Tests of factorization in D, D_s^+, and B decays

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We have tested the factorization hypothesis by comparing $B(D^0\to K^-\rho^+)$, $B(D^0\to K^-a_1^+)$, and $B(D^0\to K^-\pi^+)$ with $B(D^0\to K^-e^+\nu)$, and $B(D^0\to\pi^+\pi^-)$ with $B(D^0\to\pi^-e^+\nu)$. In all cases *except* $D^0\to K^-a_1^+$ we find that the factorization hypothesis works once final-state interactions are taken into account. $D^0\to K^-a_1^+$ remains a problem. We have also predicted $B(D_s^+\to K^0e^+\nu)$ by relating it to $B(D_s^+\to K^+\bar{K}^{*0})$ and $B(D_s^+\to(\eta,\eta')e^+\nu)$ by relating them to $B(D_s^+\to(\eta,\eta')\rho^+)$. Finally, we discuss tests of factorization in $B^0\to\pi^+\pi^-$ by relating it to $B^0\to\pi^-e^+\nu$.

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I. INTRODUCTION

In the recent past the factorization hypothesis has been tested quite extensively in two-body hadronic $b\to c$ decays of the B meson [1–4] and found to be sound within experimental errors which are typically of the order of 25%. Factorization is expected to work well in B decays into D (or D^*) and a light meson, since the latter, carrying a large momentum, escapes the region of interaction thereby minimizing the effects of final-state interactions (FSI's). In addition, in decays involving $b\to c$ transitions, the heavy quark effective theory (HQET) [5] allows one to relate form factors entering the description of $B\to D$ transition to those of $B\to D^*$ transition through a common Isgur-Wise function [5]. In the factorization model this leads one to the predictions [4,5]

$$B(B^0\to D^-\pi^+)=B(B^0\to D^{*-}\pi^+)$$

and

$$B(B^0\to D^-\rho^+)=B(B^0\to D^{*-}\rho^+)\ .$$

Though data at present are not very accurate, these relations are found to be satisfied within an experimental accuracy of 30%.

By contrast, factorization is not expected to work quite very well in D decays. To begin with, an amplitude analysis [6] of $D\to K\pi$, $K\rho$, $K^*\pi$ and $K^*\rho$ has shown that there indeed are strong FSI interference effects. These effects tend to destroy the validity of the factorization hypothesis. Second, HQET cannot be used with confidence to relate the decays involving the $D\to K$ transition to those involving $D\to K^*$ transitions due to the relative lightness of s quarks.

Recently Pham and Vu [7] have tested factorization in hadronic D decays and concluded that factorization works within a factor of 2 for some channels but fails badly for $D^0\to K^-a_1^+$ and $D^0\to K^{*-}\pi^+$ decays. We have reconsidered the modes Pham and Vu [7] had studied and, in addition, have investigated D_s^+ decays. The advantage of the latter is that the final state is populated by a single isospin mode in Cabibbo-favored hadronic decays. Hence the interference effects due to FSI's do not come into play, though inelastic FSI's could still play a role. In the decays considered by Pham and Vu [7], while we confirm their findings, we demonstrate that the disagreement between theory and experiment in many cases can be attributed to FSI's, though problems remain with $D^0\to K^-a_1^+$ for reasons we will explain later. We have also related $B(D_s^+\to(\eta,\eta')\rho^+)$ to $B(D_s^+\to(\eta,\eta')e^+\nu)$. As the factorization model has so far failed to explain [8] $D_s^+\to(\eta,\eta')\rho^+$ [and $D_s^+\to(\eta,\eta')\pi^+$] data, we suggest that $B(D_s^+\to(\eta,\eta')e^+\nu)$ be measured to test its consistency with $B(D_s^+\to(\eta,\eta')\rho^+)$ data.

Tests of factorization in B decays, performed up until now [1–4], have involved heavy$\to$heavy, $b\to c$, transitions. An important hadronic decay mode of B is $B\to\pi^+\pi^-$ whose importance lies in its dependence on the Cabibbo-Kobayashi-Maskawa (CKM) mixing element V_{ub}. Before one uses this mode to determine V_{ub}, one ought to make certain that factorization works well. To this end we have related $B^0\to\pi^+\pi^-$ to $B^0\to\pi^-e^+\nu$ in complete analogy with the discussion of $D^0\to\pi^+\pi^-$ and $D^0\to\pi^-e^+\nu$ and proposed tests of factorization in $B^0\to\pi^+\pi^-$ decay.

This paper is organized as follows. Section II discusses the factorization hypothesis and its consequences. We then move on to tests of factorization involving $D^0\to K^-\rho^+$, $K^-\pi^+$, $K^-a_1^+$, and $\pi^+\pi^-$ in Sec. III. Section IV deals with the prediction of some as yet unmeasured rates involving D_s^+ decays. In Sec. V we discuss tests of factorization in $B^0\to\pi^+\pi^-$ decay by relating it to $B^0\to\pi^-e^+\nu$. The paper concludes with Sec. VI.

II. FACTORIZATION HYPOTHESIS

Since, for the most part, we shall be considering Cabibbo-favored two-body weak decays of charmed mesons, we introduce the effective weak Hamiltonian in order to define the notation and definitions we shall adopt:

$$H_W^{\text{eff}}(\Delta c=\Delta s=-1)=\frac{G_F}{\sqrt{2}}V_{ud}V_{cs}^*[a_1(\bar{u}d)_H(\bar{s}c)_H+a_2(\bar{u}c)_H(\bar{s}d)_H]\ , \tag{1}$$

where V_{ud}, etc., are the relevant CKM mixing parameters; a_1 and a_2 are the QCD Wilson coefficients, which we will take to be $a_1=1.2$ and $a_2=-0.5$ with a 10% error in both a_1 and a_2; the notation $(\bar{u}d)_H$ stands for a color-singlet $(V-A)$ Dirac bilinear with u and d flavors. The subindex H instructs us to treat this bilinear as an interpolating hadron field. The essential feature of the factorization hypothesis is that the matrix element of H_W^{eff} between two hadronic states is given as the *product* of the two matrix elements of the currents. As an example, the matrix element for $D^0\to K^-\pi^+$ decay is given by

$$A=\frac{G_F}{\sqrt{2}}V_{ud}V_{cs}^*a_1\langle\pi^+|(\bar{u}d)_H|0\rangle\langle K^-|(\bar{s}c)_H|D^0\rangle$$
$$=i\frac{G_F}{\sqrt{2}}V_{ud}V_{cs}^*a_1f_\pi(m_D^2-m_K^2)F_0^{DK}(m_\pi^2)\ , \tag{2}$$

where the following definitions are adopted:

$$\langle\pi^+(q)|(\bar{u}d)_H|0\rangle=if_\pi q_\mu,\quad f_\pi=131\ \text{MeV} \tag{3}$$

$$\langle K^-(p)|(\bar{s}c)_H|D^0(k)\rangle$$
$$=\left[(k+p)_\mu-\frac{m_D^2-m_K^2}{q^2}q_\mu\right]F_1^{DK}(q^2)$$
$$+\frac{m_D^2-m_K^2}{q^2}q_\mu F_0^{DK}(q^2)\ , \tag{4}$$

where $q_\mu=(k-p)_\mu$ and

$$F_0(0)=F_1(0)\ . \tag{5}$$

The above matrix element can also be written as

$$\langle K^-(p)|(\bar{s}c)_H|D^0(k)\rangle=(k+p)_\mu f_+^{DK}(q^2)$$
$$+(k-p)_\mu f_-^{DK}(q^2)\ . \tag{6}$$

Equations (4) and (6) lead to the following relations between the various factors:

$$f_+(q^2)=F_1(q^2)\ ,$$
$$f_-(q^2)=\frac{m_D^2-m_K^2}{q^2}[F_0(q^2)-F_1(q^2)]\ . \tag{7}$$

The decay rate for $D^0\to K^-\pi^+$, in this example, is given by

$$\Gamma(D^0\to K^-\pi^+)=\frac{|A|^2}{8\pi m_D^2}|\mathbf{p}|\ , \tag{8}$$

where $|\mathbf{p}|$ is the center-of-mass momentum of the decay products.

In an analogous manner, the factorization model leads to the following decay amplitude for $D^0\to K^-\rho^+$:

$$\langle K^-(p),\rho^+(q)|H_W^{\text{eff}}|D^0(k)\rangle\equiv a\varepsilon^*\cdot k \tag{9}$$

with

$$a=\frac{G_F}{\sqrt{2}}V_{ud}V_{cs}^*a_1(2m_\rho)f_\rho F_1^{DK}(m_\rho^2)\ , \tag{10}$$

where f_ρ is the decay constant for $\rho^0\to e^+e^-$, which we take to be 220 MeV. The decay rate is then given by

$$\Gamma(D^0\to K^-\rho^+)=\frac{|a|^2}{8\pi m_\rho^2}|\mathbf{p}|^3\ . \tag{11}$$

The differential decay rate for the semileptonic decay $D^0\to K^-e^+\nu$ is given by [7]

$$\frac{d\Gamma}{dq^2}(D^0\to K^-e^+\nu)$$
$$=\frac{G_F^2}{192\pi^3}|V_{cs}|^2\frac{\lambda^3(m_D^2,m_K^2,q^2)}{m_D^3}|F_1^{DK}(q^2)|^2\ , \tag{12}$$

where

$$\lambda(x,y,z)=(x^2+y^2+z^2-2xy-2xz-2yz)^{1/2}$$

and, in particular, for the decay $D^0\to K^-\rho^+$, the final-state center-of-mass momentum is

$$|\mathbf{p}|=\lambda(m_D^2,m_K^2,m_\rho^2)/2m_D\ .$$

Using (10) in (11) and comparing (11) and (12) at $q^2=m_\rho^2$, one finds that in *absence of* FSI's the factorization model requires the following *local* relation:

$$\Gamma(D^0\to K^-\rho^+)_{\text{no FSI}}=6\pi^2a_1^2f_\rho^2|V_{ud}|^2$$
$$\times\frac{d\Gamma}{dq^2}(D^0\to K^-e^+\nu)\bigg|_{q^2=m_\rho^2}\ . \tag{13}$$

On comparing (8) and (12) at $q^2=m_\pi^2$ and using (2) and (5) [which allow us to use $F_0(m_\pi^2)\approx F_1(m_\pi^2)$] one finds, in *absence of* FSI's, that the factorization model also implies

$$\Gamma(D^0\to K^-\pi^+)_{\text{no FSI}}$$
$$\approx6\pi^2a_1^2f_\pi^2|V_{ud}|^2\frac{(m_D^2-m_K^2)^2}{\lambda^2(m_D^2,m_K^2,m_\pi^2)}$$
$$\times\frac{d\Gamma}{dq^2}(D^0\to K^-e^+\nu)\bigg|_{q^2=m_\pi^2}\ . \tag{14}$$

Analogous *local* relations to (13) and (14) can be written down between $\Gamma(B^0\to D^-\pi^+)$ and $\Gamma(B^0\to D^-\rho^+)$ on one hand and the semileptonic differential $d\Gamma(B^0\to D^-e^+\nu)/dq^2$ on the other [2–4].

Testing *local* relations such as (13) and (14) requires very good data to determine $d\Gamma/dq^2$ at the q^2 needed. In the absence of high precision data, one resorts to theoretical models (as has been done in tests involving B decays) to calculate the form factors in order to interpolate between data points [1–4]. An alternative test [7] of the factorization assumption is to compare the calculated ratio

$$\Gamma(D^0\to K^-\rho^+)/\Gamma(D^0\to K^-e^+\nu)$$

with the experimental one. The uncertainty in such a test comes from the assumed behavior of $F_1(q^2)$ in performing the q^2 integration in (12) to generate $\Gamma(D^0\to Ke\nu)$.

In the following section we describe several such tests of the factorization hypothesis.

III. TEST OF FACTORIZATION

A. $\Gamma(D^0 \to K^- \rho^+)$ vs $\Gamma(D^0 \to K^- e^+ \nu)$

As remarked in the previous section, one can calculate $\Gamma(D^0 \to K^- e^+ \nu)$ by using an assumed form for $F_1(q)$ in (12). If we use

$$F_1^{DK}(q^2) F_1^{DK}(0)/(1-q^2/\Lambda_1^2) , \quad (15)$$

with $\Lambda_1 = 2.11$ GeV (D_s^* mass), the semileptonic rate can be evaluated to be

$$\Gamma(D^0 \to K^- e^+ \nu) = \frac{G_F^2}{192\pi^3} |V_{cs}|^2 \frac{|F_1^{DK}(0)|^2 \Lambda_1^4}{m_D^3} \times I(m_D, m_K, \Lambda_1) , \quad (16)$$

where

$$I(m_D, m_K, \Lambda_1) \equiv \int_0^{(m_D - m_K)^2} \frac{\lambda^3(m_D^2, m_K^2, q^2)}{(q^2 - \Lambda_1^2)^2} dq^2 . \quad (17)$$

Although we have calculated this integral numerically, it can be done in a closed form. We give the formula in the Appendix.

Rewriting (11) in the same notation as (16) we get

$$\Gamma(D^0 \to K^- \rho^+) = \frac{G_F^2}{32\pi} \frac{|V_{cs}|^2 |V_{ud}|^2}{m_D^3} a_1^2 f_\rho^2 \frac{|F_1^{DK}(0)|^2 \Lambda_1^4}{(\Lambda_1^2 - m_\rho^2)^2} \times \lambda^3(m_D^2, m_K^2, m_\rho^2) . \quad (18)$$

From (16) and (18) we find [with $I(m_D, m_K, \Lambda_1) = 1.440$ GeV^4 and $a_1 = 1.2$] that the factorization hypothesis requires

$$\frac{\Gamma(D^0 \to K^- \rho^+)_{\text{no FSI}}}{\Gamma(D^0 \to K^- e^+ \nu)} = \frac{6\pi^2 a_1^2 f_\rho^2 |V_{ud}|^2 \lambda^3(m_D^2, m_K^2, m_\rho^2)}{(\Lambda_1^2 - m_\rho^2)^2 I(m_D, m_K, \Lambda_1)} = (3.0 \pm 0.6) , \quad (19)$$

where the error comes from assigning a 10% uncertainty to a_1. Experiments [9] yield

$$\left. \frac{\Gamma(D^0 \to K^- \rho^+)}{\Gamma(D^0 \to K^- e^+ \nu)} \right|_{\text{expt}} = (2.20 \pm 0.38) , \quad (20)$$

where we have treated the errors as independent. Within errors factorization model prediction for the ratio in (19) agrees with the experimental ratio. However, the following points need be made. (i) The theoretical ration in (19) does not include FSI interference effects between $I = \frac{1}{2}$ and $\frac{3}{2}$ $K\rho$ states. Nevertheless, we do not anticipate a large FSI effect for $K\rho$ decay mode since the phase difference between $I = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes, $\delta_{1/2} - \delta_{3/2}$, is known to be small [6], $(0 \pm 26)°$. Also, since an amplitude analysis for $D \to K\rho$ decays exists, one can estimate the effect of FSI as follows.

(i) Begin by writing the decay amplitude for $D^0 \to K^- \rho^+$ (and, for completeness, $D^0 \to \bar{K}^0 \rho^0$ and $D^+ \to \bar{K}^0 \rho^+$) in terms of isospin amplitudes:

$$A(D^0 \to K^- \rho^+) = \frac{1}{\sqrt{3}} (A_{3/2} e^{i\delta_{3/2}} + \sqrt{2} A_{1/2} e^{i\delta_{1/2}}) ,$$
$$A(D^0 \to \bar{K}^0 \rho^0) = \frac{1}{\sqrt{3}} (\sqrt{2} A_{3/2} e^{i\delta_{3/2}} - A_{1/2} e^{i\delta_{1/2}}) , \quad (21)$$
$$A(D^0 \to \bar{K}^0 \rho^+) = \sqrt{3} A_{3/2} e^{i\delta_{3/2}} .$$

Mark III measurements [6] yield a fit

$$\frac{A_{1/2}}{A_{3/2}} = (3.12 \pm 0.4), \quad \delta_{1/2} - \delta_{3/2} = (0 \pm 26)° . \quad (22)$$

If we use $\delta_{1/2} - \delta_{3/2} = \pm 26°$, the theoretical ratio in (19) is *lowered* by $\approx 3\%$; alternatively, one could argue that deconvoluting FSI's from the experimental data would *raise* the ratio in (20) by $\approx 3\%$, improving, thereby, the agreement between theory and experiment.

(ii) If a mass smearing is done over the ρ resonance with an energy-dependent width with an appropriate threshold behavior, the effect is to *lower* the rate [10,11], though the effect is not dramatic even for very wide resonances such as $a_1(1260)$ [10]. This is because of the fact that the lower mass region which tends to raise the rate, is suppressed by the threshold factors. (In Ref. [7] it was shown that the use of an energy-independent width also leads to a lowering of the $D^0 \to K^- \rho^+$ rate.) We would, therefore, argue that mass smearing with an energy-dependent width with an appropriate threshold behavior will *lower* the theoretical ratio in (19), leading to a better agreement between theory and experiment.

(iii) The neutral channel $D^0 \to \bar{K}^0 \rho^0$ (and analogously $D^0 \to \bar{K}^0 \pi^0$, $\bar{K}^{*0} \pi^0$, and $\bar{K}^{*0} \rho^0$) is not good for tests of factorization. This conclusion results from an examination of (21). The ratio of isospin $\frac{3}{2}$ to isospin $\frac{1}{2}$ contribution in these neutral modes is twice as large as in the charged modes and hence also the interference effects. Further, since $A_{3/2}/A_{1/2} \approx \frac{1}{3}$ in all of these decay modes [6], FSI phases affect the neutral modes much more than the charged ones.

In conclusion the "raw" test (uncorrected for FSI effects) of factorization for $D^0 \to K^- \rho^+$ is satisfactory. If the effects of FSI and mass smearing over the ρ width are taken into consideration, the agreement between theory and experiment improves.

B. $\Gamma(D^0 \to K^- a_1^+)$ vs $\Gamma(D^0 \to K^- e^+ \nu)$

The decay rate for $D^0 \to K^- a_1^+$ can be calculated in complete analogy with the case of $D^0 \to K^- \rho^+$ discussed above. Without going into details we list below some of the relations that follow from factorization hypothesis.

The analogue of the local relation (13) is

$$\Gamma(D^0 \to K^- a_1^+)_{\text{no FSI}} = 6\pi^2 a_1^2 f_a^2 |V_{ud}|^2 \times \left. \frac{d\Gamma}{dq^2}(D^0 \to K^- e^+ \nu) \right|_{q^2 = m_a^2} . \quad (23)$$

The analogue of (19) is (we have used $f_a = f_\rho = 220$ MeV)

$$\frac{\Gamma(D^0\to K^- a_1^+)_{\text{no FSI}}}{\Gamma(D^0\to K^- e^+\nu)}=\frac{6\pi^2 a_1^2 f_1^2 |V_{ud}|^2\lambda^3(m_D^2,m_K^2,m_a^2)}{(\Lambda_1^2-m_a^2)I(m_D,m_K,\Lambda_1)}=0.43\ . \tag{24}$$

On using [9]

$$B(D^0\to K^- e^+\nu)=(3.31\pm0.29)\%$$

we obtain

$$B(D^0\to K^- a_1^+)_{\text{no FSI}}=(1.4\pm0.14\pm0.28)\%\ , \tag{25}$$

where the first error comes from the error quoted in $B(D^0\to K^- e^+\nu)$ and the second from the 10% uncertainty assigned to a_1.

We need to make a few comments on the results obtained above.

(i) For the value of the Wilson coefficient a_1 we use, the estimate of $B(D^0\to K^- a_1^+)$ in (25) is not in disagreement with Pham and Vu [7]. We make more detailed comments on the numerical value of a_1 we have chosen in the Conclusion section.

(ii) As $a_1(1260)$ is a broad resonance one ought to do a mass-smearing over a_1 width. However, as discussed already, if the smearing is done with an energy-dependent width with appropriate threshold factors, the effect of mass smearing is not dramatic and is to *lower* the rate [10]. In this respect we disagree with Ref. [7] where the smearing is done with a constant width and the smeared rate $\Gamma(D^0\to K^- a_1^+)$ is larger than that in the narrow resonance approximation.

(iii) The ratio in (24) is not very sensitive to mass smearing as while the numerator favors smaller values of m_a by raising λ, the effect of the denominator is to suppress the contribution from smaller values of m_a. The opposite is the case for larger values of m_a.

(iv) Finally, the experimental value [9] of the branching ratio $B(D^0\to K^- a_1^+)=(7.4\pm1.3)\%$ is in strong disagreement with the theoretical prediction of the factorization model.

C. $\Gamma(D^0\to K^-\pi^+)$ vs $\Gamma(D^0\to K^- e^+\nu)$

In the notation adopted in the discussion so far, we can write the decay rate for $D^0\to K^-\pi^+$, Eq. (8), as

$$\Gamma(D^0\to K^-\pi^+)=\frac{G_F^2}{32\pi}\frac{|V_{cs}|^2|V_{ud}|^2}{m_D^3}a_1^2 f_\pi^2(m_D^2-m_K^2)^2 \times |F_0^{DK}(m_\pi^2)|^2\lambda(m_D^2,m_K^2,m_\pi^2)\ . \tag{26}$$

From (16) and (26) we get (with $a_1=1.2$)

$$\frac{\Gamma(D^0\to K^-\pi^+)_{\text{no FSI}}}{\Gamma(D^0\to K^- e^+\nu)}\simeq\frac{6\pi^2 a_1^2 f_\pi^2(m_D^2-m_K^2)^2|V_{ud}|^2\lambda(m_D^2,m_K^2,m_\pi^2)}{\Lambda_1^4 I(m_D,m_K,\Lambda_1)}=(1.63\pm0.33)\ . \tag{27}$$

In deriving (27) we have used $F_0(m_\pi^2)\approx F_0(0)=F_1(0)$, and the error comes from a 10% uncertainty assigned by a_1. The experimental [9] ratio is

$$\left.\frac{\Gamma(D^0\to K^-\pi^+)}{\Gamma(D^0\to K^- e^+\nu)}\right|_{\text{expt}}=(1.10\pm0.11)\ . \tag{28}$$

Although (27) and (28) appear to be in disagreement, if the known effects of FSI are folded in, theory and experiment agree as we show below.

$D\to K\pi$ decays are experimentally well studied. In the notation introduced in (21) the following parameters are known [6] for these modes:

$$\frac{A_{1/2}}{A_{3/2}}=(3.67\pm0.27),\quad \delta_{1/2}-\delta_{3/2}=(77\pm11)^\circ\ . \tag{29}$$

Using (21) for the isospin decomposition and (29) we find that the effect of FSI's is to lower the ratio in (27) to (1.27 ± 0.25) if we use $\delta_{1/2}-\delta_{3/2}=80^\circ$. Thus after the theoretical prediction of the factorization model is corrected for FSI's we find very good agreement between theory and experiment.

D. $\Gamma(D^0\to\pi^+\pi^-)$ vs $\Gamma(D^0\to\pi^- e^+\nu)$

Contrary to the cases discussed so far, these are Cabibbo-suppressed processes. Following the procedure used in the previous cases one can derive the following prediction of the factorization model:

$$\frac{\Gamma(D^0\to\pi^-\pi^+)_{\text{no FSI}}}{\Gamma(D^0\to\pi^- e^+\nu)}\simeq\frac{6\pi^2 a_1^2 f_\pi^2(m_D^2-m_\pi^2)^2|V_{ud}|^2\lambda(m_D^2,m_\pi^2,m_\pi^2)}{\Lambda_2^4 I(m_D,m_\pi,\Lambda_2)}=(1.02\pm0.2)\ , \tag{30}$$

where we have used $\Lambda_2=2.01$ GeV (D^* mass) and the calculated value $I(m_D,m_\pi,\Lambda_2)=3.45\ \text{GeV}^4$. The error in (30) results from a 10% uncertainly assigned to a_1. The experimental ratio is [9]

$$\frac{\Gamma(D^0\to\pi^+\pi^-)}{\Gamma(D^0\to\pi^- e^+\nu)}=(0.42^{+0.25}_{-0.14})\ . \tag{31}$$

The disagreement between theory (30) and experiment (31) is largely due to FSI's. Despite the fact that an amplitude analysis for $D\to\pi\pi$ decays is not yet possible (due to absent $D^+\to\pi^0\pi^+$ data) there is a fair amount of confidence [12] in the ability of the factorization model with FSI's to reproduce $D^0\to\pi^+\pi^-$ and $\pi^0\pi^0$ data [9]. It was shown in Ref. [12] that the Bauer, Stech, and Wirbel (BSW) model [13] does a fair job of reproducing $D\to\pi\pi$ data with $\delta_0-\delta_2\approx90^\circ$ [14]. The subscripts refer to the isospin of the $\pi\pi$ system. The suppression of $D^0\to\pi^+\pi^-$

rate in the BSW model with $\delta_0-\delta_2\approx 90°$ is by a factor of 0.68. Thus the effect of FSI's would be to lower the ratio in (30) to (0.69 ± 0.14) which would be in agreement with experiment (31). Better semileptonic data are clearly called for.

IV. SOME PREDICTIONS

In this section we use the factorization model to make some predictions. In order to minimize the interference effects introduced by FSI's, we limit ourselves to Cabibbo-favored decays of D_s^+ where the final hadronic states populate a single isospin, $I=1$. In the following three cases we predict the semileptonic rates by relating them to the measured two-body hadronic rates. These predictions would test the validity of factorization scheme in these processes.

A. Predict $\Gamma(D_s^+\to K^0 e^+\nu)$ from $\Gamma(D_s^+\to K^+\bar{K}^{*0})$

The semileptonic decay $D_s^+\to K^0 e^+\nu$ is a Cabibbo-suppressed decay involving the CKM mixing parameter V_{cd}. In the factorization model it can be related to the Cabibbo-favored but "color-suppressed" process $D_s^+\to K^+\bar{K}^{*0}$. In the notation of the previous section, the semileptonic differential rate and the total rate are given as

$$\frac{d\Gamma}{dq^2}(D_s^+\to K^0 e^+\nu)=\frac{G_F^2}{192\pi^3}\frac{|V_{cd}|^2}{m_{D_s}^3}\lambda^3(m_{D_s}^2,m_K^2,q^2)\times|F_1^{D_sK}(q^2)|^2 \qquad (32)$$

and

$$\Gamma(D_s^+\to K^0 e^+\nu)=\frac{G_F^2}{192\pi^3}\frac{|V_{cd}|^2}{m_{D_s}^3}|F_1^{D_sK}(0)|^2\times\Lambda_2^4 I(m_{D_s},m_K,\Lambda_2)\ , \qquad (33)$$

where $\Lambda_2=2.01$ GeV (D^* mass) and $I(m_{D_s},m_K,\Lambda_2)$ is defined in (17).

In the factorization model, with a monopole form for $F_1(q^2)$, the hadronic rate for $D_s^+\to K^+\bar{K}^{*0}$ is given by

$$\Gamma(D_s^+\to K^+\bar{K}^{*0})=\frac{G_F^2}{32\pi}|V_{cs}|^2|V_{ud}|^2\frac{a_2^2 f_{K^*}^2}{m_{D_s}^3}|F_1^{D_sK}(0)|^2\times\frac{\Lambda_2^4}{(\Lambda_2^2-m_{K^*}^2)^2}\lambda^3(m_{D_s}^2,m_K^2,m_{K^*}^2)\ . \qquad (34)$$

From (33) and (34) we find

$$\Gamma(D_s^+\to K^0 e^+\nu)=(2.33)\frac{|V_{cd}|^2}{|V_{cs}|^2|V_{ud}|^2}\Gamma(D_s^+\to K^+\bar{K}^{*0})\ , \qquad (35)$$

where we have used $a_1=-0.5$, $f_{K^*}=220$ MeV and $I(m_{D_s},m_K,\Lambda_2)=3.10\ \text{GeV}^4$. Using [9]

$$B(D_s^+\to K^+\bar{K}^{*0})=(2.6\pm 0.5)\%\ ,$$

we find (using $|V_{ud}|\approx V_{cs}|=0.975$ and $|V_{cd}|^2\approx 1-|V_{ud}|^2$)

$$B(D_s^+\to K^0 e^+\nu)\approx(0.33\pm 0.06\pm 0.06)\%\ , \qquad (36)$$

where the first error is from $B(D_s^+\to K^+\bar{K}^{*0})$ and the second from a 10% uncertainty assigned to a_2.

Factorization hypothesis also related $\Gamma(D_s^+\to K^- e^+\nu)$ to $\Gamma(D_s^+\to K^0\pi^+)$ in the following manner:

$$\frac{\Gamma(D_s^+\to K^0\pi^+)}{\Gamma(D_s^+\to K^0 e^+\nu)}=\frac{6\pi^2 a_1^2 f_\pi^2|V_{ud}|^2(m_{D_s}^2-m_K^2)^2\lambda(m_{D_s}^2,m_K^2,m_\pi^2)}{\Lambda_2^4 I(m_{D_s},m_K,\Lambda_2)}\ . \qquad (37)$$

Equation (37) leads to

$$\Gamma(D_s^+\to K^0 e^+\nu)=(0.76\pm 0.15)\Gamma(D_s^+\to K^0\pi^+)\ . \qquad (38)$$

The uncertainty is due to a 10% uncertainty assigned to a_1. At present only an upper bound exists on $\Gamma(D_s^+\to K^0\pi^+)$: $\Gamma(D_s^+\to K^0\pi^+)<0.6\%$ [9]. This upper bound used in (38) leads to a $\Gamma(D_s^+\to K^0 e^+\nu)$ consistent with that predicted in (36).

Equations (32) and (34) lead to a *local* relation

$$\Gamma(D_s^+\to K^+\bar{K}^{*0})=6\pi^2 a_2^2 f_{K^*}^2\frac{|V_{cs}|^2}{|V_{cd}|^2}|V_{ud}|^2\times\left.\frac{d\Gamma}{dq^2}(D_s^+\to K^0 e^+\nu)\right|_{q^2=m_{K^*}^2}\ , \qquad (39)$$

and with a monopole structure for the form factor $F_1(q^2)$, to a differential rate

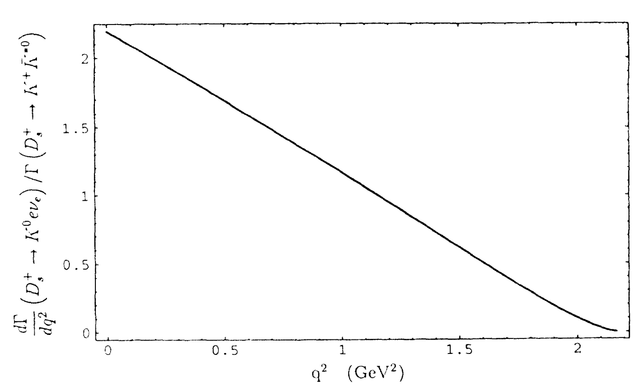

FIG. 1. Differential rate of the decay $D_s^+\to K^0 e\nu_e$ normalized with respect to the rate of the decay $D_s^+\to K^+\bar{K}^{*0}$.

$$\frac{d\Gamma}{dq^2}(D_s^+\to K^0e^+\nu)=\frac{1}{6\pi^2a_2^2f_{K^*}^2}\frac{|V_{cd}|^2}{|V_{cs}|^2|V_{ud}|^2}\times\frac{\lambda^3(m_{D_s}^2,m_K^2,q^2)}{\lambda^3(m_{D_s}^2,m_K^2,m_{K^*}^2)}\frac{(\Lambda_2^2-m_{K^*}^2)^2}{(\Lambda_2^2-q^2)^2}\times\Gamma(D_s^+\to K^+\bar{K}^{*0})\ . \quad (40)$$

This differential rate has been plotted in Fig. 1.

B. Predict $\Gamma(D_s^+\to(\eta,\eta')e^+\nu)$ from $\Gamma(D_s^+\to(\eta,\eta')\rho^+)$

Factorization model relates the semileptonic process $D_s^+\to(\eta,\eta')e^+\nu$ directly to the hadronic process $D_s^+\to(\eta,\eta')\rho^+$. It has also been shown [8] that the measurements [15] of $B(D_s^+\to\eta\rho^+)$ and $B(D_s^+\to\eta'\rho^+)$ have defied theoretical explanation based on factorization model. A measurement of the semileptonic branching ratios would further test the validity, or otherwise, of the factorization assumption.

In describing the η and η' system we use the conventions

$$|\eta\rangle=|8\rangle\cos\theta_p-|0\rangle\sin\theta_p\ ,$$
$$|\eta'\rangle=|8\rangle\sin\theta_p+|0\rangle\cos\theta_p\ , \quad (41)$$

where the flavor-singlet and -octet are defined as

$$|0\rangle=\frac{1}{\sqrt{3}}|u\bar{u}+d\bar{d}+s\bar{s}\rangle\ ,$$
$$|8\rangle=\frac{1}{\sqrt{6}}|u\bar{u}+d\bar{d}-2s\bar{s}\rangle\ , \quad (42)$$

and the mixing angle θ_p is taken to be $\simeq-19°$.

In the notation of the previous section, the differential and the total rates for the semileptonic $D_s^+\to\eta e^+\nu$ are given by

$$\frac{d\Gamma}{dq^2}(D_s^+\to\eta e^+\nu)=\frac{G_F^2}{192\pi^3}\frac{|V_{cs}|^2C_\eta^2}{m_{D_s}^3}\lambda^3(m_{D_s}^2,m_\eta^2,q^2)\times|F_1^{D_s\eta}(q^2)|^2 \quad (43)$$

and

$$\Gamma(D_s^+\to\eta e^+\nu)=\frac{G_F^2}{192\pi^3}\frac{|V_{cs}|^2C_\eta^2}{m_{D_s}^3}|F_1^{D_s\eta}(0)|^2\times\Lambda_1^4I(m_{D_s},m_\eta,\Lambda_1)\ , \quad (44)$$

where

$$C_\eta=\left[\frac{2}{3}\right]^{1/2}\left[\cos\theta_p+\frac{1}{\sqrt{2}}\sin\theta_p\right] \quad (45)$$

and $I(m_{D_s},m_\eta,\Lambda_1)$ appropriate for a monopole form factor is defined in (17). $\Lambda_1=2.11$ GeV is the mass of D_s^*.

The hadronic rate for $D_s^+\to\eta\rho^+$ in the factorization model is given by

$$\Gamma(D_s^+\to\eta\rho^+)=\frac{G_F^2}{32\pi}\frac{|V_{cs}|^2|V_{ud}|^2a_1^2f_\rho^2}{m_{D_s}^3}\times C_\eta^2|F_1^{D_s\eta}(m_\rho^2)|^2\lambda^3(m_{D_s}^2,m_\eta^2,m_\rho^2)\ . \quad (46)$$

From (43) and (46) one gets a *local* relation

$$\Gamma(D_s^+\to\eta\rho^+)=6\pi^2a_1^2f_\rho^2|V_{ud}|^2\frac{d\Gamma}{dq^2}(D_s^+\to\eta e^+\nu)\bigg|_{q^2=m_\rho^2}\ , \quad (47)$$

and from (44) and (46) we obtain

$$\Gamma(D_s^+\to\eta e^+\nu)=\frac{I(m_{D_s},m_\eta,\Lambda_1)}{6\pi^2a_1^2f_\rho^2|V_{ud}|^2}\frac{(\Lambda_1^2-m_\rho^2)^2}{\lambda^3(m_{D_s}^2,m_\eta^2,m_\rho^2)}\times\Gamma(D_s^+\to\eta\rho^+)\ . \quad (48)$$

With $a_1=1.2$, $f_\rho=220$ MeV, and $I(m_{D_s},m_\eta,\Lambda_1)=2.17$ GeV^4 we obtain

$$\Gamma(D_s^+\to\eta e\nu)=0.35\Gamma(D_s^+\to\eta\rho^+)\ . \quad (49)$$

CLEO II data [15] on $D_s^+\to\eta\rho^+$ are

$$\frac{B(D_s^+\to\eta\rho^+)}{B(D_s^+\to\phi\pi^+)}=(2.86\pm0.38^{+0.36}_{-0.38})\ .$$

This together with (49) results in

$$\frac{B(D_s^+\to\eta e\nu)}{B(D_s^+-\phi\pi^+)}=(1.00\pm0.19\pm0.20)\ . \quad (50)$$

In (50) the first error comes from the experimental value of

$$B(D_s^+\to\eta\rho^+)/B(D_s^+\to\phi\pi^+)$$

(we have combined the statistical and the systematic errors) and the second from a 10% uncertainty assigned to a_1.

An analogous treatment of the decays involving η' results in the following factorization model predictions:

$$\Gamma(D_s^+-\eta'\rho^+)=6\pi^2a_1^2f_\rho^2|V_{ud}|^2\times\frac{d}{dq^2}\Gamma(D_s^+\to\eta'e^+\nu)\bigg|_{q^2=m_\rho^2} \quad (51)$$

and

$$\Gamma(D_s^+\to\eta'e^+\nu)=\frac{I(m_{D_s},m_{\eta'},\Lambda_1)}{6\pi^2a_1^2f_\rho^2|V_{ud}|^2}\frac{(\Lambda_1^2-m_\rho^2)^2}{\lambda^3(m_{D_s}^2,m_{\eta'}^2,m_\rho^2)}\times\Gamma(D_s^+\to\eta'\rho^+)=0.35\Gamma(D_s^+\to\eta'\rho^+)\ , \quad (52)$$

where $I(m_{D_s},m_{\eta'},\Lambda_1)=0.584$ GeV^4 was used. Using

CLEO II data [15],

$$\frac{B(D_s^+\to\eta'\rho^+)}{B(D_s^+\to\phi\pi^+)}=(3.44\pm0.62^{+0.44}_{-0.46})\ , \tag{53}$$

we obtain

$$\frac{B(D_s^+\to\eta' e^+\nu)}{B(D_s^+\to\phi\pi^+)}=(1.20\pm0.27\pm0.24)\ , \tag{54}$$

where, again the first error comes from the data in (53) (statistical and systematic errors are added in quadrature) and the second error comes from a 10% uncertainty assigned to a_1. Measurements of $B(D_s^+\to(\eta,\eta')e^+\nu)$ to test (50) and (54) would be desirable to see if the factorization model adequately describes the hadronic processes $D_s^+\to(\eta,\eta')\rho^+$.

In Figs. 2 and 3 we have shown the predicted differential rates for the semileptonic processes $D_s^+\to(\eta,\eta')e^+\nu$ with a monopole structure for $F_1(q^2)$.

V. $B^0\to\pi^+\pi^-$ VS $B^0\to\pi^-e^+\nu$

The importance of the mode $B^0\to\pi^+\pi^-$ lies in its dependence on the CKM parameter V_{ub}. In analogy with the discussion of $D^0\to\pi^+\pi^-$ vs $D^0\to\pi^-e^+\nu$ in Sec. III, factorization hypothesis related $B^0\to\pi^+\pi^-$ to $B^0\to\pi^-e^+\nu$. Before using the mode $B^0\to\pi^+\pi^-$ to determine V_{ub} one ought to ensure that factorization indeed works. Testing factorization in $B^0\to\pi^+\pi^-$ decay is the topic of this section.

In analogy with the discussion of $D^0\to\pi^+\pi^-$ vs $D^0\to\pi^-e^+\nu$ in Sec. III we write

$$\Gamma(B^0\to\pi^-\pi^+)_{\text{no FSI}}\approx\frac{G_F^2}{32\pi}\frac{|V_{ud}|^2|V_{ub}|^2}{m_B^3}a_1^2f_\pi^2m_B^4 \times|f_+^{B\pi}(m_\pi^2)|^2\lambda(m_B^2,m_\pi^2,m_\pi^2) \tag{55}$$

and

$$\frac{d\Gamma}{dq^2}(B^0\to\pi^-e^+\nu)=\frac{G_F^2}{192\pi^2}\frac{|V_{ub}|^2}{m_B^3}|f_+^{B\pi}(q^2)|^2 \times\lambda^3(m_B^2,m_\pi^2,q^2)\ . \tag{56}$$

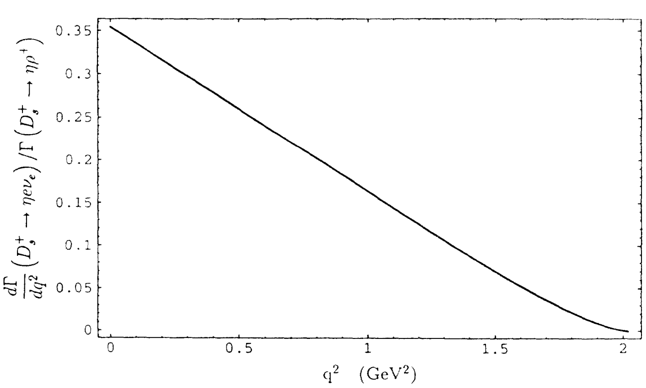

FIG. 2. Differential rate of the decay $D_s^+\to\eta e\nu_e$ normalized with respect to the rate of the decay $D_s^+\to\eta\rho^+$.

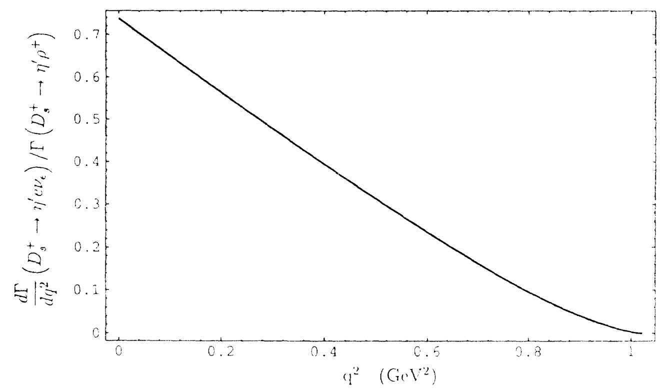

FIG. 3. Differential rate of the decay $D_s^+\to\eta' e\nu_e$ normalized with respect to the rate of the decay $D_s^+\to\eta'\rho^+$.

In (55) we have replaced $F_0^{B\pi}(m_\pi^2)$ by $F_1^{B\pi}(m_\pi^2)[=f_+^{B\pi}(m_\pi^2)]$ and $(m_B^2-m_\pi^2)^2$ by m_B^4. From (55) and (56) one obtains a *local* relation

$$\Gamma(B^0\to\pi^+\pi^-)_{\text{no FSI}}=6\pi^2|V_{ud}|^2\frac{a_1^2f_\pi^2m_B^4}{\lambda^2(m_B^2,m_\pi^2,m_\pi^2)} \times\frac{d}{dq^2}\Gamma(B^0\to\pi^-e^+\nu\left.\right|_{q^2=m_\pi^2}\ . \tag{57}$$

And assuming

$$f_+(q^2)=\frac{f_+(0)}{1-\dfrac{q^2}{\Lambda^2}}\ , \tag{58}$$

with $\Lambda=m_{B^*}=5.324$ GeV, one obtains

$$\Gamma(B^0\to\pi^-e^+\nu)=\frac{G_F^2}{192\pi^3}\frac{|V_{ub}|^2}{m_B^3}|f_+(0)|^2 \times\Lambda^4I(m_B,m_\pi,\Lambda)\ . \tag{59}$$

From (55) and (59) we get

$$\frac{\Gamma(B^0\to\pi^+\pi^-)_{\text{no FSI}}}{\Gamma(B^0\to\pi^-e^+\nu)} =\frac{6\pi^2a_1^2f_\pi^2m_B^4|V_{ud}|^2\lambda(m_B^2,m_\pi^2,m_\pi^2)}{\Lambda^4I(m_B,m_\pi,\Lambda)}\ . \tag{60}$$

A numerical evaluation with $V_{ud}=0.975$ and $I(m_B,m_\pi,\Lambda)=357.5$ GeV^4 yields (here, the Wilson coefficient a_1 is chosen to be appropriate for the b-mass scale)

$$\frac{\Gamma(B^0\to\pi^+\pi^-)_{\text{no FSI}}}{\Gamma(B^0\to\pi^-e^+\nu)}=\begin{cases}0.09 & (a_1=1.1)\ ,\\ 0.07 & (a_1=1.0)\ .\end{cases} \tag{61}$$

Individually, the processes involved in the ratio in Eq. (61) are rare due to their dependence on V_{ub}. *There is probably a large uncertainty associated with calculating $\Gamma(B^0\to\pi^-\nu)$ using a monopole ansatz for $f_+^{B\pi}(q^2)$ as the allowed range of q^2 is rather large.* However, the *local* constraint (57) does not suffer from the lack of knowledge of $f_+^{B\pi}(q^2)$. It might be a better test of the validity of the factorization assumption for the hadronic decay $B^0 \to \pi^+\pi^-$.

VI. CONCLUSION

In this paper we have tested the factorization assumption in charmed meson decays by comparing two-body hadronic decay rates for $D^0 \to K^-\rho^+, K^-\pi^+$, and $K^-a_1^+$ with the semileptonic rates for $D^0 \to K^-e^+\nu$, and that of $D^0 \to \pi^+\pi^-$ with $D^0 \to \pi^-e^+\nu$. The reason for selecting these decays was that they involved just one form factor $F_1(q^2)$ [in the case of $D^0 \to K^-\pi^+$ and $\pi^-\pi^+$ we have used $F_0(m_\pi^2) \simeq F_0(0) = F_1(0)$], which drops out in comparing the hadronic rates to the *differential* semileptonic rate for an appropriate channel at the relevant fixed value of q^2. In comparing the hadronic rates to the semileptonic ones an assumption has to be made about the q^2 dependence of $F_1(q^2)$ which we took to be a monopole.

In contrast with the decays we have studied, tests of factorization using decays such as $D^0 \to K^{*-}\pi^+$ and $K^{*-}\rho^+$ involve three form factors A_1, A_2, and V (in the notation of Ref. [13]). The dependence on V is usually very weak, leaving the rate to depend essentially on A_1 and A_2 [7]. Because of the appearance of more than one form factor in these decays the test of factorization is not "clean." For this reason we have confined our discussion only to decays involving a single form factor.

We have used a value of the Wilson coefficient a_1 somewhat larger than that in Ref. [7]. If FSI's are not used then a value $a_1 = 1.2$ with $F_0^{DK}(0)$ given by the BSW model [13] leads to $B(D^0 \to K^-\pi^+) \approx 5.7\%$, which is too large. However, inclusion of FSI's allows us to lower the theoretical estimate to within experimental bounds with $a_1 = 1.2$. All these statements are model dependent; they not only depend on the estimate of $F_0^{DK}(0)$ but also on the Wilson coefficient a_2 since the isospin $\frac{1}{2}$ and $\frac{3}{2}$ amplitudes involve both a_1 and a_2. We have judiciously allowed a 10% error in the values of a_1 and a_2.

In testing factorization we have found that the theoretical ratios

$$\Gamma(D^0 \to K^-\rho^+)/\Gamma(D^0 \to K^-e^+\nu) ,$$

$$\Gamma(D^0 \to K^-\pi^+)/\Gamma(D^0 \to K^-e^+\nu) ,$$

and

$$\Gamma(D^0 \to \pi^+\pi^-)/\Gamma(D^0 \to \pi^-e^+\nu)$$

are consistent with the factorization hypothesis once corrections due to FSI's are taken into account. Without FSI's are the "raw" ratio

$$\Gamma(D^0 \to \pi^+\pi^-)/\Gamma(D^0 \to \pi^-e^+\nu)$$

is a factor of 2 larger than the experimentally measured value, while the "raw" ratio

$$\Gamma(D^0 \to K^-\rho^+)/\Gamma(D^0 \to K^-e^+\nu)$$

is almost consistent with experiment indicating a small FSI phase which is also consistent with the amplitude analysis [6] of $D \to K\rho$ decays.

In contrast to the successes of the factorization model alluded to above, we find that this model predicts $B(D^0 \to K^-a_1^+)$ too small by a factor of 5. Perhaps factorization ought not to be expected to work well for decays involving small energy release such as $D^0 \to K^-a_1^+$. Thus aside from $D^0 \to K^-a_1^+$, we find no need to introduce annihilation terms in $D \to K\rho$ and $K\pi$ decays nor a need to introduce penguin contributions to $D \to \pi\pi$ decays.

We have also made predictions for $B(D_s^+ \to K^0e^+\nu)$ by relating it to $B(D_s^+ \to K^+\bar{K}^{*0})$ and $B(D_s^+ \to K^0\pi^+)$, and $B(D_s^+ \to (\eta,\eta')e^+\nu)$ by relating them to the measured rates $B(D_s^+ \to (\eta,\eta')\rho^+)$ since it is this process that is most directly related to $D_s^+ \to (\eta,\eta')e^+\nu$. We point out that Pham [16] has also estimated the ratio

$$B(D_s^+ \to (\eta,\eta')e^+\nu)/B(D_s^+ \to \phi\pi^+)$$

by relating it to

$$B(D^0 \to K^-e^+\nu)/B(D^0 \to K^{*-}\pi^+)$$

and finds a much smaller branching ratio for $D_s^+ \to (\eta,\eta')e^+\nu$ than we have obtained [see (50) and (54)]. The reason for this difference lies in the fact that we have directly related $D_s^+ \to (\eta,\eta')e^+\nu$ to $D_s^+ \to (\eta,\eta')\rho^+$ as both these processes involve the form factor $F_1(q^2)$. Further, experimentally $B(D_s^+ \to (\eta,\eta')\rho^+)$ are known to be quite large [15]. It will be interesting to see if eventually experiments will support a larger rate for $D_s^+ \to (\eta',\eta')e^+\nu$ as we predict, and vindicate the measurement of $B(D_s^+ \to (\eta,\eta')\rho^+)$.

Finally, we have proposed that when data become available one ought to test for validity of factorization in $B^0 \to \pi^+\pi^-$ decay by comparing it with the semileptonic process $B^0 \to \pi^-e^+\nu$. $B^0 \to \pi^+\pi^-$ (as also $B^0 \to \pi^-e^+\nu$) is a relevant decay mode for the determination of V_{ub}.

Note added in proof. E-653 has measured $D_s^+ \to (\eta,\eta')\mu^+\nu$. The total branching ratio is in agreement [17] with the predictions (50) and (54) of the factorization assumption.

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APPENDIX

Consider

$$I(M,m,\Lambda) \equiv \int_0^{q_m^2} \frac{\lambda^3(M^2,m^2,q^2)}{(q^2-\Lambda^2)^2} dq^2 , \qquad \text{(A1)}$$

with

$$\lambda(M^2,m^2,q^2) = (M^4+m^4+q^4-2M^2q^2 - 2m^2q^2-2M^2m^2)^{1/2}$$

and

$$q_m^2=(M-m)^2 \ . \qquad \text{(A2)}$$

By a change of variables, $x=\Lambda^2-q^2$, (A1) can be written as

$$I(M,m,\Lambda)\equiv\int_{\Lambda^2-q_m^2}^{\Lambda^2}\left[1+\frac{b}{x}+\frac{1}{x^2}\right]\sqrt{R}\,dx \ , \qquad \text{(A3)}$$

where

$$a=\lambda^2(M^2,m^2,\Lambda^2) \ ,$$

$$b=-2(\Lambda^2-M^2-m^2) \ ,$$

$$R=a+bx+x^2 \ . \qquad \text{(A4)}$$

The integrals involved in (A3) are standard and can be looked up in a table of integrals.

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