PHYSICAL REVIEW D VOLUME 49, NUMBER 3 1 FEBRUARY 1994

Suppression of bremsstrahlung at nonzero temperature

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(Received 3 August 1993)

The first-order bremsstrahlung emission spectrum is $\alpha d\omega/\omega$ at zero temperature. If the radiation is emitted into a region that contains a thermal distribution of photons, then the rate is multiplied by a factor $1+N(\omega)$ where $N(\omega)$ is the Bose-Einstein function. The stimulated emission changes the spectrum to $\alpha T d\omega/\omega^2$ for $\omega \ll T$. If this were correct, an infinite amount of energy would be radiated in the low frequency modes. This unphysical result indicates a breakdown of perturbation theory. The paper computes the bremsstrahlung rate to all orders of perturbation theory, neglecting the recoil of the charged particle. When the perturbation series is summed, it has a different low-energy behavior. For $\omega \ll \alpha T$, the spectrum is independent of ω and has a value proportional to $d\omega/\alpha T$.

PACS number(s): 11.10.Wx, 12.20.Ds, 25.75.+r, 41.60.−m

I. INTRODUCTION AND SUMMARY

A. Background

In studies of the quark-gluon plasma to be produced in ultrarelativistic heavy-ion collision and, more generally, in studies of field theory at finite temperature, a central concern is how the cancellation of infrared divergences affects various finite, physical quantities. The experimentally measured rates for certain low-energy processes can be significantly modified by the infrared structure.

This paper will investigate the emission of low-energy photons by a charged particle that passes through a fixed-temperature plasma. The charged particle must undergo a collision in order to radiate. If the radiated energy is much smaller than the energy transfer in the collision, then the inelastic cross section factors

$$2\omega\frac{d\sigma}{d^3k\, dq^2} \approx 2\omega\frac{dP(q^2)}{d^3k}\frac{d\sigma}{dq^2} \quad (|k\cdot q| \ll |q^2|) , \qquad (1.1)$$

with the collision cross section $d\sigma/dq^2$ independent of $k\cdot q$, where $q=p-p'$. Thus, to investigate the probability of radiation, P, the details of the hard scattering do not matter. All that matters is that the charged particle began with four-momentum p^μ and ended with four-momentum p'^μ. The radiation probability to first order in α is

$$2\omega\frac{dP_1}{d^3k} = \frac{1}{(2\pi)^3}\sum_{\rm pol}|\epsilon_\mu\cdot J^\mu|^2[1+N(\omega)] , \qquad (1.2)$$

where $\omega=|\mathbf{k}|$ and $N(\omega)=[1-\exp(\omega/T)]^{-1}$ is the Bose-Einstein function. When the radiated energy is small, the matrix element of the electromagnetic current is

$$J^\mu(k) = ie\left[\frac{p'^\mu}{p'\cdot k}-\frac{p^\mu}{p\cdot k}\right]e^{-k/2\Lambda} , \qquad (1.3)$$

where Λ is a momentum-space regulator that will be necessary later. The current results from an on-shell charge of four-momentum p radiating an on-shell photon of four-momentum k, which gives a Feynman denominator $(p-k)^2-m^2=-2p\cdot k$. For low-energy radiation, this current is valid regardless of the spin of the charged particle [1–3]. Because of the denominators in (1.3), the radiation is mostly parallel to $\mathbf{p}$ or $\mathbf{p}'$. When (1.2) is integrated over photon angles, the probability of radiating energy ω in any direction is

$$\frac{dP_1}{d\omega} = \frac{A}{\omega}[1+N(\omega)]e^{-\omega/2\Lambda} . \qquad (1.4)$$

The charged particle momenta only occur in the function A:

$$A(p\cdot p') = \frac{\alpha}{\pi}\left[\frac{1}{v}\ln\left[\frac{1+v}{1-v}\right]-2\right], \qquad (1.5)$$

where v is defined by $p\cdot p'=m^2(1-v^2)^{-1/2}$. In terms of momentum transfer Q, the limiting behaviors of A are [1,2]

$$A \to \frac{2\alpha}{3\pi}\frac{Q^2}{m^2} \quad \text{for } Q \ll m , \qquad (1.6a)$$

$$A \to \frac{2\alpha}{\pi}\ln\left[\frac{Q^2}{m^2}\right] \quad \text{for } Q \gg m . \qquad (1.6b)$$

Formula (1.4) is classical, except for photon quantum statistics which produce the factor N. The formula fails at low energies because it predicts that the total energy radiated will be infinite:

$$\int_0^{E_{\rm max}} d\omega\,\omega\frac{dP_1}{d\omega} = \infty , \qquad (1.7)$$

regardless of the value of $E_{\rm max}$. This does not occur at zero temperature, where the radiation probability is A/ω; but at nonzero temperature, the Bose-Einstein factor makes it more singular:

$$\frac{dP_1}{d\omega} \approx \frac{AT}{\omega^2} \quad (\omega \ll T) . \qquad (1.8)$$

The breakdown comes from the smallest values of ω,

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where the classical approximation (1.3) works best. The resolution of the breakdown must occur from a higher-order calculation.

B. Higher orders

To go to higher orders in α, one must quantize the electromagnetic field with the interaction

$$\mathcal{L}_I = -J^\mu(x) A_\mu(x) , \tag{1.9}$$

with J^μ the classical current (1.3) and A_μ the quantized field. This interaction will produce multiple emissions and absorptions of real photons as well as closed loops of virtual photons. For example, to second order in α there are three types of contributions to the probability of radiating energy ω: Either radiate two photons whose energy totals ω; or radiate one photon of energy $\omega+\omega'$ and absorb another whose energy is ω', giving a net energy of ω; or radiate a single photon of energy ω with a one-loop correction. The infrared divergences cancel among these processes and give a finite answer $dP_2/d\omega$ to order α^2.

The semiclassical interaction (1.9) does not conserve energy momentum: It allows the particle of momentum p^μ to radiate and still have momentum p^μ (provided the hard collision eventually deflects the charge to p'^μ). This is a sensible approximation if each photon energy (real or virtual) is much smaller than the energy of the charged particle. To maintain this consistently in higher orders, it is convenient to introduce the momentum cutoff Λ in (1.3) and require $\Lambda \ll (\mathbf{p}'^2+m^2)^{1/2}$ and $\Lambda \ll (\mathbf{p}^2+m^2)^{1/2}$.

Because the current is classical, one can compute the generating functional for the multiphoton Green's functions by performing a Gaussian functional integration. This allows one to calculate the exact multiphoton amplitudes $\mathcal{M}$. The probability of radiating a *net* energy ω to all orders in perturbation theory is

$$\frac{dP}{d\omega} = \sum_{n=1}^{\infty} \int d\Phi_1 \cdots d\Phi_n \delta(k_1^0 + \cdots + k_n^0 - \omega) \times \frac{1}{n!} \sum_{\text{pol}} |\mathcal{M}(k_1, \ldots, k_n)|^2 . \tag{1.10}$$

Positive k^0's correspond to emission of photons; negative k^0's correspond to absorption of photons. The phase space integration includes the appropriate statistical factors:

$$d\Phi = \frac{d^3k}{2k(2\pi)^3} \times \begin{cases} 1+N(k), & k^0=+k , \\ N(k), & k^0=-k . \end{cases} \tag{1.11}$$

The amplitudes $\mathcal{M}$ contain infrared divergences from the virtual photon integrations, but the integration over the real photons in (1.10) produces an infrared-finite probability. This is the Bloch-Nordsieck cancellation at $T\neq 0$. The cancellation is very delicate: It would not occur if negative k^0's (i.e., energy absorption from the heat bath) were omitted from the δ function in (1.10).

C. Summary of results

Remarkably, one can calculate the probability (1.10) with only the approximation $T \ll \Lambda$. The final answer obtained at the end of Sec. III is

$$\frac{dP}{d\omega} = \left| \Gamma\left[\frac{A}{2} + i\frac{\omega}{2\pi T}\right] \right|^2 \frac{e^{\omega/2T} e^{-|\omega|/\Lambda}}{4\pi^2 T \Gamma(A)} \left[\frac{2\pi T}{\Lambda}\right]^A . \tag{1.12}$$

Here $A \propto \alpha$ is the same function of $p\cdot p'$ as (1.5). This result also applies if ω is negative, which means that the total emitted energy is less than the total absorbed energy. For $\omega = -\tilde{\omega} < 0$, it has the property

$$\frac{dP}{d\omega} = e^{-\tilde{\omega}/T} \frac{dP}{d\tilde{\omega}} . \tag{1.13}$$

In the first-order result (1.4), this corresponds to $\exp(-\tilde{\omega}/T)[1+N(\tilde{\omega})] = N(\tilde{\omega})$. The zero-temperature limit of (1.12) is discussed in Appendix A.

At $\omega=0$ the exact result (1.12) is finite. This resolves the total energy problem that arose in (1.7). The most interesting feature of (1.12) is that it involves two dimensionless quantities: A and ω/T. To display a simpler form, it is helpful to assume $A \ll 1$. This is quite reasonable because extremely large momentum transfers are necessary to make A large (for example, $A = \frac{1}{16}$ requires $Q/m = 10^3$). Even with $A \ll 1$, the behavior of (1.12) depends on the size of ω/T. If $A\pi \ll \omega/T$, then

$$\frac{dP}{d\omega} \approx \frac{A}{\omega}[1+N(\omega)]e^{-\omega/\Lambda} \quad (A\pi T \ll \omega) , \tag{1.14}$$

which agrees with (1.4). However, this does not hold at the smallest values of ω. To approximate (1.12) for $A \ll 1$ and $\omega \ll T$, one can use $\Gamma(z) \approx 1/z$ when $|z| \ll 1$. Then

$$\frac{dP}{d\omega} \approx \frac{AT}{\omega^2 + (A\pi T)^2} \quad (\omega \ll T) . \tag{1.15}$$

Naturally, (1.15) agrees with (1.14) in the region of overlap, $A\pi T \ll \omega \ll T$. But for $\omega \ll A\pi T$ the radiation is suppressed relative to the first-order rate [4]. Figure 1

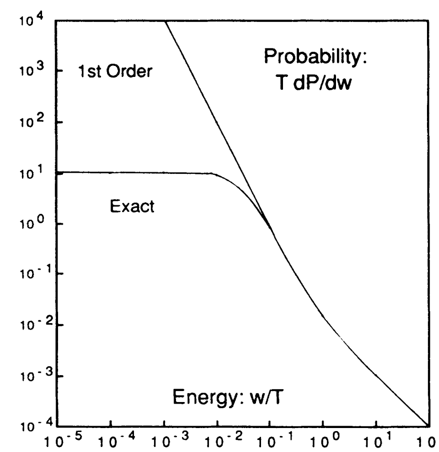

FIG. 1. Dimensionless probability $TdP/d\omega$ for a charged particle to radiate a net energy ω. The first-order result fails when $\omega < A\pi T$. For $\omega \ll A\pi T$, the exact result is constant, $TdP/d\omega \approx 1/A\pi^2$. In this plot, $A=0.01$, which corresponds to a momentum transfer $Q/m=3$.

compares (1.12) with the first-order result over a wide range of ω.

Naturally, the net radiated energy is finite,

$$\int_0^T d\omega\, \omega \frac{dP}{d\omega} \approx AT \ln\left[\frac{1}{A\pi}\right] , \qquad (1.16)$$

and numerically small. In the denominator of (1.15), the quantity $A\pi T$ appears to be some type of thermal mass of order e^2T. However, the following discussion will argue that $A\pi T = \Gamma/2$, where Γ is the radiation-damping rate.

It is necessary to emphasize several points. (i) At $T=0$ the exact bremsstrahlung probability does not enjoy this damping. It is peculiar to $T \neq 0$. (ii) Whenever $T>0$ the suppression will occur for a charge of any energy. Whether the charge is relativistic or nonrelativistic alters the value of A, but the plateau shown in Fig. 1 will always exist.

D. Interpretation of the suppression

One can understand the suppression of low-frequency bremsstrahlung as the radiation reaction brought about by unitarity. First, we examine the failure of the first-order bremsstrahlung probability by writing it as

$$\frac{dP_1}{d\omega} = \frac{|F_1(\omega)|^2}{2\pi} ,$$
$$F_1(\omega) = \sqrt{2\pi A}\left[\frac{1+N(\omega)}{\omega}\right]^{1/2} e^{-|\omega|/2\Lambda} . \qquad (1.17)$$

Let the Fourier transform of this be

$$\tilde{F}_1(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} F_1(\omega) . \qquad (1.18)$$

Then the bremsstrahlung probability has a time dependence

$$\frac{dP_1}{dt} = |\tilde{F}_1(t)|^2 . \qquad (1.19)$$

As $t \to \infty$, the Fourier transform of (1.17) behaves like

$$\tilde{F}_1(t) \to -i\sqrt{\Gamma} , \qquad (1.20)$$

where $\Gamma \equiv 2\pi AT$. Therefore $dP_1/dt \to \Gamma$ as $t\to\infty$. It is obviously unphysical for the radiation probability to remain constant after an infinitely long time. This problem is familiar in elementary atomic physics calculations of radiative transitions (e.g., $2p \to 1s+\gamma$ in hydrogen). The resolution is that at higher orders the transition amplitude usually falls exponentially with time. Hence a reasonable guess is that higher-order calculations should replace (1.20) by

$$\tilde{F}(t) \stackrel{?}{=} -i\theta(t)\sqrt{\Gamma} e^{-\Gamma t/2} . \qquad (1.21)$$

The Fourier transform to frequency gives

$$F(\omega) \stackrel{?}{=} \frac{\sqrt{\Gamma}}{\omega + i\Gamma/2} . \qquad (1.22)$$

The guess gives a radiation probability

$$\frac{dP}{d\omega} \stackrel{?}{=} \frac{1}{2\pi}\frac{\Gamma}{\omega^2+(\Gamma/2)^2} , \qquad (1.23)$$

which coincides with (1.15) in the weak-coupling, low-energy limit. Note that this radiation damping does not come from modifications to the charged particle trajectory, as would be the case in multiple scattering [5].

The remainder of the paper describes the calculation that yields (1.12). Section II formulates the bremsstrahlung problem in detail, and Sec. III performs the necessary integrations. Appendix A discusses the $T=0$ limit of (1.12) and its relation to conventional $T=0$ calculations. Appendix B discusses the relation of (1.12) to some previous work of mine.

II. THERMAL BREMSSTRAHLUNG

When charged particles are described by classical currents J^μ, quantizing the radiation fields becomes elementary. The generating functional for multiphoton Green's functions is

$$Z(J) = \int D[A] \exp\left[i\int_C d^4x(\mathcal{L}_0 - J\cdot A)\right] , \qquad (2.1)$$

where C is a contour in the complex x^0 plane that incorporates the temperature. For a real-time formulation [6–8], this gives

$$Z(J) = \exp\left[\frac{-i}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k) D_{\mu\nu}(k) J^\nu(-k)\right] , \qquad (2.2)$$

where the finite-temperature propagator is

$$D_{\mu\nu}(k) = -g_{\mu\nu}\left[\frac{1}{k^2+i\epsilon} - i2\pi\delta(k^2)N\right] , \qquad (2.3)$$

with $N = [\exp(|\mathbf{k}|/T)-1]^{-1}$. The amplitude for one real photon is

$$\mathcal{M}(k) = -ik^2\epsilon_\mu(k)\frac{\delta Z(J)}{\delta J^\mu(-k)}(2\pi)^4$$
$$= \epsilon_\mu(k) J^\mu(k) Z(J) . \qquad (2.4)$$

This amplitude describes emission if $k^0>0$, absorption, if $k^0<0$. The amplitude for n photons is

$$\mathcal{M}(k_1,k_2,\ldots,k_n) = Z(J)\prod_{l=1}^{n}[\epsilon_\mu(k_l)J^\mu(k_l)] . \qquad (2.5)$$

To compute probabilities, one must square the amplitude and integrate over photon momenta. For the photon phase space, it is convenient to employ the notation

$$d\Phi_l = \frac{d^4k_l}{(2\pi)^3}\delta(k_l^2)[\theta(k_l^0)+N(|\mathbf{k}_l|)] . \qquad (2.6)$$

This correctly weights photon emission with the statistical factor $1+N$ and photon absorption with the factor N. The probability of radiating a net energy ω is

$$\frac{dP}{d\omega} = \sum_{n=1}^{\infty}\int d\Phi_1\cdots d\Phi_n \delta(k_1^0+\cdots+k_n^0-\omega)$$
$$\times \frac{1}{n!}\sum_{\text{pol}}|\mathcal{M}(k_1,\ldots,k_n)|^2 . \qquad (2.7)$$

The polarization sums give

$$\sum_{\rm pol}|\mathcal{M}(k_1,\ldots,k_n)|^2=|Z(J)|^2\prod_{l=1}^{n}J_\mu(k_l)J^\mu(k_l)\ . \tag{2.8}$$

It is important to note that the current (1.3) has the properties

$$\begin{aligned}J_\mu(k)J^{\mu *}(k)&=J_\mu(k)J^\mu(-k)\\&=-J_\mu(k)J^\mu(k)<0\ .\end{aligned} \tag{2.9}$$

Virtual photons. The integration over all virtual photons is contained in the multiplicative factor $|Z(J)|^2$. Using (2.2) and (2.3) gives

$$\begin{aligned}&|Z(J)|^2=\exp(V)\ ,\\&V=-\int\frac{d^3k}{2k(2\pi)^3}J_\mu(k)J^\mu(k)[1+2N].\end{aligned} \tag{2.10}$$

Because $J\sim 1/k$, this integral contains both linear and logarithmic divergences in the infrared.

Real photons. The δ function that constrains the real photon energies can be represented as

$$\delta(k_1^0+\cdots+k_n^0-\omega)=\int_{-\infty}^{\infty}\frac{dz}{2\pi}e^{-i\omega z}e^{i(k_1^0+\cdots+k_n^0)z}\ . \tag{2.11}$$

In (2.7) each integration over d^4k_l will give the same function of z:

$$R(z)=\int d\Phi\, e^{ik^0z}\sum_{\rm pol}|\epsilon_\mu(k)J^\mu(k)|^2\ . \tag{2.12}$$

More explicitly this is

$$R(z)=\int\frac{d^3k}{2k(2\pi)^3}J_\mu(k)J^\mu(k)([1+N]e^{ikz}+Ne^{-ikz})\ . \tag{2.13}$$

This is the contribution of the real photons. Because $J\sim 1/k$, this integral has both linear and logarithmic infrared divergence.

The probability (2.7) is

$$\begin{aligned}\frac{dP}{d\omega}&=|Z(J)|^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}e^{-i\omega z}\sum_{n=1}^{\infty}\frac{[R(z)]^n}{n!}\\&=\int_{-\infty}^{\infty}\frac{dz}{2\pi}e^{-i\omega z}\exp[\bar{R}(z)]\end{aligned} \tag{2.14}$$

(the $n=0$ term has no Fourier transform at $\omega\neq 0$), and $\bar{R}$ is

$$\begin{aligned}\bar{R}(z)&=R(z)+V\\&=\int\frac{d^3k}{2k(2\pi)^3}J_\mu(k)J^\mu(k)\\&\quad\times([1+N]e^{ikz}+Ne^{-ikz}-[1+2N])\ .\end{aligned} \tag{2.15}$$

Each term in $\bar{R}$ has a clear interpretation. Recall that z is the variable conjugate to the net energy ω. In (2.15) the term proportional to $1+N$ represents the stimulated emission of energy, the term proportional to N represents the absorption of energy, and the subtracted term $1+2N$ represents all virtual photons (emitted and absorbed). At small k, $J^\mu\sim 1/k$ and $N\sim 1/k$. Expanding out the $\exp(\pm ikz)$ for small k shows that the linearly divergent terms dk/k^2 cancel and the logarithmically divergent terms dk/k cancel. Thus $\bar{R}$ is a completely finite function. [See (3.15) for $\bar{R}$.]

III. EXPLICIT INTEGRATION

A. Angular integration

The remaining task is to perform the integrations necessary for the probability (2.14). The first step is to write

$$\bar{R}(z)=\int_0^\infty\frac{dk}{k}Ae^{-k/\Lambda}([1+N]e^{ikz}+Ne^{-ikz}-[1+2N])\ , \tag{3.1}$$

where A contains the angular integration

$$Ae^{-k/\Lambda}=\frac{k^2}{2}\int\frac{d\Omega}{(2\pi)^3}J_\mu(k)J^\mu(k)\ . \tag{3.2}$$

Using $k^\mu=k(1,\hat{\mathbf{k}})$ in the current (1.3) gives

$$\begin{aligned}J_\mu(k)J^\mu(k)=\frac{e^2}{k^2}&\left[\frac{2(E'E-\mathbf{p}'\cdot\mathbf{p})}{(E'-\mathbf{p}'\cdot\hat{\mathbf{k}})(E-\mathbf{p}\cdot\hat{\mathbf{k}})}\right.\\&\left.-\frac{m^2}{(E'-\mathbf{p}'\cdot\hat{\mathbf{k}})^2}-\frac{m^2}{(E-\mathbf{p}\cdot\hat{\mathbf{k}})^2}\right]\\&\times e^{-k/\Lambda}\ .\end{aligned} \tag{3.3}$$

The factors of k completely cancel so that A is independent of k. This is the same angular integration that arises in lowest-order bremsstrahlung and is described in textbooks [2]. It turns out that A is a rather messy function of momentum transfer Q. It is a bit simpler to express it in terms of $\sigma=p'\cdot p/m^2$:

$$A=\frac{2\alpha}{\pi}\left[\frac{\sigma}{\sqrt{\sigma^2-1}}\ln(\sigma+\sqrt{\sigma^2-1})-1\right]\ . \tag{3.4}$$

As noted by Weinberg [9], it is simplest to express A in terms of the relative velocity v of the final particle in the rest frame of the initial particle (or vice versa) defined by $p'\cdot p=m^2(1-v^2)^{-1/2}$ so that

$$A(p\cdot p')=\frac{\alpha}{\pi}\left[\frac{1}{v}\ln\left[\frac{1+v}{1-v}\right]-2\right]\ . \tag{3.5}$$

B. Integration over k

To perform the k integration necessary for (3.1), separate $\bar{R}$ into a temperature-independent part and a temperature-dependent part:

$$\bar{R}(z)=\bar{R}_0(z)+\bar{R}_T(z)\ , \tag{3.6a}$$

$$\bar{R}_0(z)=A\int_0^\infty\frac{dk}{k}(e^{ikz}-1)\exp(-k/\Lambda)\ , \tag{3.6b}$$

$$\bar{R}_T(z)=2A\int_0^\infty\frac{dk}{k}\frac{\cos(kz)-1}{\exp(k/T)-1}\exp(-k/\Lambda)\ . \tag{3.6c}$$

Both $\bar{R}_0$ and $\bar{R}_T$ are infrared finite. For $\bar{R}_0(z)$, expand the integrand in powers of z:

$$\bar{R}_0(z)=A\sum_{n=1}^{\infty}\frac{(iz)^n}{n!}\int_0^{\infty}dk\,k^{n-1}e^{-k/\Lambda}\,. \tag{3.7}$$

The displayed integrals over k each give $(n-1)!\Lambda^n$. The sum on n is elementary:

$$\bar{R}_0(y)=-A\ln[1-i\Lambda z]\,. \tag{3.8}$$

For $\bar{R}_T(z)$, put $k=2Tx$ so that

$$\bar{R}_T(z)=A\int_0^{\infty}\frac{dx}{x}\frac{\cos(2Tzx)-1}{\sinh(x)}e^{-x[1+2T/\Lambda]}\,. \tag{3.9}$$

This is finite at $x=0$, but it is convenient to multiply the integrand by an additional convergence factor x^{μ} in order to use [10]

$$\int_0^{\infty}dx\,x^{\mu-1}\frac{e^{-\beta x}}{\sinh(x)}=2^{1-\mu}\Gamma(\mu)\zeta\left[\mu,\frac{\beta+1}{2}\right]\,. \tag{3.10}$$

This gives

$$\bar{R}_T(z)=\lim_{\mu\to 0}A2^{1-\mu}\Gamma(\mu)\mathrm{Re}(\zeta[\mu,\hat{q}]-\zeta[\mu,q])\,, \tag{3.11}$$

where $\hat{q}=q+iTz$ and $q=1+T/\Lambda$. Now expand ζ in a Taylor series about $\mu=0$ using [10]

$$\zeta[\mu,q]|_{\mu=0}=-q+\tfrac{1}{2}\,, \tag{3.12a}$$

$$\left.\frac{d\zeta[\mu,q]}{d\mu}\right|_{\mu=0}=\ln[\Gamma(q)]-\tfrac{1}{2}\ln(2\pi)\,. \tag{3.12b}$$

When the limit $\mu\to 0$ is taken in (3.11), the result is

$$\bar{R}_T(z)=A\ln\left[\frac{\Gamma(q+iTz)\Gamma(q-iTz)}{\Gamma(q)\Gamma(q)}\right]\,. \tag{3.13}$$

$\bar{R}(z)$ is the sum of (3.8) and (3.13).

C. Integration over z

The last integration to perform is the Fourier transform

$$\frac{dP}{d\omega}=\int_{-\infty}^{\infty}\frac{dz}{2\pi}e^{-i\omega z}\exp[\bar{R}(z)]\,, \tag{3.14}$$

where the integrand is

$$\exp[\bar{R}(z)]=\left[\frac{\Gamma(q+iTz)\Gamma(q-iTz)}{(1-i\Lambda z)\Gamma(q)\Gamma(q)}\right]^{A} \tag{3.15}$$

and $q=1+T/\Lambda$. Since A is noninteger, (3.15) has an infinite number of branch cuts in the variable z. For $\omega>0$ (corresponding to net emission of energy by the charge), one can close the z contour in the lower half-plane. Then there is a branch point at $z_0=-i/\Lambda$ and an infinite set of branch points at $z_n=-i/\Lambda-in/T$ for $n=1,2,3,\ldots$ that come from the poles of $\Gamma(q-iTz)$. Choose the branch cuts to run from z_n to $z_n+\infty$ as shown in Fig. 2 so that

$$\frac{dP}{d\omega}=\sum_{n=0}^{\infty}\int_{z_n}^{z_n+\infty}\frac{dz}{2\pi}e^{-i\omega z}\mathrm{Disc}_n\{\exp[\bar{R}(z)]\}\,. \tag{3.16}$$

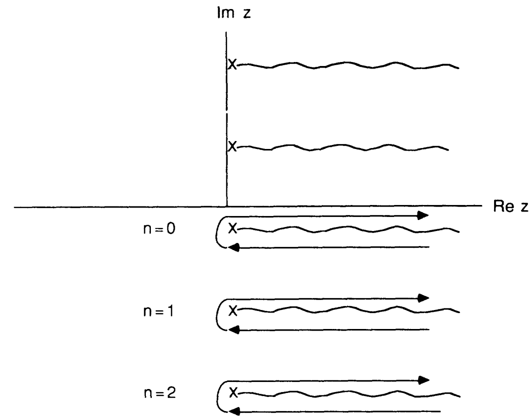

FIG. 2. Location of the branch cuts of the function $\exp[\bar{R}(z)]$ at $z=-i/\Lambda-in/T$. For $\omega>0$, the z integration contour may be wrapped around the branch cuts in the lower half-plane.

The discontinuity across the nth branch cut is

$$\mathrm{Disc}_n\{\exp[\bar{R}(z)]\}=[1-e^{-i2\pi A}]\exp[\bar{R}(z)]\,. \tag{3.17}$$

Then put $z=z_n+r$ where r is real:

$$\frac{dP}{d\omega}=[1-e^{-i2\pi A}]\sum_{n=0}^{\infty}e^{-i\omega z_n}\times\int_0^{\infty}\frac{dr}{2\pi}e^{-i\omega r}\exp[\bar{R}(z_n+r)]\,. \tag{3.18}$$

The value of the function along the branch cut is

$$\exp[\bar{R}(z_n+r)]=\left[C_n(r)\frac{\pi T e^{i\pi(-n+1/2)}}{\Lambda\sinh(\pi Tr)]}\right]^{A}\,, \tag{3.19a}$$

$$C_n(r)=\frac{\Gamma(1+2T/\Lambda+n+iTr)}{\Gamma(1+n+iTr)[\Gamma(1+T/\Lambda)]^2}\,. \tag{3.19b}$$

At this stage it is necessary to make the approximation $T\ll\Lambda$ so that $C_n(r)\to 1$. This allows the summation on n in (3.18) to be done:

$$\frac{dP}{d\omega}=\frac{e^{\omega/2T}e^{-\omega/\Lambda}\sin(A\pi)}{\sin(A\pi/2-i\omega/2T)}\left[\frac{\pi T}{\Lambda}\right]^{A}\frac{I(\omega)}{2\pi}\,, \tag{3.20}$$

$$I(\omega)=\int_0^{\infty}dr\,e^{-i\omega r}[\sinh(\pi Tr)]^{-A}\,. \tag{3.21}$$

This is a known integral [10]:

$$I(\omega)=\frac{\Gamma(1-A)\Gamma(A/2+i\omega/2\pi T)}{\pi T2^{1-A}\Gamma(1-A/2+i\omega/2\pi T)}\,. \tag{3.22}$$

Using the reflection property of the Γ function gives the final result quoted in (1.12):

$$\frac{dP}{d\omega}=\frac{e^{\omega/2T}e^{-\omega/\Lambda}}{\Gamma(A)4\pi^2T}\left[\frac{2\pi T}{\Lambda}\right]^{A}\left|\Gamma\left[\frac{A}{2}+i\frac{\omega}{2\pi T}\right]\right|^2\,. \tag{3.23}$$

ACKNOWLEDGMENTS

It is a pleasure to thank M. Gyulassy, S. Weinberg, and F. Wilczek for their comments. This work was supported in part by the U.S. National Science Foundation under Grant No. PHY-9213734.

APPENDIX A: ZERO-TEMPERATURE LIMIT

At zero temperature (1.12) has the simple behavior

$$\frac{dP}{d\omega}=\frac{1}{\omega}\left[\frac{\omega}{\Lambda}\right]^{A}\frac{e^{-\omega/\Lambda}}{\Gamma(A)} \tag{A1}$$

and easily satisfies

$$\int_0^{\infty} d\omega\frac{dP}{d\omega}=1 \; . \tag{A2}$$

Since all detectors have some energy threshold for detection, physical probabilities must include an integration over the below-threshold photons. If the detector measures total radiant energy deposition (as in a calorimeter) above $E_{\min}$, then the appropriate probability is

$$\sum_{n=1}^{\infty}\int\prod_{l=1}^{n}\left[\frac{d^2k_l}{2k_l(2\pi)^3}\right]\theta\left[E_{\min}-\sum_j k_j\right]\frac{|\mathcal{M}_n|^2}{n!} \; . \tag{A3}$$

This probability is the integral of (A1)

$$\int_0^{E_{\min}} d\omega\frac{dP}{d\omega}=\left[\frac{E_{\min}}{\Lambda}\right]^{A}\frac{1}{\Gamma(1+A)} \; , \tag{A4}$$

with $\exp(-\omega/\Lambda)$ neglected.

The more familiar situation is a detector that is sensitive to single photons each of which has energy above some threshold $E_{\min}$. In that case the single θ function in (A3) is replaced by a product:

$$\theta(E_{\min}-k_1)\theta(E_{\min}-k_2)\cdots\theta(E_{\min}-k_n) \; . \tag{A5}$$

This calculation is discussed by Weinberg [9] and gives a probability

$$\left[\frac{E_{\min}}{\Lambda}\right]^{A} b(A) \; , \tag{A6}$$

where $b(A)\approx 1-(\pi A)^2/12+\cdots$ is a different function than the $1/\Gamma(1+A)$ that occurs in (A4).

APPENDIX B: COMMENT ON A PREVIOUS CALCULATION

In a previous paper on the semiclassical approximation [11], I computed the residual effects of infrared cancellation on processes which occur in a finite-volume heat bath but with no photons detected outside the heat bath. The absence of high-energy photons means none were radiated by the charge. The absence of low-energy photons, which have a short mean free path, is unavoidable because all those radiated by the charge would be thermalized in the heat bath and lose their identity. Consequently, there is a threshold energy ϵ that depends on the size of the system. In the notation of Sec. II, the photons with $k<\epsilon$ contribute

$$R_\epsilon(z)=\int_0^{\epsilon}\frac{d^3k}{2k(2\pi)^3}J_\mu(k)J^\mu(k)([1+N]e^{ikz}+Ne^{-ikz}) \; . \tag{B1}$$

There is no constraint on the total energy ω, and so the probability of no detected photons is

$$\int_0^{\infty} d\omega\frac{dP_\epsilon}{d\omega}=\exp[\bar{R}_\epsilon(0)] \; , \tag{B2}$$

where

$$\bar{R}_\epsilon(0)=V+R_\epsilon(0) = -A\int_\epsilon^{\infty}\frac{dk}{k}[1+2N]e^{-k/\Lambda} \; . \tag{B3}$$

If $\epsilon \ll T$, then $\bar{R}_\epsilon(0)\approx -2AT/\epsilon$. However, for reasonable mean free paths it invariably turns out that $\epsilon > T$, which made these thermal corrections small.

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