0^2}\mathbf{s}_1\cdot\mathbf{s}_2\left[\frac{8}{9}+\ln 2-\frac{7}{2}\ln\frac{k^2}{m^2}\right] . \tag{2}$$

We have used the Gupta-Radford renormalization scheme [5]. Equation (2) will agree with the corresponding equation of Ref. [1] in the Gupta-Radford scheme if we replace p_0^2 by m^2.

In order to use Eq. (2) for calculating the hyperfine splitting we note that the expectation value of the spin operator $\mathbf{s}_1\cdot\mathbf{s}_2$ in the triplet and the singlet quarkonium states can be written as

$$\langle \mathbf{s}_1\cdot\mathbf{s}_2\rangle = \tfrac{1}{2}\langle (s^2 - s_1^2 - s_2^2)\rangle = \tfrac{1}{2}[s(s+1)-\tfrac{3}{2}] \,, \tag{3}$$

where $s=1$ for the triplet state and $s=0$ for the singlet state. So if we denote the hyperfine mass splitting of the P states,

$$\Delta M_P = M_{\text{c.o.g.}}^{^3P_J} - M^{^1P_1} \,, \tag{4}$$

in first order perturbation theory it will be given by

$$\Delta M_P=\frac{1}{(2\pi)^3}\int\int d^3p\,d^3p'\phi^*(\mathbf{p}')\phi(\mathbf{p})V(\mathbf{p},\mathbf{p}')\ . \quad (5)$$

In Eq. (5), $V(\mathbf{p},\mathbf{p}')$ is given by

$$V(\mathbf{p},\mathbf{p}')=\frac{K_1}{m^2+\mathbf{p}^2}+\frac{K_2}{m^2+\mathbf{p}^2}\ln|\mathbf{p}-\mathbf{p}'|^2\ , \quad (6)$$

where

$$K_1=\left[\frac{32\pi}{9}\alpha_2-\frac{208}{27}\alpha_s^2-\frac{8\ln 2}{3}\alpha_s^2+\frac{8}{27}\alpha_s^2(33-2n_f)\ln(\mu^2)-\frac{28}{3}\alpha_s^2\ln(m^2)\right] \quad (7)$$

and

$$K_2=-\frac{4}{27}\alpha_s^2(3-4n_f)\ . \quad (8)$$

Substituting Eq. (6) in Eq. (5) we find that the contribution of the first term in Eq. (6) to the integral of Eq. (5) is zero since $\int\phi(\mathbf{p}')d^3p'=0$ for P states. So,

$$\Delta M_P=\frac{K_2}{(2\pi)^3}\int\int d^3p\,d^3p'\frac{\phi^*(\mathbf{p}')\phi(\mathbf{p})}{m^2+\mathbf{p}^2}\ln|\mathbf{p}-\mathbf{p}'|^2\ . \quad (9)$$

Just as in Ref. [1], the expression for ΔM_P in Eq. (9) does not depend on the QCD renormalization scale μ or the renormalization scheme, since the expression for K_2 in Eq. (8) is independent of the renormalization scheme. After doing the angular integrations in Eq. (9),

$$\Delta M_P=\frac{1}{(2\pi)^3}\frac{4\pi\alpha_s^2}{9}(1-\tfrac{4}{3}n_f)\times\int dp\,dp'\frac{\phi^*(p')\phi(p)}{m^2+p^2}\times\left\{pp'(p^2+p'^2)+\frac{1}{2}(p^2-p'^2)^2\ln\left|\frac{p-p'}{p+p'}\right|\right\}\ . \quad (10)$$

Equation (10) is very similar to the expression in Ref. [1], except for the fact that $1/m^2$ is replaced by $1/(m^2+\mathbf{p}^2)$ under the integral sign in our Eq. (10). The difference in the factors $1/(2\pi)^3$ and $1/(2\pi)^6$ comes from the different normalizations of the wave functions. We have normalized the wave function $\phi(\mathbf{p})$ to unity whereas in Ref. [1] they have normalized it to $(2\pi)^3$.

The two-dimensional integral in Eq. (10) is done numerically. For this we had to know the radial momentum wave function $\phi(p)$. In order to calculate $\phi(p)$ we used the unperturbed Hamiltonian of Gupta, Radford, and Repko [6]:

$$H_0=2\sqrt{\mathbf{p}^2+m^2}+Ar+C-\frac{4}{3}\frac{\alpha_s}{r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E]\right]\ , \quad (11)$$

where γ_E is the Euler constant. The Hamiltonian of Eq. (11) is also the spin-independent part of the nonsingular Hamiltonian used by Gupta, Repko, and Suchyta [7] (GRS). We found the eigenfunctions $\phi(\mathbf{p})$ of this Hamiltonian by the variational method [5]. Our trial wave function has the form

$$\phi(\mathbf{p})=i\left[\frac{2}{\pi}\right]^{1/2}\frac{R^2}{p}Y_{1m}(\hat{p})\sum_{n=1}^{9}A_n\Gamma(n+1)(\cos\theta)^{n+2}\times\left\{(n+2)\cos(n+2)\theta-\frac{\sin(n+2)\theta}{pR}\right\}\ , \quad (12)$$

where $\theta=\arctan(pR)$ and R is the parameter in the trial wave function in coordinate space:

$$\psi(\mathbf{r})=\sum_{n=1}^{9}A_n(r/R)^n e^{-r/R}Y_{1M}(\hat{r})\ . \quad (13)$$

R is determined by applying the virial theorem and the coefficients A_i are determined by the variational method. For the mass m of the charmed quark, following Ref. [8], we used

$$m=1.919\ \text{GeV}\ .$$

Our numerical result is

$$\Delta M_P=M_{\text{c.o.g.}}^{^3P_J}-M^{^1P_1}=-0.63\ \text{MeV}\ . \quad (14)$$

This is very close to the experimental value [3] and to the result quoted in Ref. [1]. Moreover, if we replace p_0^2 of Eq. (2) by $p_0'^2$ or by p_0p_0', choices which are equivalent for on-shell quarks, we find the change of Eq. (14) to be insignificant.

We also would like to make an important comment on the improved quasistatic approximation as utilized by Gupta [4]. The approximation he used had two ingredients. First the scattering operator was expanded in powers of $\mathbf{p}^2/p_0^2$ and only terms of first order in $\mathbf{p}^2/p_0^2$ were kept, just as we have done above. Then he made the further assumption that

$$\mathbf{p}^2=\tfrac{1}{4}\mathbf{k}^2+\tfrac{1}{4}\mathbf{s}^2\simeq\tfrac{1}{4}\mathbf{k}^2\ , \quad (15)$$

where

$$\mathbf{k}^2=|\mathbf{p}-\mathbf{p}'|^2 \quad (16)$$

and

$$\mathbf{s}^2=|\mathbf{p}+\mathbf{p}'|^2\ . \quad (17)$$

So in this approximation we can write

$$p_0^2=m^2+\mathbf{p}^2\simeq m^2+\tfrac{1}{4}\mathbf{k}^2\ . \quad (18)$$

With this approximation, all the terms in the Hamiltonian become nonsingular in coordinate space and a nonperturbative variational calculation becomes possible with the full Hamiltonian. This was the motivation behind

this approximation. If we also adopted this approximation for the spin-spin interaction, the first term in Eq. (6) would have contributed a finite result to the integral in Eq. (5) and in fact the linear term in α_s (in the expression for the hyperfine interaction) would give the most dominant contribution. This happens to be +3.07 MeV with our wave function, whereas the contribution from the α_s^2 term is only −0.88 MeV. Even the sign of the hyperfine splitting then changes. We also used the nonsingular Hamiltonian of Gupta, Repko, and Suchyta [7] obtained by means of the improved quasistatic approximation of Gupta [4] in a nonperturbative variational calculation to obtain the hyperfine splitting ΔM_p of the P states. Now all the terms in the nonsingular Hamiltonian, including the spin-independent, tensor, and spin-orbit terms, contribute to the hyperfine splitting since the wave functions for the triplet and the singlet states are different in a nonperturbative variational calculation using the full Hamiltonian. The mass splitting ΔM_p for charmonium turns out to be −2.32 MeV. Even though it has the right sign as the experimental number [3] its magnitude is too large. Perhaps it suggests that the variable $\mathbf{s}^2$ of Eq. (15) is generally of the same magnitude as $\mathbf{k}^2$ in charmonium and its neglect may be a poor approximation, at least in the spin-spin interaction.

We also calculated the electric dipole ($E1$) one-photon transition rate for the process $^1P_1 \rightarrow {}^1S_0 + \gamma$ for charmonium. The $E1$ decay rate for this process is given by the formula [8]

$$\Gamma_{E1} = \frac{8}{27}\alpha(\omega_{BA}^I)^2 \omega_{BA} J_1^2 \left[1 + \left[\frac{2}{3}\right]^{1/2} \frac{k}{10m}\frac{J_2}{J_1}\right], \quad (19)$$

where α is the fine structure constant,

$$\omega_{BA}^I = M(^1P_1) - M(^1S_0), \quad (20)$$

$$\omega_{BA} = \omega_{BA}^I - \frac{(\omega_{BA}^I)^2}{2M_{^1S_0}}, \quad (21)$$

$$J_1 = \int_0^\infty R_{^1S_0} R_{^1P_1} r^3 dr, \quad (22)$$

$$J_2 = \int_0^\infty R_{^1S_0} \frac{dR_{^1P_1}}{dr} r^4 dr, \quad (23)$$

where $R_{^1S_0}$ and $R_{^1P_1}$ are the radial wave functions of the $^1S_0$ and the $^1P_1$ states which were calculated nonperturbatively by the variational method in the GRS model [7]. In Eq. (19) the second term in the large brackets represents the finite-size corrections. By Seigert's theorem [9], the $E1$ transition operator in the position vector form [10] takes into account most of the relativistic corrections of relative order $\langle v^2/c^2 \rangle$ except for finite-size effects and the effects of any possible anomalous magnetic moments of the quark. The effect of any reasonable anomalous magnetic moment of the quark on the $E1$ decay rate is found to be extremely small and so we neglected it in our calculations. So most of the relativistic corrections will be included when we use the eigenfunctions for the full Hamiltonian (which includes relativistic corrections) to calculate the integral J_1. Because of the nonsingular nature of the GRS Hamiltonian [7] we can calculate these eigenfunctions nonperturbatively by a variational calculation. In calculating these wave functions for charmonium we used the GRS scalar-exchange model [7].

In estimating the numerical value of the $E1$ decay rate of the $^1P_1$ state in charmonium we took the photon energy to be 521 MeV. We find the decay rate to be about 630 keV. The finite-size correction coming from the second term on the right-hand side of Eq. (19) is about −16 keV. We should also mention that Bodwin, Braaten, and Lepage [11] have calculated the radiative decay rate of the $^1P_1$ state in charmonium using a QCD factorization formula and heavy quark effective field theory. They get a decay rate ranging from 300 to 700 keV where the range of values is due to a combination of experimental errors in the branching ratios and total widths of the χ_{c1} and χ_{c2} states, the uncertainty in α_s, and an estimate of theoretical uncertainty which includes v^2/c^2 corrections and higher order perturbative QCD corrections. Whether we accept their value or our result from a potential model calculation, the predicted decay rate seems to be a rather large fraction of the measured upper limit of the total $^1P_1$ width [3]. So it is probably the dominant decay mode for the $^1P_1$ state of charmonium. The detection of this photon will confirm the discovery of this state.

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