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Unstable particle mixing and CP violation in weak decays

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We discuss unstable particle mixing in *CP*-violating weak decays. It is shown that for a completely degenerate system unstable particle mixing does not introduce a *CP*-violating partial rate difference, and that when the mixings are small only the off-diagonal mixings are relevant. Also, in the absence of mixing, unstable particle wave function renormalization does not introduce any additional effect. An illustrative example is given to heavy scalar decays with arbitrary mixing.

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I. INTRODUCTION

The smallness of the K_L-K_S mass difference allows us to have access to rare processes such as *CP* violation. Up to now the only established experimental evidence of *CP* violation comes from the mixing of the unstable particles K^0 and $\bar{K}^0$ [1].

Earlier studies of unstable particle mixing followed two physically equivalent paths. One is due to Weisskopf and Wigner [2], in which one introduces an effective complex mass matrix. The evolution of the system is determined by the standard time-dependent Hamiltonian formalism [3]. The other is due to Sachs [4], in which one studies the dynamics of the complex pole of the kaon field propagator. The Hamiltonian method is expressed directly in terms of the measured quantities and is therefore more transparent from a phenomenological viewpoint. On the other hand, the propagator method arises naturally in the context of quantum field theory, and hence is more easily adapted to fundamental gauge theories of weak interactions. Both approaches are phenomenological, having difficulties handling ultraviolet divergences arising from higher-order corrections. In spite of these fundamental difficulties, the phenomenological formalisms have been very successful. They provide the standard descriptions for the study of unstable particle mixing.

The advent of renormalizable gauge theory provides a connection between the parameters of a phenomenological formalism and the parameters of a given fundamental theory. In this paper we would like to study these connections for unstable particle mixing in some detail, focusing on *CP*-violating processes. We will adopt an approach that combines the two methods mentioned above.

Instead of introducing a complete renormalization prescription, our immediate goal is more modest. In the next section we discuss some general properties of *S*-matrix elements in the presence of unstable particle mixing. The results of this analysis turn out to be very useful for simplifying Feynman diagram calculations.

In Sec. III we study the relationship between the unstable particle mixing and antiparticle mixing. For simplicity, we only focus on scalars. A simple formula valid for small mixings is derived for *CP*-violating partial rate differences.

The formalisms developed in Secs. II and III are applied to a simple example of baryogenesis by heavy scalar decay. The results are shown to agree with the published results obtained directly from Feynman diagram calculations. This part is presented in Sec. IV, followed by a discussion in Sec. V of large mixing and renormalization. Our conclusion is presented in Sec. VI. We give two appendixes to present some technical details: one discusses the renormalization of unstable particle mixing and the other shows how to diagonalize an arbitrary $n\times n$ complex matrix.

II. GENERAL FORMALISM

Consider the weak decay of a set of unstable particles ϕ_a produced at $t=0$, where the index $a=1,2,\ldots$ labels different flavor of ϕ. Suppose the lowest-order amplitude of ϕ_a decaying into a final state $|F_f\rangle$ is given by

$$T_{fa}=\langle F_f|H_{\rm weak}|\phi_a\rangle \ . \tag{1}$$

By *CPT* invariance, to first order of $H_{\rm weak}$ a replacement of F_f and ϕ_a by their antiparticles $\bar{F}_f$ and $\bar{\phi}_a$ corresponds to change T_{fa} to T^*_{fa}. By the superposition principle, the weak amplitudes at a later time t are

$$T_{fa}(t)=\sum_{b,c} T_{fb}V^{-1}_{bc}V_{ca}e^{-i\omega_c t} \ , \tag{2}$$

$$\bar{T}_{fa}(t)=\sum_{b,c} T^*_{fb}\bar{V}^{-1}_{bc}\bar{V}_{ca}e^{-i\omega_c t} \ , \tag{3}$$

where V and $\bar{V}$ are the mixing matrices

$$|\phi_a\rangle=V_{ca}|\phi'_c\rangle \ , \tag{4}$$

$$|\bar{\phi}_a\rangle=\bar{V}_{ca}|\bar{\phi}'_c\rangle \ , \tag{5}$$

and $|\phi'_c\rangle$, $|\bar{\phi}'_c\rangle$ are the eigenstates of an effective Hamiltonian, which is correct up to second order in $H_{\rm weak}$. $|\phi'_c\rangle$ and $|\bar{\phi}'_c\rangle$ will be referred to as the eigenstate of propagation hereafter. The eigenvalue of $|\phi'_c\rangle$ and $|\bar{\phi}'_c\rangle$ is $\omega_c=m_c-i\gamma_c/2$ in the rest frame ϕ'_c, where m_c and γ_c may be interpreted, respectively, as the mass and width of ϕ'_c and $\bar{\phi}'_c$.

If $|F_f\rangle$ ($|\bar{F}_f\rangle$) belongs to the unstable particle set, then T_{fb} and $\bar{T}_{fb}$ are zero unless $f=b$, and Eqs. (2) and (3) are useful for the study of time distribution of CP asymmetry in oscillation [5,6]. Except for situations in which the particles are stable or the mass and decay matrices commute, V and $\bar{V}$ are, in general, not unitary. If both CPT and CP are conserved, $V=\bar{V}$ and Eqs. (2) and (3) are the natural generalizations of the known formalism [7] describing the so-called "mix-and-decay" phenomena. The formalism of [7] has been employed in the study of unstable neutrino oscillation [8,9]. For practical purpose the mixing matrix has so far been approximated as unitary. Such an approximation is not always justifiable.

Our main interest is in the unstable particle mixing effect in the time-integrated rate difference

$$\Delta_{fa}=\Gamma(\phi_a \to F_f)-\Gamma(\bar{\phi}_a \to \bar{F}_f) . \tag{6}$$

It should be pointed out that, in addition to the mixing, the rate difference may also depend on final-state interactions, and it is not always possible to separate them if these two effects are comparable. A discussion on particle mixing is always warranted, however, unless its effect is negligible. For recent discussions on the final-state interaction effect see Refs. [10–15].

In order for Δ_{fa} to be nonzero it is necessary that CP be violated and to have significant nontrivial CP-conserving phases. According to Eqs. (2) and (3), a CP-conserving phase may arise either from the mixing matrix V or $\bar{V}$ or/and from the evolution phase $e^{-i\omega_c t}$. For a completely degenerate system, i.e., $\omega_a=\omega$, the evolution phase factors

$$T_{fa}(t)=e^{-i\omega t}T_{fa} , \tag{7}$$

$$\bar{T}_{fa}(t)=e^{-i\omega t}T_{fa}^{*} . \tag{8}$$

Although a nonreal T_{fa} implies CP violation, the contribution to the rate difference from mixing is seen to vanish because $|T_{fa}(t)|^2=|\bar{T}_{fa}(t)|^2$.

This result has an important implication in searching for mechanisms for baryogenesis [16]. It has previously been suggested [17] that a degenerate unstable system might provide a resonance enhancement to the CP-violating partial rate difference from mixing. Our analysis shows that the outcome would be the opposite if the degeneracy is complete. Instead of a resonance enhancement, we expect that contributions to the generation of baryon number asymmetry from unstable particle mixing is highly suppressed whenever the particles in question have nearly equal mass and lifetime, and is zero in the complete degenerate limit.

One special example of a completely degenerate system is that the set contains only one particle, i.e., $a=b=c=1$. In that case (2) and (3) show that particle instability itself does not contribute to the CP-violating partial rate difference. The same statement applies to systems with an arbitrary number of unmixed particles. In terms of perturbation theory these results imply that, in the absence of mixing, unstable particle wave function renormalization does not affect Δ_{fa}.

Had we considered the rate difference of the eigenstates of propagation, i.e., $\Delta'_{f1}\equiv\Gamma(\phi'_1\to F_f)-\Gamma(\bar{\phi}'_1\to\bar{F}_f)$, we would have reached a different conclusion [18]. This is evident by dropping the matrix V (not V^{-1}) from (2) and $\bar{V}$ from (3). Phenomenologically, neglecting V and $\bar{V}$ in (2) and (3) corresponds to ignoring mixing in particle production. As pointed out earlier [4,19], the propagation eigenstates cannot be regarded as physical in the sense that they cannot be directly produced nor detected [20]. As a result, Δ'_{fa} is not a physical observable.

Another important feature of Eqs. (2) and (3) is that, when the mixings are small, the diagonal mixings and phases are irrelevant for Δ_{fa}. Indeed, for small mixings, we expand V and $\bar{V}$ as

$$V_{ca}=e^{i\alpha_c}[\delta_{ca}+\Delta V_{ca}] , \tag{9}$$

$$\bar{V}_{ca}=e^{i\bar{\alpha}_c}[\delta_{ca}+\Delta \bar{V}_{ca}] , \tag{10}$$

where the diagonal phases α_c and $\bar{\alpha}_c$ are real. The elements in ΔV and $\Delta\bar{V}$ are assumed to be much smaller than unity. To first order of ΔV and $\Delta\bar{V}$ we have

$$T_{fa}(t)=e^{-i\omega_a t}T_{fa}+\sum_b T_{fb}\Delta V_{ba}[e^{-i\omega_b t}-e^{-i\omega_a t}] , \tag{11}$$

$$\bar{T}_{fa}(t)=e^{-i\omega_a t}T_{fa}^{*}+\sum_b T_{fb}^{*}\Delta \bar{V}_{ba}[e^{-i\omega_b t}-e^{-i\omega_a t}] . \tag{12}$$

The second terms in Eqs. (11) and (12) vanish for diagonal mixings. Therefore, neglecting off-diagonal mixings the CP-conserving phase $e^{-i\omega_a t}$ factors and $|T_{fa}(t)|^2-|\bar{T}_{fa}(t)|^2=0$. Hence, for small mixings the diagonal phases α_a, $\bar{\alpha}_a$, and $\omega_a t$ do not enter into the determination of Δ_{fa}; only the off-diagonal mixings, i.e., $b\neq a$, are relevant. This is similar to a result of Wolfenstein's for final-state interactions [6]. Thus, for the calculation of Δ_{fa} one does not need to consider flavor-conserving one-particle-reducible diagrams.

III. RELATIONS BETWEEN PARTICLE AND ANTIPARTICLE MIXINGS

CPT invariance provides a relationship between V and $\bar{V}$. In quantum mechanics this would be obtained by studying the Hamiltonian. The existence of the "standard model" underlines the usefulness of a field-theoretical analysis. In field theory the relation between V and $\bar{V}$ can be easily obtained from particle propagator. Consider situations in which ϕ_a is a scalar. At tree level, the propagator of ϕ_a is

$$i\tilde{\Delta}_{ba}^{(0)}(P^2)=i[P^2-m_a^{(0)2}]^{-1}\delta_{ba} , \tag{13}$$

where $m_a^{(0)}$ is the bare mass of ϕ_a, and we have implicitly assumed that $|\phi_a\rangle$ is an eigenstate of zeroth order of H_{weak}. Including one-loop corrections the following changes occur in (13):

$$m_a^{(0)2}\delta_{ba}\to\mathcal{M}_{ba}^2 , \tag{14}$$

where $\mathcal{M}^2$ is the square of an effective complex mass matrix,

$$\mathcal{M}^2_{ba}=\tilde{\mathcal{M}}^2_{ba}-i\tilde{\Gamma}^2_{ba}\ , \tag{15}$$

in which $\tilde{\mathcal{M}}$ is the effective mass matrix and $\tilde{\Gamma}^2=\frac{1}{2}(\tilde{\mathcal{M}}\Gamma+\Gamma\tilde{\mathcal{M}})$, where Γ is the decay matrix, and $\Gamma^2/4$ has been neglected. Both $\tilde{\mathcal{M}}$ and $\tilde{\Gamma}^2$ are Hermitian, but $\mathcal{M}^2$ is not.

For a given P^2, $\mathcal{M}^2$ can be diagonalized by a transformation

$$[Q\mathcal{M}^2Q^{-1}]_{ba}=(m_a^2-im_a\gamma_a)\delta_{ba}\ , \tag{16}$$

where Q is a complex matrix. The one-loop regularized propagator can be written as

$$i\tilde{\Delta}^{(1)}_{ba}(P^2)=iQ^{-1}_{bc}[P^2-m_c^2+im_c\gamma_c]^{-1}Q_{ca}\ , \tag{17}$$

where for simplicity we have neglected terms of order γ^2. Evidently, $i[P^2-m_c^2+im_c\gamma_c]^{-1}$ is the propagator of ϕ'_c. It follows that

$$V_{ba}=Q_{ba}\ , \tag{18}$$

$$\bar{V}_{ba}=Q^{-1}_{ab}\ , \tag{19}$$

and thus the relationship between $\bar{V}$ and V is

$$\bar{V}_{ba}=V^{-1}_{ab}\ . \tag{20}$$

For stable particles $\mathcal{M}^2$ is Hermitian, Q is unitary, and hence so are V and $\bar{V}$, and (20) reduces to the known result $\bar{V}=V^*$. Substituting (20) into (3) one sees that the time-dependent mixing matrix in the antiparticle decay is $\sum_c V^{-1}_{ac}e^{-i\omega_c t}V_{cb}$, which differs from that in the particle decay (2) by exchanging the indices a and b (a consequence of time reversal).

It is important that $\mathcal{M}^2$ is momentum dependent. This is necessary if the orthonormality conditions are to be maintained for both $|\phi_a\rangle$ and $|\phi'_a\rangle$ [21]. In practice, this does not introduce any additional complication, as P^2 is always fixed by the on-shell condition once the initial state is specified.

We now focus on a case of special interest, small mixing. By small mixing we mean that (1) the width differences of the particles are much smaller than their mass differences and (2) the off-diagonal elements in $\mathcal{M}^2$ can be treated as a perturbation. In that case ΔV and $\Delta\bar{V}$, defined by (9) and (10), may be separated into their dispersive and absorptive parts:

$$\Delta V_{ba}=\Delta V^{(D)}_{ba}+i\Delta V^{(A)}_{ba}\ , \tag{21}$$

$$\Delta\bar{V}_{ba}=\Delta\bar{V}^{(D)}_{ba}+i\Delta\bar{V}^{(A)}_{ba}\ . \tag{22}$$

Phenomenologically, $\Delta V^{(D)}_{ba}$ and $\Delta V^{(A)}_{ba}$ correspond to the mixings arising from the mass and decay matrices, respectively. The Hermiticity of the mass and decay matrices then implies

$$\Delta V^{(D,A)}_{ba}=-\Delta V^{(D,A)*}_{ab}\ , \tag{23}$$

$$\Delta\bar{V}^{(D,A)}_{ba}=-\Delta\bar{V}^{(D,A)*}_{ab}\ . \tag{24}$$

The solution to (20) satisfying the constraints of (23) and (24) is

$$\Delta V^{(D,A)}_{ba}=\Delta\bar{V}^{(D,A)*}_{ba}\ . \tag{25}$$

This is a much simplified version of (20), valid for small mixings.

With (25) one can have a simple expression for Δ_{fa}. From (11), (12), and (25) we find that the time-differentiated CP-violating partial rate difference is

$$|T_{fa}(t)|^2-|\bar{T}_{fa}(t)|^2=4\sum_b\rho_b\{\sin\beta_b e^{2\,\mathrm{Im}\omega_a t}-\sin[\mathrm{Re}(\omega_a-\omega_b)t+\beta_b]e^{\mathrm{Im}(\omega_a+\omega_b)t}\}\ , \tag{26}$$

where

$$\rho_b=\sqrt{[\mathrm{Im}(\Delta V^{(A)}_{ba}T^*_{fa}T_{fb})]^2+[\mathrm{Im}(\Delta V^{(D)}_{ba}T^*_{fa}T_{fb})]^2}\ , \tag{27}$$

$$\tan\beta_b=\mathrm{Im}[\Delta V^{(A)}_{ba}T^*_{fa}T_{fb}]/\mathrm{Im}[\Delta V^{(D)}_{ba}T^*_{fa}T_{fb}]\ . \tag{28}$$

In the absence of CP violation $\Delta V^{(D,A)}$ and T_{fa} are real, $\rho_b=0$ and hence Eq. (26) vanishes. As pointed out earlier, Eq. (26) shows explicitly that for diagonal mixing, i.e., $b=a$, the phase makes no contribution. If the mass and decay matrices commute, $\Delta V^{(A)}=0$ and (26) reduces to

$$|T_{fa}(t)|^2-|\bar{T}_{fa}(t)|^2=-4\sum_b\mathrm{Im}[\Delta V^{(D)}_{ba}T^*_{fa}T_{fb}]\sin[\mathrm{Re}(\omega_a-\omega_b)t]e^{\mathrm{Im}(\omega_a+\omega_b)t}\ . \tag{29}$$

The result given by (29) can also be obtained from a general formalism developed recently by Gronau and Rosner [22] for a 2×2 system.

Within our approximation all oscillatory terms are integrated to zero. Hence,

$$\Delta_{fa}=\int d\Omega\int_0^\infty\frac{dt}{\tau_a}[|T_{fa}(t)|^2-|\bar{T}_{fa}(t)|^2]$$
$$=4\int d\Omega\sum_{b\neq a}\mathrm{Im}[\Delta V^{(A)}_{ba}T^*_{fa}T_{fb}]\ , \tag{30}$$

where $1/\tau_a=-2\,\mathrm{Im}\omega_a$ is the width of the particle and $\int d\Omega$ represents a phase-space sum. Equation (30) shows that for the calculation of Δ_{fa} one only needs to consider off-diagonal mixings in the decay matrix.

For small mixings it is easy to show that

$$\sum_f\Delta_{fa}=0\ . \tag{31}$$

This relation follows from unitarity of an S matrix, which in the present situation implies

$$\sum_f T^*_{fb}T_{fa}|_{P^2=m_a^2}\propto\tilde{\Gamma}^2_{ba}\propto\Delta V^{(A)}_{ba}\ . \tag{32}$$

Thus, $\sum_{f,b\neq a}\Delta V^{(A)}_{ab}T^*_{af}T_{bf}$ is real and Eq. (31) follows as a consequence. Since the final-state interaction contribution to the total rate difference of a particle and its antipaticle is known to vanish [6], the result of (31) is in accordance with the usual expectation that a particle and its antiparticle have the same lifetime [23].

IV. AN ILLUSTRATIVE EXAMPLE FOR SMALL MIXING

In this section we show how to apply (30) to model calculations. Consider a system containing two heavy scalars $S_{a,\alpha}$ ($a=1,2$) and α as a color index. The lowest-order interaction of the system is given by

$$\mathcal{L}_1=G_{1a}\bar{u}^c_{R,\alpha}d_{R,\beta}S_{a,\gamma}\epsilon^{\alpha\beta\gamma}+G_{2a}\bar{u}_{R,\alpha}e^c_R S_{a,\alpha}+\text{H.c.}\ , \tag{33}$$

where u, d, and e^c are the charged fermions of the first generation, which are considered as massless. This interaction can arise from an SU(5) grand unified theory (GUT) with two five-plets of Higgs fields. Since the two final states into which the heavy scalars can decay have different baryon number, (33) provides an interaction for baryogenesis via heavy scalar decays.

For convenience, we choose as a basis $S_{a,\alpha}$ the mass eigenstate fields and select $|F_1\rangle=|\bar{d}_R u^c_R\rangle$ and $|F_2\rangle=|e_R\bar{u}^c_R\rangle$. The lowest-order transition matrix for $|S_{1,2}\rangle\rightarrow|F_{1,2}\rangle$ is

$$T=C\begin{bmatrix}\sqrt{2}G_{11} & \sqrt{2}G_{12}\\ G_{21} & G_{22}\end{bmatrix}\ , \tag{34}$$

where C is an overall normalization constant. The factor $\sqrt{2}$ in T_{11} and T_{12} is introduced to account for the effect of summing over the indices of $\epsilon^{\alpha\beta\gamma}$ in calculating the squares of the elements.

To determine $\Delta V^{(A)}$ we consider corrections to the scalar propagators to second order of $\mathcal{L}_1$. A simple calculation shows that the regularized elements in the complex-mass-matrix square (15) are (remember $\tilde{M}^2$ and $\tilde{\Gamma}^2$ are momentum dependent)

$$\tilde{M}^2_{ba}=m_a^{(0)2}\delta_{ba}-\frac{P^2}{16\pi^2}[2G^*_{1b}G_{1a}+G^*_{2b}G_{2a}]\times\left[\frac{2}{4-n}+\frac{3}{2}-\gamma_E-\ln\frac{P^2}{4\pi\mu_0^2}\right]\ , \tag{35}$$

$$\tilde{\Gamma}^2_{ba}=\frac{P^2}{16\pi}[2G^*_{1b}G_{1a}+G^*_{2b}G_{2a}]\ , \tag{36}$$

where n is the dimension of regularization, μ_0^2 is the ultraviolet cutoff, and γ_E is Euler's number. Following the standard technique [3] we find that the matrix which diagonalizes $\mathcal{M}^2$ according to (16) is

$$V=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}e^{-i\delta} & 0\\ 0 & e^{i\delta}\end{bmatrix}\ , \tag{37}$$

where $\theta=\text{Re}\theta+i\,\text{Im}\theta$ is complex,

$$\tan2\theta=2\frac{\hat{M}^2+i|\tilde{\Gamma}^2_{12}|}{(\tilde{M}^2_{11}-\tilde{M}^2_{22})-i(\tilde{\Gamma}^2_{11}-\tilde{\Gamma}^2_{22})}\ , \tag{38}$$

with the real quantity

$$\hat{M}^2=(P^2/16\pi^2)|2G^*_{11}G_{12}+G^*_{21}G_{22}|\times\left[\frac{2}{4-n}+\frac{3}{2}-\gamma_E-\ln(P^2/4\pi\mu_0^2)\right]\ .$$

The phase δ is real and determined by

$$\delta=\tfrac{1}{2}\arg[2G^*_{11}G_{12}+G^*_{21}G_{22}]\ . \tag{39}$$

The eigenvalues are

$$m^2_{1,2}-im_{1,2}\gamma_{1,2}=\tfrac{1}{2}\{(\tilde{M}^2_{11}+\tilde{M}^2_{22})-i(\tilde{\Gamma}^2_{11}+\tilde{\Gamma}^2_{22})\pm\sqrt{[(\tilde{M}^2_{11}-\tilde{M}^2_{22})-i(\tilde{\Gamma}^2_{11}-\tilde{\Gamma}^2_{22})]^2+4[\hat{M}^2+i|\tilde{\Gamma}^2_{12}|]^2}\}\ . \tag{40}$$

In the small mixing limit, i.e.,

$$|\tilde{M}^2_{11}-\tilde{M}^2_{22}|\gg|\tilde{\Gamma}^2_{11}-\tilde{\Gamma}^2_{22}|,|\tilde{M}^2_{12}|,|\tilde{\Gamma}^2_{12}|\ , \tag{41}$$

one has from (37) that

$$\Delta V^{(A)}=\begin{bmatrix}0 & -\text{Im}\theta e^{i2\delta}\\ \text{Im}\theta e^{-i2\delta} & 0\end{bmatrix}\ , \tag{42}$$

where

$$\text{Im}\theta e^{i2\delta}=[\text{Im}\theta e^{-i2\delta}]^*=\frac{P^2[2G^*_{11}G_{12}+G^*_{21}G_{22}]}{16\pi[\tilde{M}^2_{11}-\tilde{M}^2_{22}]}\ . \tag{43}$$

Substituting (34) and (42) into (30) and making use of the relation $m^{(0)2}_{1,2}=\tilde{M}^2_{11,22}=m^2_{1,2}$, which is valid to zeroth order in $\mathcal{L}_1$, we obtain, for $S_1\rightarrow F_{1,2}$, in which $P^2=m_1^2$,

$$\Delta_{11}=-\Delta_{21}=\frac{\Omega_1}{2\pi}\frac{m_1^2}{m_1^2-m_2^2}\text{Im}[G^*_{11}G_{12}G_{21}G^*_{22}]\ , \tag{44}$$

where $\Omega_1=m_1/16\pi$ is a phase-space factor for S_1. Also, for the S_2 decays we have $P^2=m_2^2$ and

$$\Delta_{12}=-\Delta_{22}=\frac{\Omega_2}{2\pi}\frac{m_2^2}{m_2^2-m_1^2}\text{Im}[G_{11}G^*_{12}G^*_{21}G_{22}]\ , \tag{45}$$

where $\Omega_2=m_2/16\pi$. These results are in complete agreement with those obtained from a direct Feynman diagram calculation [25]. Contributions to Δ_{fa} from vertex corrections can be found in [24,26,27,25].

V. AN ILLUSTRATIVE EXAMPLE FOR LARGE MIXING

In the small mixing limit (41) one can simply use the regularized (but not renormalized) formalism to compute Δ_{fa}. The unphysical quantities in $\tilde{M}^2$ do not enter. However, to go beyond this limit we must introduce a renormalization prescription to remove the divergences. A pedagogic introduction illustrating how this may be done for the example discussed above is given in Appendix A.

From a phenomenological viewpoint, the mixing phenomena under consideration is determined by the renormalized interaction Lagrangian [28]

$$\mathcal{L}_I = g_{1a}\bar{u}^c_{R,\alpha}d_{R,\beta}S_{a,\gamma}\epsilon^{\alpha\beta\gamma} + g_{2a}\bar{u}_{R,\alpha}e^c_R S_{a,\alpha} + \text{H.c.} \ , \tag{46}$$

where g_{fa} $(f=1,2)$ are the renormalized couplings in the weak eigenstate basis. The square of a renormalized complex mass matrix is

$$\mathcal{M}^2 = \begin{bmatrix} \tilde{M}^2_{11} & \tilde{M}^2_{12} \\ \tilde{M}^{2*}_{12} & \tilde{M}^2_{22} \end{bmatrix} - i \begin{bmatrix} \tilde{\Gamma}^2_{11} & \tilde{\Gamma}^2_{12} \\ \tilde{\Gamma}^{2*}_{12} & \tilde{\Gamma}^2_{22} \end{bmatrix} , \tag{47}$$

where $\tilde{M}^2_{ba}$ and g_{fa} are the parameters of the model determined experimentally (at some scale). The parameters in $\tilde{\Gamma}^2_{ba}$ are calculable:

$$\tilde{\Gamma}^2_{ba} = \frac{P^2}{16\pi^2}[2g^*_{1b}g_{1a} + g^*_{2b}g_{2a}] \ . \tag{48}$$

A convenient form for the matrix which diagonalizes (47) according to (16) is

$$V = \begin{bmatrix} \cos\theta & -\sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & \cos\theta \end{bmatrix} . \tag{49}$$

Again, the mixing angle θ is complex:

$$\tan^2 2\theta = 4\frac{[\tilde{M}^2_{12} - i\tilde{\Gamma}^2_{12}][\tilde{M}^{2*}_{12} - i\tilde{\Gamma}^{2*}_{12}]}{[(\tilde{M}^2_{11} - \tilde{M}^2_{22}) - i(\tilde{\Gamma}^2_{11} - \tilde{\Gamma}^2_{22})]^2} \ . \tag{50}$$

The phase ϕ is also complex determined by

$$e^{-i2\phi} = \frac{\tilde{M}^{2*}_{12} - i\tilde{\Gamma}^{2*}_{12}}{\tilde{M}^2_{12} - i\tilde{\Gamma}^2_{12}} \ . \tag{51}$$

The eigenvalues are

$$\begin{aligned} \omega^2_{1,2} &= \left[m_{1,2} - \frac{i}{2}\gamma_{1,2} \right]^2 \\ &= \tfrac{1}{2}[(\tilde{M}^2_{11} + \tilde{M}^2_{22}) - i(\tilde{\Gamma}^2_{11} + \tilde{\Gamma}^2_{22})] \pm \tfrac{1}{2}\sqrt{[(\tilde{M}^2_{11} - \tilde{M}^2_{22}) - i(\tilde{\Gamma}^2_{11} - \tilde{\Gamma}^2_{22})]^2 + 4(\tilde{M}^2_{12} - i\tilde{\Gamma}^2_{12})(\tilde{M}^{2*}_{12} - i\tilde{\Gamma}^{2*}_{12})} \ . \end{aligned} \tag{52}$$

If $\arg\tilde{M}^2_{12} = \arg\tilde{\Gamma}^2_{12}$, $\phi = 2\arg\tilde{M}^2_{12} = 2\delta$ is real. Compared to the regularized formula (37), one sees that Eq. (49) differs from (37) only by a diagonal phase matrix. This difference has no physical significance. The diagonal phase matrix can be removed by a suitable choice of phase convention.

Let us now turn back to Δ_{fa}. Equations (48)–(52) enable us to compute Δ_{fa} for arbitrary mixing angle and phase. The results are only limited by the validity of perturbation expansion of $\mathcal{L}_I$. Here one should use the general formulas (2), (3), and (20). In applying Eqs. (49)–(52) one should also be very careful whenever the parameters are near the branch cuts in the complex parameter space.

As an illustration we consider that the initial (renormalized) state is an eigenstate of mass matrix, i.e., $\tilde{M}^2_{12} \to 0$, but not an eigenstate of the decay matrix. In that case ϕ is real and

$$\sin 2\theta e^{-i\phi} = i\frac{P^2}{8\pi|\omega^2_1 - \omega^2_2|^2}(\omega^{2*}_1 - \omega^{2*}_2)[2g_{11}g^*_{12} + g_{21}g^*_{22}] \ . \tag{53}$$

We find

$$\begin{aligned} \Delta_{11} &= -\Delta_{21} \\ &= \frac{\Omega_1 m_1^2 \operatorname{Im}[g^*_{11}g_{12}g_{21}g^*_{22}]}{2\pi|\omega^2_1 - \omega^2_2|^2}\int_0^\infty \frac{dt}{\tau_1}\operatorname{Re}[(\omega^2_1 - \omega^2_2)(c^2 e^{-i\omega_1 t} + s^2 e^{-i\omega_2 t})(e^{i\omega^*_1 t} - e^{i\omega^*_2 t})] \ , \end{aligned} \tag{54}$$

where $c = \cos\theta$ and $s = \sin\theta$. Also,

$$\begin{aligned} \Delta_{12} &= -\Delta_{22} \\ &= \frac{\Omega_2 m_2^2 \operatorname{Im}[g_{11}g^*_{12}g^*_{21}g_{22}]}{2\pi|\omega^2_2 - \omega^2_1|^2}\int_0^\infty \frac{dt}{\tau_2}\operatorname{Re}[(\omega^2_2 - \omega^2_1)(c^2 e^{-i\omega_2 t} + s^2 e^{-i\omega_1 t})(e^{i\omega^*_2 t} - e^{i\omega^*_1 t})] \ . \end{aligned} \tag{55}$$

In (54) and (55) the mixing angle θ is arbitrary.

In the limit in which the mixing angle is small due to $|\tilde{\Gamma}^2_{11} - \tilde{\Gamma}^2_{22}| \gg |\tilde{M}^2_{11} - \tilde{M}^2_{22}|, |\tilde{\Gamma}^2_{12}|$, Eqs. (54) and (55) are simplified to

$$\Delta_{11}=-\Delta_{21}=\frac{\Omega_1}{2\pi}\frac{m_1^2}{m_1^2-m_2^2}\,\mathrm{Im}[g_{11}^*g_{12}g_{21}g_{22}^*]\left[\frac{m_1^2-m_2^2}{\tilde\Gamma_{11}^2-\tilde\Gamma_{22}^2}\right]^2 , \tag{56}$$

$$\Delta_{12}=-\Delta_{22}=\frac{\Omega_2}{2\pi}\frac{m_2^2}{m_2^2-m_1^2}\,\mathrm{Im}[g_{11}g_{12}^*g_{21}^*g_{22}]\left[\frac{m_1^2-m_2^2}{\tilde\Gamma_{11}^2-\tilde\Gamma_{22}^2}\right]^2 . \tag{57}$$

Compared to (44) and (45) (after renormalization G_{fa} are replaced by g_{fa}), these results are further suppressed by a ratio $(m_1^2-m_2^2)^2/(\tilde\Gamma_{11}^2-\tilde\Gamma_{22}^2)^2 \ll 1$. In particular, all the CP-violating partial rate differences vanish in the limit $m_1=m_2$. This is in contrast with what one might have expected, based upon a naive extrapolation from (44) and (45). Since Δ_{fa} depends on the imaginary part of the mixing angle (30), it must vanish whenever $\mathrm{Im}\theta=0$. For the case at hand, θ is real when $\tilde M_{12}^2=0$ and $m_1=m_2$.

VI. CONCLUSION

We have discussed a formalism for unstable particle mixing based upon one-loop renormalization of field theory, with emphasis on its applications to CP-violating physical processes. Among various interesting results, we have found that, for a completely degenerate system, unstable particle mixing does not contribute to the CP-violating partial rate difference. In particular, in the absence of mixing, unstable particle wave function renormalization does not introduce any additional effect for the CP asymmetry. When the mixing is small we show that only the off-diagonal mixings and phases are relevant for CP violation.

We have used a simple example to show how to apply the formalisms developed in this paper in model calculations for arbitrary mixing. The basic steps for renormalizing unstable particle mixing are also outlined.

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APPENDIX A

It is most convenient to introduce a renormalization prescription in the weak eigenstate basis. We parametrize the interaction Lagrangian as

$$\mathcal{L}_I=g_{1a}^{(0)}\bar u_{R,\alpha}^c d_{R,\beta}S_{a,\gamma}^{(0)}\epsilon^{\alpha\beta\gamma}+g_{2a}^{(0)}\bar u_{R,\alpha}e_R^c S_{a,\alpha}^{(0)}+\mathrm{H.c.} , \tag{A1}$$

where $g_{fa}^{(0)}$ $(f=1,2)$ are the weak eigenstate bare couplings. The one-loop regularized inverse propagator can then be written as

$$\tilde\Delta_{ba}^{-1}(P^2,\mu^2)=P^2\delta_{ba}-m_{ba}^{(0)2}+\Sigma_{ba}^R(P^2,\mu^2) +\frac{1}{16\pi^2}[2g_{1b}^*g_{1a}+g_{2b}^*g_{2a}]\left[P^2\left[f_0-1-\ln\frac{\mu^2}{\mu_0^2}\right]+\mu^2+i\pi P^2\theta(P^2)\right] , \tag{A2}$$

where $m_{ba}^{(0)2}$ is the square of the bare mass matrix,

$$f_0=\frac{2}{4-n}+\frac{3}{2}-\gamma_E+\ln 4\pi ,$$

g_{fa} $(f=1,2)$ is the renormalized couplings (to the lowest order $g_{fa}=g_{fa}^{(0)}$) and θ is the θ function. $\Sigma_{ba}^R(P^2,\mu^2)$ is the renormalized self-energy:

$$\Sigma_{ba}^R(P^2,\mu^2)=\frac{1}{16\pi^2}[2g_{1b}^*g_{1a}+g_{2b}^*g_{2a}]\left[(P^2-\mu^2)+P^2\ln\frac{P^2}{\mu^2}+i\pi P^2[\theta(P^2)-\theta(\mu^2)]\right] , \tag{A3}$$

which has the standard features

$$\Sigma_{ba}^R(\mu^2,\mu^2)=0 , \tag{A4}$$

$$\frac{\partial}{\partial P^2}\Sigma_{ba}^R(P^2,\mu^2)|_{P^2=\mu^2}=0 , \tag{A5}$$

and μ^2 is the subtraction point. In practice, it is convenient to choose μ^2 be the invariant mass square of the initial particle.

Following the standard method the renormalized inverse propagator is

$$\tilde\Delta_{R,ba}^{-1}(P^2,\mu^2)=P^2\delta_{ba}-\tilde M_{ba}^2(\mu^2)+i\tilde\Gamma_{ba}^2(P^2)+\Sigma_{ba}^R(P^2,\mu^2) , \tag{A6}$$

where $\tilde{\Gamma}^2_{ba}(P^2)$ is given by (48) and

$$\tilde{M}^2_{ba}(\mu^2)=m^{(0)2}_{ba}-\frac{\mu^2}{16\pi^2}[2g^*_{1b}g_{1a}+g^*_{2b}g_{2a}]$$
$$-\frac{1}{16\pi^2}\sum_c\{[2g^*_{1b}g_{1c}+g^*_{2b}g_{2c}]\tilde{M}^2_{ca}+\tilde{M}^2_{bc}[2g^*_{1c}g_{1a}+g^*_{2c}g_{2a}]\}\left[f_0-1-\ln\frac{\mu^2}{\mu_0^2}\right] \tag{A7}$$

is the square of the renormalized mass matrix. The renormalized weak eigenstate fields are related to the bare fields by

$$S^{(0)}_{a,\alpha}=\sum_b\left[\delta_{ba}-\frac{1}{16\pi^2}[2g^*_{1b}g_{1a}+g^*_{2b}g_{2a}]\left[f_0-1-\ln\frac{\mu^2}{\mu_0^2}\right]\right]S_{b,\alpha}\,. \tag{A8}$$

The divergencies in the interaction Lagrangian associated with wave function renormalization and vertex corrections are finally absorbed into coupling constant renormalization, i.e., $g^{(0)}_{fa}\to g_{fa}$.

Once the renormalized parameters are determined at a given scale μ^2 by experiments, the separation of the divergent term f_0 from P^2 and μ^2 in (60), (64), and (65) allows us to predict these parameters at any other scale.

APPENDIX B

In this appendix we outline the basic ideas of diagonalizing an arbitrary $n\times n$ complex matrix $\mathcal{H}$. $\mathcal{H}$ can always be diagonalized by a biunitary transformation, i.e., $[V_L\mathcal{H}V_R^\dagger]_{ba}=\lambda_b\delta_{ba}$, where V_L and V_R are unitary and the eigenvalues λ_a are real. However, this procedure is not useful for us unless $V_LV_R^\dagger=1$; otherwise the kinetic energy part of the Lagrangian will not remain diagonal under such a transformation.

Methods for diagonalizing a 2×2 complex non-Hermitian matrix are known [3]. We have employed one of them in our analysis in Sec. V. Here we extend them to arbitrary cases. Consider an arbitrary $n\times n$ complex matrix $\mathcal{H}$. In general, $\mathcal{H}$ has $2n^2$ parameters. It is convenient to separate $\mathcal{H}$ in to a Hermitian and an anti-Hermitian part:

$$\mathcal{H}=M-i\Gamma\,, \tag{B1}$$

where

$$M=M^\dagger=\frac{\mathcal{H}+\mathcal{H}^\dagger}{2}\,, \tag{B2}$$

$$\Gamma=\Gamma^\dagger=i\frac{\mathcal{H}-\mathcal{H}^\dagger}{2}\,. \tag{B3}$$

In the study of unstable particle mixing M and Γ are the mass and decay matrices, respectively.

Let us start from simple cases involving real M and Γ. These correspond to situations in which the mixed unstable-particle system conserves *CP*. Now, $\mathcal{H}$ is symmetric. It has $n(n+1)$-independent parameters. We can diagonalize such an $\mathcal{H}$ by an orthogonal transformation [29]

$$O\mathcal{H}O^T=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\,, \tag{B4}$$

where $OO^T=1$. O differs from a usual orthogonal matrix by having complex mixing angles, i.e., $\theta_i=\mathrm{Re}\theta_i+\mathrm{Im}\theta_i$ $[i=1,\dots,n(n-1)/2]$. It should be emphasized that a complex mixing angle is not equivalent to a real mixing angle times a real phase. Indeed, $|\cos\theta|$ could be bigger than 1 if θ is complex, whereas $|\cos\theta e^{i\alpha}|\le 1$ if both θ and α are real. Nevertheless, trigonometric functions with complex angles have very similar properties, such as $\sin^2\theta+\cos^2\theta=1$, etc., as the elementary trigonometric functions. This is the advantage of introducing complex missing angles.

The $n(n+1)$ parameters of $\mathcal{H}$ determine the n complex eigenvalues, which have $2n$ parameters and the $n(n-1)/2$ complex angles involving $n(n-1)$ parameters.

This idea can be extended to situations in which both M and Γ are complex, provided the phases involved in the diagonalization are chosen as complex as well [30].

Let us first review the diagonalization of a Hermitian matrix H:

$$UHU^{-1}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\,, \tag{B5}$$

where λ_i are real. $U^{-1}=U^\dagger$ is unitary with $n(n-1)/2$ real angles and $n(n+1)/2$ real phases. However, not all phases of U are determined by (B5). To be specific, we write

$$U=KU'\,, \tag{B6}$$

where U' is a *reduced* unitary matrix with $n(n-1)/2$ angles and $n(n-1)/2$ phase, K is diagonal with n phases:

$$K=\begin{bmatrix}e^{i\alpha_1} & & 0\\ & \ddots & \\ 0 & & e^{i\alpha_n}\end{bmatrix}\,. \tag{B7}$$

Clearly, if U diagonalizes H so does its reduced matrix U'.

An arbitrary $\mathcal{H}$ has $2n^2$ parameters: among them, n^2 in M and the rest in Γ. The transformation that leaves the unity matrix invariant and diagonalizes $\mathcal{H}$ is

$$V\mathcal{H}V^{-1}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}, \tag{B8}$$

where λ_i are complex. In practice, we may parametrize V in terms of a reduced unitary matrix, such as U' discussed above, but change all the mixing angles and phases into complex variables. Now, the $2n^2$ parameters determine the n complex eigenvalues which involve $2n$ parameters, the $n(n-1)/2$ complex mixing angles [with $n(n-1)$ parameters], and the $n(n-1)/2$ complex phases, which also have $n(n-1)$ parameters.

For example, for a 2×2 $\mathcal{H}$, V has one complex mixing angle and one complex phase [see (49)]. For a 3×3 $\mathcal{H}$, V has three complex mixing angles and three complex mixing phases. It may be parametrized as

$$V=\begin{bmatrix}1 & 0 & 0\\ 0 & c_2 & s_2e^{i\alpha_2}\\ 0 & s_2e^{-i\alpha_2} & -c_2\end{bmatrix}\begin{bmatrix}c_1 & s_1e^{i\alpha_1} & 0\\ -s_1e^{-i\alpha_1} & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & c_3 & s_3e^{i\alpha_3}\\ 0 & -s_3e^{-i\alpha_3} & c_3\end{bmatrix}, \tag{B9}$$

where $c_i=\cos\theta_i$ and $s_i=\sin\theta_i$ $(i=1,2,3)$. θ_i and α_i are complex.

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