Inclusive semileptonic B and polarized Λ_b decays from QCD

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(Received 12 August 1993)

The differential decay spectrum $d\Gamma/dE_e dq^2$ for the semileptonic decay of an unpolarized hadron containing a b quark and the decay spectrum $d\Gamma/dE_e dq^2 d\cos\theta$ for polarized Λ_b decay are computed to second order in the $1/m_b$ expansion. Most of the $1/m_b^2$ corrections have a simple physical interpretation, which is discussed in detail. The implications of the results for the determination of V_{ub} are discussed. The decay spectra for the semileptonic decay of hadrons containing a c quark are also given. There is a subtlety in the use of the equations of motion at order $1/m_b^2$ in the heavy quark expansion which is explained.

PACS number(s): 13.20.He, 12.38.Bx, 13.20.Fc, 13.30.Ce

I. INTRODUCTION

The inclusive lepton spectrum from semileptonic B decays has undergone intensive experimental and theoretical study. An understanding of it provides information on the weak mixing angles V_{cb} and V_{ub}. Recently there has been considerable progress in understanding the theory of these decays. Chay, Georgi, and Grinstein [1] showed that inclusive semileptonic $B \to X e \bar{\nu}_e$ decay can be treated in a fashion similar to deep-inelastic scattering. Using a two-step process that consisted of first using an operator product expansion and then a transition to the heavy quark effective theory, they showed that $d\Gamma/dq^2 dE_e$ [$q^2 = (p_e + p_\nu)^2$] when suitable averaged over E_e is calculable. Their leading-order result agrees with the free b-quark decay picture. The full power of their method becomes apparent when corrections to the leading-order result are discussed. These corrections are of two types: perturbative $\alpha_s(m_b)$ corrections and nonperturbative corrections suppressed by powers of m_b. Chay, Georgi, and Grinstein pointed out that there are no nonperturbative corrections of order $\Lambda_{\rm QCD}/m_b$. In this paper, we will compute the $\Lambda_{\rm QCD}^2/m_b^2$ corrections. The order $\alpha_s(m_b)$ corrections have been computed previously [2–5]. The results of this paper can be combined with Refs. [2–5] to give the inclusive semileptonic decay including all corrections to order $\Lambda_{\rm QCD}^2/m_b^2$ and $\alpha_s(m_b)$.

At leading order in $\alpha_s(m_b)$ and $\Lambda_{\rm QCD}/m_b$ the differential $B \to X_u e \bar{\nu}_e$ semileptonic decay rate is

$$\frac{d\Gamma}{dq^2 dE_e} = \frac{|V_{ub}|^2 G_F^2 m_b^2}{8\pi^3} \left[\frac{2E_e}{m_b} - \frac{q^2}{m_b^2}\right]\left[1 + \frac{q^2}{m_b^2} - \frac{2E_e}{m_b}\right]$$
$$\times \theta(2E_e m_b - q^2) . \tag{1.1}$$

In the approach of Chay, Georgi, and Grinstein the b-quark mass appearing in Eq. (1.1) has a precise meaning that is provided by the heavy quark effective theory. This is necessary for the statement that there are no $\Lambda_{\rm QCD}/m_b$

corrections to have content. In the heavy quark effective theory the strong interactions of a bottom quark with four-velocity v are given by the Lagrange density [6]

$$\mathcal{L} = \bar{b}_v(iv \cdot D)b_v + \cdots . \tag{1.2}$$

In Eq. (1.2) the bottom quark field b_v satisfies the constraint $\rlap{/}{v}b_v = b_v$ and the ellipsis denote terms suppressed by powers of $1/m_b$. The relationship between the b-quark field in the effective theory and the "full QCD" b-quark field is

$$b_v = \frac{1 + \rlap{/}{v}}{2}e^{im_b v \cdot x}b + \cdots , \tag{1.3}$$

where the ellipsis again denote terms suppressed by powers of $1/m_b$. It is the same m_b that appears in Eq. (1.3) that is used in Eq. (1.1). With the heavy quark effective theory given by Eq. (1.2) (i.e., no mass term for b_v) m_b is a physical quantity that can, at least in principle, be determined experimentally. For example, the form factors for the exclusive decay $\Lambda_b \to \Lambda_c e \bar{\nu}_e$ depend on $\bar{\Lambda} = M_{\Lambda_b} - m_b$ [7] and the determination of $\bar{\Lambda}$ from a detailed study of this decay together with the measured Λ_b mass gives the b-quark mass to be used in Eq. (1.1).

Bigi, Shifman, Uraltsev, and Vainshtein [8] have performed an analysis of the $\Lambda_{\rm QCD}^2/m_b^2$ nonperturbative corrections to the lepton energy spectrum $d\Gamma/dE_e$ in semileptonic B-meson decay. Bigi *et al.* found that these corrections are determined by the two local matrix elements $\langle B(v)|\bar{b}_v(iD)^2 b_v|B(v)\rangle/2m_b^2$ and $\langle B(v)|g\bar{b}_v\sigma^{\mu\nu}G_{\mu\nu}b_v|B(v)\rangle/4m_b^2$. The latter is fixed by the measured value of the $B^* - B$ mass difference.

In this paper we extend the results of Bigi *et al.* and Chay, Georgi, and Grinstein and compute the $\Lambda_{\rm QCD}^2/m_b^2$ corrections to the fully differential decay distribution $d\Gamma/dq^2 dE_e$ for an unpolarized hadron H_b containing a b quark. We also consider inclusive polarized semileptonic decay for the special case of the Λ_b. Λ_b's produced in Z^0 decays are expected to be polarized [9]. The differential

decay distribution for a polarized Λ_b has the form

$$\frac{d\Gamma}{dq^2 dE_e d\cos\theta} = A(E_e, q^2) + B(E_e, q^2)\cos\theta , \quad (1.4)$$

where θ is the angle between the electron direction and the Λ_b spin in the rest frame of the Λ_b. For reasons similar to those given by Chay, Georgi, and Grinstein in the case of B-meson decay, there are no Λ_{QCD}/m_b nonperturbative corrections to the differential decay distribution in Eq. (1.4). The Λ_{QCD}^2/m_b^2 corrections to $A(E_e, q^2)$ are similar to those computed for $d\Gamma/dq^2 dE_e$ in B-meson decay. One simply replaces B-meson matrix elements by Λ_b matrix elements and sets the Λ_b matrix element of $\bar{b}_v g \sigma^{\mu\nu} G_{\mu\nu} b_v$ to zero. However, we find that the Λ_{QCD}^2/m_b^2 corrections to $B(E_e, q^2)$ are not characterized by just $\langle \Lambda_b(v,s)|\bar{b}_v(iD)^2 b_v|\Lambda_b(v,s)\rangle/2m_b^2$. The normalization of $B(E_e, q^2)$ involves another order Λ_{QCD}^2/m_b^2 correction which arises because the heavy quark spin is renormalized at order $1/m_b^2$.

We examine the physical interpretation of the Λ_{QCD}^2/m_b^2 corrections. Most of the corrections [i.e., all of those involving the matrix element of $\bar{b}_v(iD)^2 b_v$ and some of those involving the matrix element of $\bar{b}_v g \sigma^{\mu\nu} G_{\mu\nu} b_v$] can be interpreted as arising from the fact that in the bound state the b quark has an effective mass that differs from m_b and an effective four-velocity that differs from v^μ. These differences arise, for example, from the motion of the b quark in the hadron rest frame. Corrections of this type are similar in spirit to those included in models for inclusive semileptonic B-meson decay [2-5].

Section II contains a discussion of the kinematics relevant for semileptonic H_b and polarized Λ_b decay. The operator product expansion and the transition to the heavy quark effective theory are discussed in Sec. III. This section contains a lengthy discussion of the computation. Readers not interested in the details are advised to skip this section entirely. Section IV contains a brief discussion on the use of equations of motion in time-ordered products. It clears up some confusion on this subject that occurred in the previous literature. Section V gives the differential decay rates for unpolarized H_b semileptonic decay and polarized Λ_b decay. Section VI is concerned with the physical interpretation of the Λ_{QCD}^2/m_b^2 corrections. Section VII gives differential decay rates for unpolarized hadrons H_c containing a c quark and polarized Λ_c semileptonic decay. Section VIII discusses the prediction for $d\Gamma(B \to X_u e \bar{\nu}_e)/dq^2 dE_e$ near the boundary of the Dalitz plot. This region is important for the determination of V_{ub}. Particular attention is paid to the size of the region of E_e that must be averaged over before experimental results can be compared with theory. Numerical estimates and plots of the lepton spectrum are given in Sec. IX, and concluding remarks are given in Sec. X.

II. KINEMATICS

The semileptonic decay of a b quark is due to the weak Hamiltonian density

$$H_W = -V_{jb}\frac{4G_F}{\sqrt{2}}\bar{q}_j \gamma^\mu P_L b \; \bar{e}\gamma_\mu P_L \nu_e$$

$$= -V_{jb}\frac{4G_F}{\sqrt{2}} J_j^\mu J_{l\mu} , \quad (2.1)$$

where P_L is the left-handed projection operator $\frac{1}{2}(1-\gamma_5)$. J_j^μ and J_l^μ are the hadronic and leptonic currents, respectively. The final quark q_j can be either a u or a c quark. The inclusive differential decay rate for a hadron H_b containing a b quark to decay semileptonically, $H_b \to X_{u,c} e \bar{\nu}_e$ is determined by the hadronic tensor

$$W_j^{\mu\nu} = (2\pi)^3 \sum_X \delta^4(p_{H_b} - q - p_X)$$

$$\times \langle H_b(v,s)|J_j^{\mu\dagger}|X\rangle\langle X|J_j^\nu|H_b(v,s)\rangle , \quad (2.2)$$

where $j = u, c$. The spin-J hadron state $|H_b(v,s)\rangle$ is normalized to v^0, instead of to the usual relativistic normalization of $2M_{H_b} v^0$ as this is more convenient for the heavy quark expansion. $W^{\mu\nu}$ can be expanded in terms of five form factors if one spin averages over the initial state:

$$W^{\mu\nu} = -g^{\mu\nu}W_1 + v^\mu v^\nu W_2 - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta W_3$$

$$+ q^\mu q^\nu W_4 + (q^\mu v^\nu + q^\nu v^\mu)W_5 , \quad (2.3)$$

where v is the velocity of the initial hadron, defined by

$$p_{H_b} = M_{H_b} v^\mu . \quad (2.4)$$

W_1 and W_2 have mass dimension -1, W_3 and W_5 have mass dimension -2, and W_4 has dimension -3. (The form factor W_6 of Ref. [1] vanishes by time-reversal invariance.) The form factors are functions of the invariants q^2 and $q \cdot v$, and will also depend on the initial hadron H_b and the final quark mass m_j. The difference between the heavy quark mass m_b and the hadron mass M_{H_b} will be important in our analysis, so we have chosen to write the form factors in terms of q rather than the rescaled $\hat{q} = q/m_b$ used in Ref. [1].[1] The spin-averaged differential semileptonic decay rate is

$$\frac{d\Gamma}{dq^2 dE_e dE_\nu} = \frac{|V_{jb}|^2 G_F^2}{2\pi^3}[W_1 q^2 + W_2(2E_e E_\nu - \tfrac{1}{2}q^2)$$

$$+ W_3 q^2(E_e - E_\nu)] , \quad (2.5)$$

where E_e and E_ν are the electron and neutrino energies in the H_b rest frame, q^2 is the invariant mass of the lepton pair, and the kinematic variables are to be integrated over the region $q^2 \leq 4E_e E_\nu$. The terms proportional to q^μ or q^ν in Eq. (2.3) do not contribute to the decay rate if one neglects the electron mass.

The polarized Λ_b has in addition to the five form factors in Eq. (2.3), nine spin-dependent form factors which are

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$$W_S^{\mu\nu} = -q \cdot s[\,-g^{\mu\nu}G_1 + v^\mu v^\nu G_2 - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta G_3$$

$$+ q^\mu q^\nu G_4 + (q^\mu v^\nu + q^\nu v^\mu)G_5\,]$$

$$+ (s^\mu v^\nu + s^\nu v^\mu)G_6 + (s^\mu q^\nu + s^\nu q^\mu)G_7$$

$$+ i\epsilon^{\mu\nu\alpha\beta}v_\alpha s_\beta G_8 + i\epsilon^{\mu\nu\alpha\beta}q_\alpha s_\beta G_9 \ . \tag{2.6}$$

The identity

$$g^{\mu\nu}\epsilon^{\alpha\beta\lambda\sigma} - g^{\mu\alpha}\epsilon^{\nu\beta\lambda\sigma} + g^{\mu\beta}\epsilon^{\alpha\nu\lambda\sigma} - g^{\mu\lambda}\epsilon^{\nu\alpha\beta\sigma} + g^{\mu\sigma}\epsilon^{\nu\alpha\beta\lambda} = 0$$

has been used to eliminate terms in $W_S^{\mu\nu}$ of the form $i[q^\mu\epsilon^{\nu\alpha\beta\lambda}v_\alpha q_\beta s_\lambda - (\mu \rightarrow \nu)]$ and $i[v^\mu\epsilon^{\nu\alpha\beta\lambda}v_\alpha q_\beta s_\lambda - (\mu \rightarrow \nu)]$. The form factors G_4, G_5, G_7 do not contribute to the decay rate if the lepton mass is neglected. The differential decay rate is

$$\frac{d\Gamma}{dq^2 dE_e dE_\nu d\cos\theta} = \cdots + \frac{|V_{jb}|^2 G_F^2}{4\pi^3}\cos\theta\{\ [G_1 q^2 + G_2[2E_e E_\nu - \tfrac{1}{2}q^2] + G_3 q^2|(E_e - E_\nu)|\,](E_e + E_\nu - q^2/2E_e)$$

$$+ G_6(q^2 - 4E_e E_\nu) - G_8 q^2 - G_9 q^2 (E_e - E_\nu + q^2/2E_e)\} \ , \tag{2.7}$$

where θ is the angle between the electron three-momentum and the Λ_b spin vector in the Λ_b's rest frame. The ellipsis in Eq. (2.7) denotes the part independent of $\cos\theta$ and is one-half the expression in Eq. (2.5) so that integration over $\cos\theta$ reproduces the unpolarized decay rate.

The form factors in $W^{\mu\nu}$ and $W_S^{\mu\nu}$ are given by the discontinuities across a cut of the amplitudes $T^{\mu\nu}$ and $T_S^{\mu\nu}$:

$$T^{\mu\nu} = -i\int d^4x\, e^{-iq\cdot x}\frac{1}{2J+1}\sum_s \langle H_b(v,s)|T[J^{\mu\dagger}(x)J^\nu(0)]|H_b(v,s)\rangle$$

$$= -g^{\mu\nu}T_1 + v^\mu v^\nu T_2 - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta T_3 + q^\mu q^\nu T_4 + (q^\mu v^\nu + q^\nu v^\mu)T_5 \ . \tag{2.8}$$

It is easy to see that $\mathrm{Im}\,T^{\mu\nu} = -\pi W^{\mu\nu}$ by inserting a complete set of states between the currents. $T_S^{\mu\nu}$ for the polarized Λ_b is defined similarly and has nine additional spin-dependent form factors:

$$T_S^{\mu\nu} = -q \cdot s[\,-g^{\mu\nu}S_1 + v^\mu v^\nu S_2 - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta S_3$$

$$+ q^\mu q^\nu S_4 + (q^\mu v^\nu + q^\nu v^\mu)S_5\,]$$

$$+ (s^\mu v^\nu + s^\nu v^\mu)S_6 + (s^\mu q^\nu + s^\nu q^\mu)S_7$$

$$+ i\epsilon^{\mu\nu\alpha\beta}v_\alpha s_\beta S_8 + i\epsilon^{\mu\nu\alpha\beta}q_\alpha s_\beta S_9 \ . \tag{2.9}$$

The analytic structure of $T^{\mu\nu}$ (or $T_S^{\mu\nu}$) as a function of $q \cdot v$ for fixed q^2 was given in Ref. [1] and is shown in Fig. 1. There is a cut from

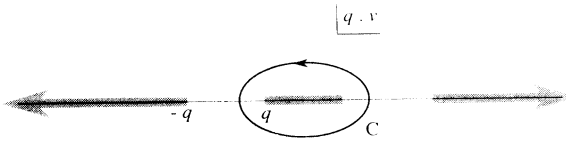$$\sqrt{q^2} \le q \cdot v \le \left[\frac{1}{2M_{H_b}}\right](M_{H_b}^2 + q^2 - M_j^2) \ ,$$

FIG. 1. The analytic structure of $T^{\mu\nu}$ in the complex $q \cdot v$ plane for fixed timelike q^2. The contour C encircles the cut relevant for semileptonic b decay. The other cuts extend to infinity, and correspond to other physical processes. Here $q \equiv \sqrt{q^2}$.

where M_j is the mass of the lightest hadron containing the final-state quark q_j. The discontinuity across this cut (which will be called the physical cut) gives $W^{\mu\nu}$ for semileptonic H_b decay. In addition there are cuts along the real axis for

$$M_{H_b}q \cdot v \ge \tfrac{1}{2}[(2M_{H_b} + M_j)^2 - q^2 - M_{H_b}^2\,]$$

corresponding to the physical process $e\bar{\nu}_e H_b \rightarrow X$ (where X contains two b quarks), and for $v \cdot q \le -\sqrt{q^2}$ corresponding to the physical process $e^+\nu_e H_b \rightarrow X$. There is no cut for $-\sqrt{q^2} \le q \cdot v \le \sqrt{q^2}$, since $(q^0)^2 > q^2$ for any physical state, so that, in general, the physical cut is separated from the other two cuts along the real axis. (This disagrees with Ref. [1].) For $M_j = 0$ and $q^2 = M_{H_b}^2$ the two cuts on the positive real axis are not separated. The end of the physical cut at $v \cdot q = M_{H_b}$ coincides with the beginning of the second cut. Similarly for $q^2 = 0$ the cut on the negative real axis ends at the same point the physical cut begins.

The amplitude $T^{\mu\nu}$ can be computed in QCD perturbation theory away from the cuts along the real axis. Provided q^2 is not too near zero or $M_{H_b}^2$ (for $q_j = u$), the value of the amplitude in the physical region can be obtained by performing a contour integration along the closed contour C shown in Fig. 1 that stays away from the cuts in the complex $q \cdot v$ plane.

III. THE OPERATOR PRODUCT EXPANSION

The amplitude $T^{\mu\nu}$ can be computed reliably in perturbative QCD in a region that is far from the cuts and therefore free of infrared singularities. The time-ordered product

$$-i \int d^4 e^{-iq\cdot x} T(J^{\mu\dagger} J^\nu) \qquad (3.1)$$

can be computed using an operator product expansion in terms of operators involving b-quark fields in the heavy quark effective field theory. The coefficients of the operators in the operator product expansion are determined by evaluating the matrix element of the time-ordered product between quark and gluon states. Once the operator product expansion has been computed, one can compute $T^{\mu\nu}$ by taking the matrix element of the operator product expansion between hadron states. We will compute the matrix element between unpolarized hadron states to determine the form factors T_i, and we will take the matrix element between polarized Λ_b states to determine the spin-dependent form factors S_i. The operator product expansion can be written as an expansion in inverse powers of m_b. The expansion of $T^{\mu\nu}$ to order $1/m_b^2$ can be written in terms of gauge-invariant operators of dimension less than or equal to five. The operators will involve the b field, covariant derivatives D, and the gluon field strength tensor $G^{\mu\nu}$. The coefficients of the operators involving b and D are determined by taking quark matrix elements of Eq. (3.1), and the coefficient of operators involving $G^{\mu\nu}$ are determined by taking gluon matrix elements of Eq. (3.1). We only calculate the form factors T_{1-3}, S_{1-3}, S_6, S_8, and S_9 which are the only ones that contribute to semileptonic b decay when the mass of the lepton in the final state is neglected.

The quark matrix element of Eq. (3.1) between b-quark states with momentum $m_b v + k$ is

$$\frac{1}{(m_b v - q + k)^2 - m_j^2 + i\epsilon}$$
$$\times \bar{u} \gamma^\mu P_L (m_b \not{v} - \not{q} + \not{k} + m_j) \gamma^\nu P_L u , \qquad (3.2)$$

from Fig. 2(a), where u is the quark spinor. The crossed diagram of Fig. 2(b) has no singularities inside the contour C of Fig. 1, and does not contribute to semileptonic b decay. The matrix element Eq. (3.2) can be simplified using the identity

$$\gamma^\mu \gamma^\alpha \gamma^\nu = g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i\epsilon^{\mu\nu\alpha\beta} \gamma_\beta \gamma_5 .$$

The $1/m_b$ expansion is given by expanding Eq. (3.2) in a power series. The momentum q can be of order m_b, but k is only of order Λ_{QCD}. Thus, each factor of k in the expansion of Eq. (3.2) corresponds to a $1/m_b$ suppression. The factors of k in the matrix element will become factors of iD in the operator product expansion, so each factor of iD corresponds to a $1/m_b$ suppression.
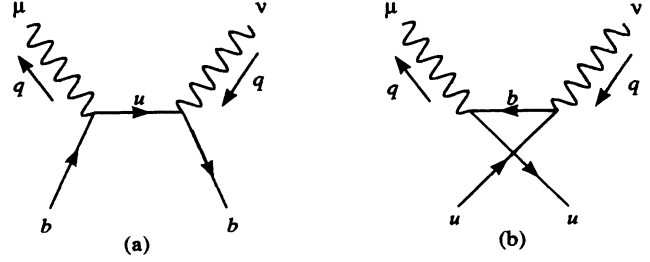

FIG. 2. The leading term in the operator product expansion. (b) does not contribute to semileptonic b decay.

A. The k^0 terms

The order k^0 term in the expansion of Eq. (3.2) is

$$\frac{1}{\Delta_0} \bar{u} \{ (m_b v - q)^\mu \gamma^\nu + (m_b v - q)^\nu \gamma^\mu$$
$$- (m_b \not{v} - \not{q}) g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta} (m_b v - q)_\alpha \gamma_\beta \} P_L u , \qquad (3.3)$$

where

$$\Delta_0 = (m_b v - q)^2 - m_j^2 + i\epsilon . \qquad (3.4)$$

The matrix elements of the operators $\bar{b}\gamma^\lambda b$ and $\bar{b}\gamma^\lambda\gamma_5 b$ between b-quark states are $\bar{u}\gamma^\lambda u$ and $\bar{u}\gamma^\lambda\gamma_5 u$, respectively, so the operator product expansion is obtained by replacing $\bar{u}$ and u in Eq. (3.3) by the fields $\bar{b}$ and b, respectively:

$$\frac{1}{\Delta_0} \{ (m_b v - q)^\mu g^{\nu\lambda} + (m_b v - q)^\nu g^{\mu\lambda}$$
$$- (m_b v - q)^\lambda g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\lambda} (m_b v - q)_\alpha \} \bar{b}\gamma_\lambda P_L b . \qquad (3.5)$$

The spin-averaged matrix element

$$\sum_s \langle H_b(v,s) | \bar{b}\gamma^\lambda\gamma_5 b | H_b(v,s) \rangle$$

between hadrons H_b of velocity v is zero. The matrix element

$$\langle H_b(v,s) | \bar{b}\gamma^\lambda b | H_b(v,s) \rangle = v^\lambda ,$$

to all orders in $1/m_b$, since the b-quark number is an exact symmetry of QCD. [This relation is true because we have normalized our hadron states to v^0, and defined v in Eq. (2.4) to be the velocity of the hadron H_b.] Taking the matrix element of the operator product expansion between spin-averaged H_b states, and comparing with Eq. (2.8) gives the contribution to $T^{\mu\nu}$ of k^0 terms in the expansion of Eq. (3.2):

$$T_1^{(0)} = \frac{1}{2\Delta_0}(m_b - q\cdot v) ,$$

$$T_2^{(0)} = \frac{1}{\Delta_0}m_b , \qquad (3.6)$$

$$T_3^{(0)} = \frac{1}{2\Delta_0} .$$

The k^0 terms in the operator product expansion give b-quark operators that have zero derivatives. In principle,

these operators could produce contributions to $T^{\mu\nu}$ of higher order in $1/m_b$ because of $1/m_b$ corrections to the matrix element of the operator between hadron states. However, the only matrix element we need is the zero-momentum transfer matrix element of a conserved current, which has no $1/m_b$ corrections.

The matrix element of the operator $\bar{b}\gamma^\lambda\gamma_5 b$ between polarized Λ_b states does not vanish. At leading order (in $1/m_b$) it is equal to the spin vector s^λ. However, the axial-vector current is a generator of heavy quark symmetry [10] that is broken by $1/m_b$ terms in the effective Lagrangian. Thus, the matrix element of the axial-vector current between polarized Λ_b states is not equal to the spin four-vector. The first correction to the matrix element is of second order in symmetry breaking [11], so we write

$$\langle \Lambda_b(v,s)|\bar{b}\gamma^\lambda\gamma_5 b|\Lambda_b(v,s)\rangle = (1+\epsilon_b)\bar{u}(v,s)\gamma^\lambda\gamma_5 u(v,s)$$

$$= (1+\epsilon_b)s^\lambda , \qquad (3.7)$$

where ϵ_b is a parameter which is of order Λ_{QCD}^2/m_b^2, and is defined by Eq. (3.7). Substituting this into Eq. (3.5) gives the spin-dependent form factors from the k^0 terms:

$$S_1^{(0)} = -\tfrac{1}{2}(1+\epsilon_b)\frac{1}{\Delta_0} , \quad S_8^{(0)} = \tfrac{1}{2}(1+\epsilon_b)\frac{m_b}{\Delta_0} ,$$

$$S_6^{(0)} = -\tfrac{1}{2}(1+\epsilon_b)\frac{m_b}{\Delta_0} , \quad S_9^{(0)} = -\tfrac{1}{2}(1+\epsilon_b)\frac{1}{\Delta_0} , \qquad (3.8)$$

$$S_2^{(0)} = S_3^{(0)} = 0 .$$

B. The order k^1 terms: Spin-averaged case

The linear terms in k in Eq. (3.2),

$$\frac{1}{\Delta_0}\bar{u}\{k^\mu\gamma^\nu + k^\nu\gamma^\mu - g^{\mu\nu}\slashed{k} - i\epsilon^{\mu\nu\alpha\beta}k_\alpha\gamma_\beta\}P_L u - \frac{2k\cdot(m_b v - q)}{\Delta_0^2}\bar{u}\{(m_b v - q)^\mu\gamma^\nu + (m_b v - q)^\mu\gamma^\nu - (m_b\slashed{v} - \slashed{q})g^{\mu\nu}$$

$$- i\epsilon^{\mu\nu\alpha\beta}(m_b v - q)_\alpha\gamma_\beta\}P_L u , \qquad (3.9)$$

produce operators in the operator product expansion with one derivative, of the form $\bar{b}\gamma^\lambda iD^\tau b$ and $\bar{b}\gamma^\lambda\gamma_5 iD^\tau b$. The matrix elements of these operators need to be computed to first order in $1/m_b$, since they contribute to terms that are already suppressed by $1/m_b$. Unlike the current operator $\bar{b}\gamma^\lambda b$ in the k^0 terms, these operators will have $1/m_b$ corrections to their matrix elements, and so can contribute terms to $T^{\mu\nu}$ at order $1/m_b$ and $1/m_b^2$. The matrix element of $\bar{b}\gamma^\lambda\gamma_5 iD^\tau b$ vanishes between spin-averaged H_b states. Its contribution to the polarized Λ_b decay amplitude is discussed in the next subsection. The matrix element of $\bar{b}\gamma^\lambda iD^\tau b$ can be computed in a $1/m_b$ expansion using the heavy quark effective theory.

The b quark is represented by the velocity-dependent b-quark field b_v in the heavy quark effective theory. Since the terms linear in k are already of order $1/m_b$, we only need the relation between $b(x)$ and $b_v(x)$ to first order in $1/m_b$ [12],

$$b(x) = e^{-im_b v\cdot x}\left[1 + \frac{iD\!\!\!\!/}{2m_b}\right]b_v(x) . \qquad (3.10)$$

The QCD Lagrangian for the b quark in the heavy quark effective theory is

$$\mathcal{L} = \bar{b}_v iv\cdot D b_v + \bar{b}_v \frac{(iD)^2}{2m_b}b_v$$

$$- Z_b\bar{b}_v \frac{gG_{\alpha\beta}\sigma^{\alpha\beta}}{4m_b}b_v + O\left[\frac{1}{m_b^2}\right] , \qquad (3.11)$$

where Z_b is a renormalization factor, with $Z_b(\mu = m_b) = 1$. The operator $\bar{b}_v(iD)^2 b_v/2m_b$ is not renormalized because of reparametrization invariance [13].

Equation (3.10) gives the expansion of the operator in the effective theory:

$$\bar{b}\gamma^\lambda iD^\tau b = \bar{b}_v\gamma^\lambda iD^\tau b_v + \bar{b}_v\frac{-iD\!\!\!\!/}{2m_b}\gamma^\lambda iD^\tau b_v$$

$$+ \bar{b}_v\gamma^\lambda iD^\tau\frac{iD\!\!\!\!/}{2m_b}b_v$$

$$= v^\lambda\bar{b}_v iD^\tau b_v + \frac{1}{m_b}\bar{b}_v iD^{(\lambda}iD^{\tau)}b_v$$

$$- \frac{1}{2m_b}\bar{b}_v gG^{\alpha\tau}\sigma_\alpha{}^\lambda b_v , \qquad (3.12)$$

where we have used the relation $\bar{b}_v\gamma^\lambda b_v = v^\lambda\bar{b}_v b_v$ (which follows from the constraint $\slashed{v}b_v = b_v$) and the commutator $[D^\mu, D^\nu] = igG^{\mu\nu}$. In Eq. (3.12) the parentheses around indices denote that they are symmetrized.

The spin-averaged matrix element of the first term of Eq. (3.12) must have the form

$$\frac{1}{2J+1}\sum_s\langle H_b(v,s)|\bar{b}_v iD^\tau b_v|H_b(v,s)\rangle = Av^\tau , \qquad (3.13)$$

where A is a constant to be determined. Contracting both sides of Eq. (3.13) with v^τ gives

$$A = \langle H_b(v,s)|\bar{b}_v iD\cdot v b_v|H_b(v,s)\rangle . \qquad (3.14)$$

The coefficient A is zero at lowest order in $1/m_b$, since $(v \cdot D)b_v = 0$ is the lowest-order equation of motion in the heavy quark effective theory. Thus, the order k terms make no contribution to $T^{\mu\nu}$ at order $1/m_b$. The order

k^0 terms also did not contribute to $T^{\mu\nu}$ at order $1/m_b$, so there is no $1/m_b$ correction to $T^{\mu\nu}$. This reproduces the result of Ref. [1]. The parameter A is nonzero at first order in $1/m_b$,

$$A = -\left\langle H_b(v,s) \left| \left[\bar{b}_v \frac{(iD)^2}{2m_b} b_v - Z_b \bar{b}_v \frac{gG_{\alpha\beta}\sigma^{\alpha\beta}}{4m_b} b_v \right] \right| H_b(v,s) \right\rangle , \tag{3.15}$$

using the equations of motion from the Lagrangian, Eq. (3.11), to order $1/m_b$. This $1/m_b$ value of A contributes to $T^{\mu\nu}$ at order $1/m_b^2$.

It is useful to define the dimensionless parameters

$$E_b \equiv -\left\langle H_b(v,s) \left| \left[\bar{b}_v \frac{(iD)^2}{2m_b^2} b_v - Z_b \bar{b}_v \frac{gG_{\alpha\beta}\sigma^{\alpha\beta}}{4m_b^2} b_v \right] \right| H_b(v,s) \right\rangle ,$$

$$K_b \equiv -\left\langle H_b(v,s) \left| \bar{b}_v \frac{(iD)^2}{2m_b^2} b_v \right| H_b(v,s) \right\rangle , \tag{3.16}$$

$$G_b \equiv Z_b \left\langle H_b(v,s) \left| \bar{b}_v \frac{gG_{\alpha\beta}\sigma^{\alpha\beta}}{4m_b^2} b_v \right| H_b(v,s) \right\rangle ,$$

to characterize the $1/m_b^2$ corrections, with $E_b = K_b + G_b$. G_b, K_b, and E_b can be thought of as the average value of the spin energy, the kinetic energy, and the total energy of the b quark in the hadron H_b, in units of m_b. All three parameters are expected to be order Λ_{QCD}^2/m_b^2. The operators in Eq. (3.16) are renormalization point independent, since the μ dependence of Z_b cancels the μ dependence of $\bar{b}_v gG_{\alpha\beta}\sigma^{\alpha\beta}b_v$. The anomalous dimensions of the operators only affect the relations between the parameters for different heavy quarks. For example, the parameters for b and c quarks are related by

$$m_c^2 K_c = m_b^2 K_b ,$$
$$m_c^2 G_c / Z_c = m_b^2 G_b / Z_b . \tag{3.17}$$

Since $Z_c(m_c) = 1$ and $Z_b(m_b) = 1$, the ratio Z_b/Z_c is given by the scaling of $\bar{b}_v gG_{\alpha\beta}\sigma^{\alpha\beta}b_v$ between the scales m_c and m_b, $Z_b/Z_c = [\alpha_s(m_b)/\alpha_s(m_c)]^{9/25}$ [12]. The gluon operator $G^{\mu\nu}$ in the operator product expansion occurs through the equations of motion, and through the commutator $[D^\mu, D^\nu] = igG^{\mu\nu}$. We will use E_b to parametrize the matrix elements of the gluon operators obtained using the equations of motion, and G_b to parametrize the matrix elements of the gluon operators obtained from $[D^\mu, D^\nu]$. This distinction will be useful in Sec. VI. With this convention,

$$A = m_b E_b , \tag{3.18}$$

and the matrix element of the first term of Eq. (3.13) can now be written as

$$\frac{1}{2J+1} \sum_s \langle H_b(v,s) | v^\lambda \bar{b}_v iD^\tau b_v | H_b(v,s) \rangle = m_b E_b v^\lambda v^\tau . \tag{3.19}$$

The matrix element of the second term in Eq. (3.12) must have the form

$$\frac{1}{2J+1} \sum_s \langle H_b(v,s) | \bar{b}_v iD^{(\lambda} iD^{\tau)} b_v | H_b(v,s) \rangle$$
$$= (B_1 g^{\lambda\tau} + B_2 v^\lambda v^\tau) . \tag{3.20}$$

This term has two covariant derivatives, so we only need its matrix element to lowest order in $1/m_b$. The lowest-order equation of motion $(v \cdot D)b_v = 0$ implies that $B_1 + B_2 = 0$. Taking the trace and comparing with Eq. (3.16) gives

$$\frac{1}{2J+1} \sum_s \langle H_b(v,s) | \bar{b}_v iD^{(\lambda} iD^{\tau)} b_v | H_b(v,s) \rangle$$
$$= -\frac{2m_b^2 K_b}{3} (g^{\lambda\tau} - v^\lambda v^\tau) . \tag{3.21}$$

The operator $\bar{b}_v gG_{\alpha\tau}\sigma^{\alpha\lambda} b_v$ in Eq. (3.12) is renormalized at a scale $\mu = m_b$, since that is the scale at which the operator product expansion has been performed. We can therefore multiply the operator by the renormalization factor Z_b, since $Z_b(m_b) = 1$. This makes the operator renormalization group invariant, and includes the QCD scaling of the operator between m_b and μ. The matrix element of the third term in Eq. (3.12) must have the form

$$\frac{1}{2J+1} \sum_s Z_b \langle H_b(v,s) | \bar{b}_v gG^{\alpha\tau}\sigma_\alpha{}^\lambda b_v | H_b(v,s) \rangle$$
$$= (C_1 g^{\lambda\tau} + C_2 v^\lambda v^\tau) . \tag{3.22}$$

Contracting both sides with v^τ, and using $\bar{b}_v \sigma^{\alpha\tau} v_\tau b_v = 0$ gives $C_1 + C_2 = 0$. The trace gives

$$\frac{1}{2J+1}\sum_s Z_b \langle H_b(v,s)|\bar{b}_v gG^{\alpha\tau}\sigma_\alpha{}^\lambda b_v|H_b(v,s)\rangle$$

$$=\frac{4m_b^2 G_b}{3}(g^{\lambda\tau}-v^\lambda v^\tau), \quad (3.23)$$

on comparing with Eq. (3.16).

Substituting Eqs. (3.19), (3.21), and (3.23) into Eq. (3.12), and substituting the result into Eq. (3.9) gives

$$T_1^{(1)}=m_b E_b\left[\frac{1}{2\Delta_0}-\frac{(m_b-q\cdot v)^2}{\Delta_0^2}\right]$$
$$+\frac{2m_b}{3}(K_b+G_b)\left[-\frac{1}{2\Delta_0}+\frac{q^2-(q\cdot v)^2}{\Delta_0^2}\right],$$

$$T_2^{(1)}=m_b E_b\left[\frac{1}{\Delta_0}-2\frac{m_b(m_b-q\cdot v)}{\Delta_0^2}\right]$$
$$+\frac{2m_b}{3}(K_b+G_b)\left[\frac{1}{\Delta_0}+\frac{2m_b q\cdot v}{\Delta_0^2}\right], \quad (3.24)$$

$$T_3^{(1)}=-m_b E_b\left[\frac{m_b-q\cdot v}{\Delta_0^2}\right]$$
$$-\frac{2m_b}{3}(K_b+G_b)\left[\frac{m_b-q\cdot v}{\Delta_0^2}\right].$$

C. The order k^1 terms: Polarized Λ_b case

The spin-dependent form factors arising from the order k terms in the operator product expansion of Eq. (3.9) are obtained by taking the matrix element of the operator $\bar{b}\gamma^\lambda\gamma_5 iD^\tau b$ between polarized Λ_b states. This operator did not contribute to the spin-averaged matrix element between unpolarized H_b states discussed in the previous section. The method used to evaluate the matrix element is similar to that used for the operator $\bar{b}\gamma^\lambda iD^\tau b$ in the previous subsection. The operator can be written in terms of the field b_v of the effective theory:

$$\bar{b}\gamma^\lambda\gamma_5 iD^\tau b=\bar{b}_v\gamma^\lambda\gamma_5 iD^\tau b_v+\frac{1}{2m_b}\bar{b}_v i\overleftarrow{D}\!\!\!\!/\,\gamma^\lambda\gamma_5 iD^\tau b_v$$
$$+\frac{1}{2m_b}\bar{b}_v\gamma^\lambda\gamma_5 iD^\tau iD\!\!\!\!/ b_v. \quad (3.25)$$

The matrix element of the first term of Eq. (3.25) between polarized Λ_b states has to have the form

$$\langle\Lambda_b(v,s)|\bar{b}_v\gamma^\lambda\gamma_5 iD^\tau b_v|\Lambda_b(v,s)\rangle=A\bar{u}\gamma^\lambda\gamma_5 u\, v^\tau \quad (3.26)$$

by heavy quark spin symmetry. Contracting both sides with v^τ and using the equations of motion determines $A=m_b E_b$, where E_b is defined in Eq. (3.16). The matrix

element of the second and third terms in Eq. (3.25) can be simplified by neglecting terms proportional to $G^{\alpha\beta}$. The operator $G^{\alpha\beta}$ vanishes in any matrix element between Λ_b states at zero recoil, since the light degrees of freedom in the Λ_b have spin zero. [That is why spin symmetry can be used in Eq. (3.26) even though we are including effects of order $1/m_b$.] The only vector that can be constructed using the light degrees of freedom is v^μ, and it is not possible to construct a tensor that is antisymmetric in two indices from a single vector.

The $1/m_b$ terms in Eq. (3.25) can be simplified using γ-matrix algebra and neglecting $G^{\alpha\beta}$ to give

$$\frac{1}{m_b}\bar{b}_v iD^{(\alpha}iD^{\tau)}(v_\alpha\gamma^\lambda-v^\lambda\gamma_\alpha)\gamma_5 b_v, \quad (3.27)$$

which is equal to

$$-\frac{1}{m_b}\bar{b}_v iD^{(\alpha}iD^{\tau)}v^\lambda\gamma_\alpha\gamma_5 b_v, \quad (3.28)$$

using the equations of motion. The matrix element of Eq. (3.28) between polarized Λ_b states is

$$-\frac{1}{m_b}\langle\Lambda_b(v,s)|\bar{b}_v iD^{(\alpha}iD^{\tau)}v^\lambda\gamma_\alpha\gamma_5 b_v|\Lambda_b(v,s)\rangle$$

$$=\tfrac{2}{3}m_b K_b v^\lambda\bar{u}\gamma^\tau\gamma_5 u, \quad (3.29)$$

using heavy quark spin symmetry, and the matrix element Eq. (3.21). Equations (3.26), (3.27), and (3.29) imply that one can make the substitution

$$\bar{b}\gamma^\lambda\gamma_5 iD^\tau b\to m_b E_b s^\lambda v^\tau+\tfrac{2}{3}m_b K_b v^\lambda s^\tau, \quad (3.30)$$

in Eq. (3.9) to obtain the spin-dependent form factors

$$S_1^{(1)}=\frac{1}{\Delta_0^2}(m_b-q\cdot v)(m_b E_b+\tfrac{2}{3}m_b K_b),$$

$$S_2^{(1)}=\frac{4}{3\Delta_0^2}m_b^2 K_b,$$

$$S_3^{(1)}=\frac{2}{3\Delta_0^2}m_b K_b,$$

$$\quad (3.31)$$

$$S_6^{(1)}=-\frac{1}{2\Delta_0}(m_b E_b+\tfrac{2}{3}m_b K_b)+\frac{1}{\Delta_0^2}(m_b-q\cdot v)m_b^2 E_b,$$

$$S_8^{(1)}=\frac{1}{2\Delta_0}(m_b E_b-\tfrac{2}{3}m_b K_b)-\frac{1}{\Delta_0^2}(m_b-q\cdot v)m_b^2 E_b,$$

$$S_9^{(1)}=\frac{1}{\Delta_0^2}(m_b-q\cdot v)m_b E_b.$$

D. The order k^2 terms

The order k^2 terms in Eq. (3.2) are

$$-\frac{2k\cdot(m_b v-q)}{\Delta_0^2}\bar{u}\{k^\mu\gamma^\nu+k^\nu\gamma^\mu-g^{\mu\nu}k\!\!\!/-i\epsilon^{\mu\nu\alpha\beta}k_\alpha\gamma_\beta\}P_L u$$

$$+\left[4\frac{[k\cdot(m_b v-q)]^2}{\Delta_0^3}-\frac{k^2}{\Delta_0^2}\right]\bar{u}\{(m_b v-q)^\mu\gamma^\nu+(m_b v-q)^\nu\gamma^\mu-(m_b v\!\!\!/-q\!\!\!/)g^{\mu\nu}-i\epsilon^{\mu\nu\alpha\beta}(m_b v-q)_\alpha\gamma_\beta\}P_L u. \quad (3.32)$$

The matrix element of Eq. (3.32) between unpolarized hadrons H_b can be written in terms of the operator $\bar{b}\gamma^\lambda iD^{(\alpha}iD^{\beta)}b$. This matrix element is needed to lowest order in $1/m_b$, so b can be replaced by the heavy quark field b_v, and γ^λ replaced by v^λ. The matrix element needed is

$$\frac{1}{2J+1}\sum_s \langle H_b(v,s)|\bar{b}_v iD^{(\alpha}iD^{\beta)}b_v|H_b(v,s)\rangle$$

$$= -\frac{2m_b^2 K_b}{3}(g^{\alpha\beta}-v^\alpha v^\beta) , \quad (3.33)$$

using Eq. (3.21). Substituting Eq. (3.33) into Eq. (3.32) gives the contribution of the k^2 terms to $T^{\mu\nu}$:

$$T_1^{(2)} = -\tfrac{1}{3}m_b^2 K_b(m_b - q\cdot v)\left\{\frac{4}{\Delta_0^3}[q^2-(q\cdot v)^2]-\frac{3}{\Delta_0^2}\right\} ,$$

$$T_2^{(2)} = -\tfrac{2}{3}m_b^3 K_b\left\{\frac{4}{\Delta_0^3}[q^2-(q\cdot v)^2]-\frac{3}{\Delta_0^2}\right\}$$
$$+\tfrac{4}{3}m_b^2 K_b\frac{v\cdot q}{\Delta_0^2} , \quad (3.34)$$

$$T_3^{(2)} = -\tfrac{1}{3}m_b^2 K_b\left\{\frac{4}{\Delta_0^3}[q^2-(q\cdot v)^2]-\frac{3}{\Delta_0^2}\right\}+\tfrac{2}{3}m_b^2 K_b\frac{1}{\Delta_0^2} .$$

The spin-dependent contribution to $T^{\mu\nu}$ for polarized Λ_b states is given in terms of the matrix element

$$\langle \Lambda_b(v,s)|\bar{b}_v\gamma^\lambda\gamma_5 iD^{(\alpha}iD^{\beta)}b_v|\Lambda_b(v,s)\rangle$$

$$= -\tfrac{2}{3}m_b^2 K_b(g^{\alpha\beta}-v^\alpha v^\beta)s^\lambda , \quad (3.35)$$

using Eq. (3.21) and heavy quark spin symmetry. This gives the order k^2 contribution to the spin-dependent form factors:

$$S_1^{(2)} = \tfrac{1}{3}m_b^2 K_b\left\{\frac{4}{\Delta_0^3}[q^2-(q\cdot v)^2]-\frac{5}{\Delta_0^2}\right\} ,$$

$$S_6^{(2)} = \tfrac{1}{3}m_b^3 K_b\left\{\frac{4}{\Delta_0^3}[q^2-(q\cdot v)^2]-\frac{3}{\Delta_0^2}-\frac{2q\cdot v}{\Delta_0^2}\right\} ,$$

$$S_8^{(2)} = \tfrac{1}{3}m_b^2 K_b\left\{-\frac{4m_b}{\Delta_0^3}[q^2-(q\cdot v)^2]+\frac{3m_b}{\Delta_0^2}+\frac{2q\cdot v}{\Delta_0^2}\right\} ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.36)$$

$$S_9^{(2)} = \tfrac{1}{3}m_b^2 K_b\left\{\frac{4}{\Delta_0^3}[q^2-(q\cdot v)^2]-\frac{5}{\Delta_0^2}\right\} ,$$

$$S_2^{(2)} = S_3^{(2)} = 0 .$$

E. The one-gluon matrix element

Finally, one needs to compute the one-gluon matrix element of Eq. (3.1) given in Fig. 3 to determine the coefficient of the $G^{\alpha\beta}$ operators. The one-gluon matrix element can be expanded in a power series in the momentum p of the external gluon. The terms of order p^0 are identical to the one-gluon matrix element of the operators we have already found, with the gluon field from the covariant derivative $D = \partial + igA$. The terms linear in p are the matrix element of the operator:

$$\frac{g}{2\Delta_0^2}\bar{b}G^{\alpha\beta}\epsilon_{\alpha\beta\lambda\sigma}(m_b v - q)^\lambda[g^{\mu\sigma}\gamma^\nu + g^{\nu\sigma}\gamma^\mu - g^{\mu\nu}\gamma^\sigma + i\epsilon^{\mu\sigma\nu\tau}\gamma_\tau]P_L b_v . \quad (3.37)$$

This is a dimension five operator, so we need its matrix element to lowest order in $1/m_b$. The fields b can be replaced by the heavy quark fields b_v. The matrix elements needed to evaluate the one-gluon contribution to $T^{\mu\nu}$ for a spin-averaged hadron H_b are

$$\frac{1}{2J+1}\sum_s \langle H_b(v,s)|\bar{b}_v gG^{\alpha\beta}\gamma^\lambda b_v|H_v(v,s)\rangle \quad (3.38)$$

and

$$\frac{1}{2J+1}\sum_s \langle H_b(v,s)|\bar{b}_v gG^{\alpha\beta}\gamma^\lambda\gamma_5 b_v|H_b(v,s)\rangle . \quad (3.39)$$

The matrix element in Eq. (3.38) can be simplified by replacing the γ matrix between b_v fields by v. The resultant matrix element must vanish because there is no antisymmetric tensor in the indices α and β that can be constructed out of the single four-vector v. The matrix element in Eq. (3.39) must have the form

$$\frac{1}{2J+1}\sum_s \langle H_b(v,s)|\bar{b}_v gG^{\alpha\beta}\gamma^\lambda\gamma_5 b_v|H_b(v,s)\rangle = N\epsilon^{\alpha\beta\lambda\tau}v_\tau . \quad (3.40)$$

The constant N can be evaluated by contracting both sides of Eq. (3.40) with $\epsilon_{\alpha\beta\lambda\rho}v^\rho$ to give

$$\frac{1}{2J+1}\sum_s \epsilon_{\alpha\beta\lambda\rho}v^\rho\langle H_b(v,s)|\bar{b}_v gG^{\alpha\beta}\gamma^\lambda\gamma_5 b_v|H_b(v,s)\rangle = -6N . \quad (3.41)$$
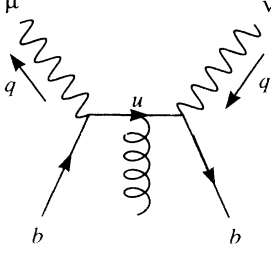
FIG. 3. The one-gluon matrix element.

Using the spinor identity

$$\epsilon_{\alpha\beta\lambda\rho} v^\rho \bar{b}_v \gamma^\lambda \gamma_5 b_v = -\bar{b}_v \sigma_{\alpha\beta} b_v ,$$

and Eq. (3.16) gives

$$\frac{1}{2J+1} \sum_s \langle H_b(v,s) | \bar{b}_v g G^{\alpha\beta} \gamma^\lambda \gamma_5 b_v | H_b(v,s) \rangle$$

$$= \tfrac{2}{3} m_b^2 G_b \epsilon^{\alpha\beta\lambda\tau} v_\tau . \quad (3.42)$$

Substituting Eq. (3.42) into (3.37) gives the contribution of the one-gluon operator to $T^{\mu\nu}$:

$$T_1^{(g)} = -\tfrac{1}{3} m_b^2 G_b \frac{m_b - q\cdot v}{\Delta_0^2} ,$$

$$T_2^{(g)} = \tfrac{2}{3} m_b^2 G_b \frac{m_b}{\Delta_0^2} , \quad (3.43)$$

$$T_3^{(g)} = -\tfrac{1}{3} m_b^2 G_b \frac{1}{\Delta_0^2} .$$

The gluon terms do not contribute to polarized Λ_b decay since $G^{\alpha\beta}$ has vanishing matrix element at zero recoil between Λ_b states.

F. Summary

The final expressions for $T^{\mu\nu}$ and $T_5^{\mu\nu}$ to order $1/m_b^2$ are obtained by combining Eqs. (3.6), (3.8), (3.24), (3.31), (3.34), (3.36), and (3.43):

$$T_1 = \frac{1}{2\Delta_0} (m_b - q\cdot v)(1 + X_b)$$

$$+ \tfrac{2}{3} m_b(K_b + G_b) \left[-\frac{1}{2\Delta_0} + \frac{q^2 - (q\cdot v)^2}{\Delta_0^2} \right]$$

$$+ \frac{m_b E_b}{2\Delta_0} - \frac{1}{3} m_b^2 G_b \frac{m_b - q\cdot v}{\Delta_0^2} ,$$

$$T_2 = \frac{m_b}{\Delta_0} (1 + X_b)$$

$$+ \tfrac{2}{3} m_b(K_b + G_b) \left[\frac{1}{\Delta_0} + \frac{2m_b q\cdot v}{\Delta_0^2} \right] + \frac{m_b E_b}{\Delta_0}$$

$$+ \frac{4}{3} m_b^2 K_b \frac{q\cdot v}{\Delta_0^2} + \frac{2}{3} m_b^2 G_b \frac{m_b}{\Delta_0^2} ,$$

$$T_3 = \frac{1}{2\Delta_0} (1 + X_b) - \tfrac{2}{3} m_b(K_b + G_b) \frac{m_b - q\cdot v}{\Delta_0^2}$$

$$+ \frac{2}{3} m_b^2 K_b \frac{1}{\Delta_0^2} - \frac{1}{3} m_b^2 G_b \frac{1}{\Delta_0^2} ,$$

$$S_1 = \frac{1}{\Delta_0^3} \{ \tfrac{4}{3} m_b^2 K_b [q^2 - (q\cdot v)^2] \}$$

$$- \frac{5 m_b K_b q\cdot v}{3\Delta_0^2} - \frac{1}{2}(1 + \epsilon_b) \frac{1}{\Delta_0} ,$$

$$S_2 = \frac{4}{3\Delta_0^2} m_b^2 K_b , \quad (3.44)$$

$$S_3 = \frac{2}{3\Delta_0^2} m_b K_b ,$$

$$S_6 = \frac{1}{\Delta_0^3} \{ \tfrac{4}{3} m_b^3 K_b [q^2 - (q\cdot v)^2] \}$$

$$- \frac{5 m_b^2 K_b q\cdot v}{3\Delta_0^2} - \frac{5 m_b K_b}{6\Delta_0} - \frac{1}{2}(1 + \epsilon_b) \frac{m_b}{\Delta_0} ,$$

$$S_8 = -\frac{1}{\Delta_0^3} \{ \tfrac{4}{3} m_b^3 K_b [q^2 - (q\cdot v)^2] \}$$

$$+ \frac{5 m_b^2 K_b q\cdot v}{3\Delta_0^2} + \frac{m_b K_b}{6\Delta_0} + \frac{1}{2}(1 + \epsilon_b) \frac{m_b}{\Delta_0} ,$$

$$S_9 = \frac{1}{\Delta_0^3} \{ \tfrac{4}{3} m_b^2 K_b [q^2 - (q\cdot v)^2] \}$$

$$- \frac{K_b}{\Delta_0^2} [m_b q\cdot v + \tfrac{2}{3} m_b^2] - \frac{1}{2}(1 + \epsilon_b) \frac{1}{\Delta_0} ,$$

where

$$X_b = -2 \frac{m_b E_b}{\Delta_0} (m_b - q\cdot v)$$

$$- \frac{8}{3} m_b^2 \frac{K_b}{\Delta_0^2} [q^2 - (q\cdot v)^2] + 2 m_b^2 \frac{K_b}{\Delta_0} . \quad (3.45)$$

The expressions in Eq. (3.44) can be simplified using the identity $E_b = K_b + G_b$. There is an additional simplification in Λ_b decay, where $G_b = 0$.

IV. TIME-ORDERED PRODUCTS AND THE EQUATIONS OF MOTION

In the above computation, we used the equation of motion to order $1/m_b$. Another method commonly used in the literature is to treat the $1/m_b$ terms in the Lagrangian, Eq. (3.11), as a perturbation, so that the equations of motion are $(v\cdot D)b_b = 0$. The $1/m_b$ terms in the Lagrangian then give terms that are time-ordered products of operators with $1/m_b$ terms in the Lagrangian. For example, the matrix element $\langle H_b(v,s) | \bar{b}_v i D^\tau b_v | H_b(v,s) \rangle$ [that was needed in Sec. III, see Eq. (3.19)] is zero using $(v\cdot D)b_v = 0$. However, one now gets an additional contribution:

$$\left\langle H_b(v,s) \left| T \left\{ \overline{b}_v iD^\tau b_v(x) i \int d^4y \frac{\mathcal{L}_1(y)}{2m_b} \right\} \right| H_b(v,s) \right\rangle \ ,$$

$$(4.1)$$

where the $1/m_b$ terms in the Lagrangian are denoted by $\mathcal{L}_1/2m_b$, using the notation of Ref. [14]. The matrix element, Eq. (4.1) is equal to Av^τ, where A is the constant of proportionality. Contracting both sides with v^τ gives

$$A = \left\langle H_b(v,s) \left| T \left\{ \overline{b}_v (iv \cdot D) b_v(x) i \int d^4y \frac{\mathcal{L}_1(y)}{2m_b} \right\} \right| H_b(v,s) \right\rangle \ .$$

$$(4.2)$$

$\mathcal{L}_1(y)$ is a sum of operators of the form $\overline{b}_v(y) X b_v(y)$, so that the time-ordered product in Eq. (4.2) is

$$\int dy \ T\{\overline{b}_v(iv \cdot D) b_v(x) \overline{b}_v X b_v(y)\}$$

$$= \int dy \ T\{\overline{b}_v(iv \cdot D) iS(x,y) X b_v(y)\} \ , \quad (4.3)$$

where $S(x,y)$ is the b-quark propagator, which satisfies the Green's-function equation $(iv \cdot D) S(x,y) = \delta(x,y)$. This converts the T product into the local operator $\overline{b}_v X b_v$, so that A is given by

$$A = -\left\langle H_b(v,s) \left| \frac{\mathcal{L}_1(y)}{2m_b} \right| H_b(v,s) \right\rangle \ ,$$

$$(4.4)$$

the same answer as that obtained by using the equations of motion to order $1/m_b$. Using the equations of motion to order $1/m_b$ is equivalent to treating the $1/m_b$ corrections to the Lagrangian as insertions, provided one takes into account that the heavy quark propagator does not satisfy the equation of motion, since $(iv \cdot D) S(x,y) = \delta(x,y)$.

V. DECAY DISTRIBUTIONS
FOR HADRONS CONTAINING A b QUARK

The amplitude $W^{\mu\nu}$ can be determined by performing a contour integration of $T^{\mu\nu}$ from Eq. (3.44) around the contour C of Fig. 1. This contour integral is trivial to do,

and is equivalent to taking the imaginary part in $T^{\mu\nu}$ directly by making the replacements

$$\frac{1}{\Delta_0} \rightarrow \delta((m_b v - q)^2 - m^2) \ ,$$

$$\frac{1}{\Delta_0^2} \rightarrow -\delta'((m_b v - q)^2 - m^2) \ , \quad (5.1)$$

$$\frac{1}{\Delta_0^3} \rightarrow \frac{1}{2} \delta''((m_b v - q)^2 - m^2) \ ,$$

in Eq. (3.44). This is very different from the analogous calculation in deep-inelastic scattering. In our problem, there are nontrivial functions of q and v, multiplying a few operators, so that the amplitude has the form $f_i(q^2, q \cdot v)\langle O \rangle_i$. The QCD corrections multiply each operator by an anomalous dimension,

$$f_i(q^2, q \cdot v)\langle O \rangle_i \rightarrow \lambda_i f_i(q^2, q \cdot v)\langle O \rangle_i \ ,$$

so the imaginary part of $T^{\mu\nu}$ is related directly to the imaginary part of f_i. In deep-inelastic scattering, the different powers of $q \cdot v$ in the expansion of f_i would each be multiplied by the anomalous dimension of a different twist-two operator, so that the imaginary part of $T^{\mu\nu}$ is related to the imaginary part of f_i by a nontrivial convolution.

One can now compute the inclusive lepton spectrum by substituting Eq. (3.44) and Eq. (5.1) into Eq. (2.5), and integrating over E_ν. The kinematic region is determined by the mass of the decaying hadron M_{H_b}. The δ functions in W_i restrict the integration to that given by the parton model kinematics determined by the quark mass m_b. In QCD, one can prove [15] that $M_{H_b} - m_b \equiv \overline{\Lambda} > 0$, so that the parton model kinematic region is contained within the hadron kinematic region. Thus, there is no dependence on the hadron mass M_{H_b} through the limits on the region of integration, and the lepton spectrum is determined only in terms of the quark mass m_b. This would not be the case if $\overline{\Lambda}$ were negative. Evaluating the E_ν integral gives the decay distribution for $H_b \rightarrow X e \overline{\nu}_e$:

$$\frac{1}{\Gamma_b} \frac{d\Gamma}{dy \, d\hat{q}^2} = \theta(z)\{ 12(y - \hat{q}^2)(1 + \hat{q}^2 - \rho - y) + 12E_b(2\hat{q}^4 - 2\hat{q}^2\rho + y - 2\hat{q}^2 y + \rho y)$$

$$+ 8K_b(2\hat{q}^2 - \hat{q}^4 + \hat{q}^2\rho - 3y) + 8G_b(-\hat{q}^2 + 2\hat{q}^4 - 2\hat{q}^2\rho - 2y - 2\hat{q}^2 y + \rho y)\}$$

$$+ \delta(z)\frac{1}{y^2}\{ 12E_b\hat{q}^2(y - \hat{q}^2)(-\hat{q}^2 + 2y - y^2) + 4K_b(-\hat{q}^6 + 9\hat{q}^4 y - 6\hat{q}^2 y^2 - 2\hat{q}^4 y^2 - \hat{q}^2 y^4 + y^5)$$

$$+ 8\hat{q}^2 G_b(y - \hat{q}^2)(-\hat{q}^2 + y + y^2)\} + K_b\delta'(z)\frac{4\hat{q}^2}{y^3}(y^2 - \hat{q}^2)^2(y - \hat{q}^2) \ ,$$

$$(5.2)$$

where

$$z = 1 + \hat{q}^2 - \rho - \frac{\hat{q}^2}{y} - y \ ,$$

$$(5.3)$$

$$\hat{q}^2 = \frac{q^2}{m_b^2}, \quad \rho = \frac{m_j^2}{m_b^2}, \quad y = \frac{2E_e}{m_b} \quad , \tag{5.4}$$

and

$$\Gamma_b = |V_{jb}|^2 G_F^2 \frac{m_b^5}{192\pi^3} \quad . \tag{5.5}$$

The curve $z = 0$ is an edge of the Dalitz plot in the $\hat{q}^2$-y plane. Integrating Eq. (5.2) with respect to $\hat{q}^2$ gives the inclusive lepton spectrum:

$$\frac{1}{\Gamma_b} \frac{d\Gamma}{dy} = \left\{ 2(3-2y)y^2 - 6y^2\rho - \frac{6y^2\rho^2}{(1-y)^2} + \frac{2(3-y)y^2\rho^3}{(1-y)^3} \right\} + E_b \left\{ 4(3-y)y^2 + \frac{12y^2\rho^2}{(1-y)^3} - \frac{4(6-4y+y^2)y^2\rho^3}{(1-y)^4} \right\}$$

$$+ K_b \left\{ -\frac{4y^2(9+2y)}{3} + \frac{4y^2(2y^2-2y-3)\rho^2}{(1-y)^4} + \frac{4y^2(18-10y+5y^2-y^3)\rho^3}{3(1-y)^5} \right\}$$

$$+ G_b \left\{ -\frac{4y^2(15+2y)}{3} + \frac{8y^2(3-2y)\rho}{(1-y)^2} + \frac{12y^2\rho^2}{(1-y)^2} + \frac{8y^2(-6+4y-y^2)\rho^3}{3(1-y)^4} \right\} \quad . \tag{5.6}$$

This agrees with the result of Bigi et al. Integrating Eq. (5.6) with respect to y gives the total decay rate

$$\frac{1}{\Gamma_b}\Gamma = \{1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2\ln\rho\} + E_b\{5 - 24\rho + 24\rho^2 - 8\rho^3 + 3\rho^4 - 12\rho^2\ln\rho\}$$

$$+ K_b\{-6 + 32\rho - 24\rho^2 - 2\rho^4 + 24\rho^2\ln\rho\} + G_b\{-2 + 16\rho - 16\rho^3 + 2\rho^4 + 24\rho^2\ln\rho\} \quad , \tag{5.7}$$

which also agrees with Bigi et al.

The decay rate for $b \to u$ is given by Eqs. (5.2)–(5.7) in the limit that $\rho \to 0$. This limit must be taken carefully because of the $1/(1-y)$ singularities in Eq. (5.6). The resulting expressions are

$$\frac{1}{\Gamma_b} \frac{d\Gamma}{dy\,d\hat{q}^2} = \theta(z)\{12(y-\hat{q}^2)(1+\hat{q}^2-y) + 12E_b(2\hat{q}^4+y-2\hat{q}^2y) + 8K_b(2\hat{q}^2-\hat{q}^4-3y) + 8G_b(-\hat{q}^2+2\hat{q}^4-2y-2\hat{q}^2y)\}$$

$$+ \delta(z)\frac{1}{y^2}\{12E_b\hat{q}^2(y-\hat{q}^2)(-\hat{q}^2+2y-y^2) + 4K_b(-\hat{q}^6+9\hat{q}^4y-6\hat{q}^2y^2-2\hat{q}^4y^2-\hat{q}^2y^4+y^5)$$

$$+ 8\hat{q}^2G_b(y-\hat{q}^2)(-\hat{q}^2+y+y^2)\} + \delta'(z)K_b\frac{4\hat{q}^2}{y^3}(y^2-\hat{q}^2)^2(y-\hat{q}^2) \quad , \tag{5.8}$$

where now

$$z = 1 + \hat{q}^2 - \frac{\hat{q}^2}{y} - y \quad , \tag{5.9}$$

$$\frac{1}{\Gamma_b} \frac{d\Gamma}{dy} = \{2(3-2y)y^2\} + E_b\{4(3-y)y^2 + 2\delta(1-y)\} + K_b \left\{ -\frac{4y^2(9+2y)}{3} - \frac{4}{3}\delta(1-y) + \frac{2}{3}\delta'(1-y) \right\}$$

$$+ G_b \left\{ -\frac{4y^2(15+2y)}{3} + \frac{16}{3}\delta(1-y) \right\} \quad , \tag{5.10}$$

and

$$\Gamma = \Gamma_b(1 + 5E_b - 6K_b - 2G_b) \quad . \tag{5.11}$$

The results, Eqs. (5.10) and (5.11), agree with Bigi et al. The δ' function in Eq. (5.10) is present because the parton model decay distribution $\propto 2(3-2y)y^2$ does not vanish at the end point. This will be explained in more detail in the next section.

The decay distributions for a polarized Λ_b have the form $A + B\cos\theta$. The coefficient A is half the value of the corresponding decay distribution for an unpolarized Λ_b given in Eqs. (5.2)–(5.11). The results can be simplified for Λ_b by setting $E_b = K_b$ and $G_b = 0$. The coefficients of the $\cos\theta$ terms are

$$\frac{1}{\Gamma_b}\frac{d\Gamma}{dy\,d\hat{q}^2 d\cos\theta} = \cdots + \left[\frac{\theta(z)}{y}\{\,6(1+\epsilon_b)(y-\hat{q}^2)(-2\hat{q}^2+y+\hat{q}^2 y -\rho y -y^2)\right.$$

$$+2K_b(-6\hat{q}^4+12\hat{q}^2 y +4\hat{q}^4 y -4\hat{q}^2\rho y -3y^2-6\hat{q}^2 y^2+3\rho y^2)\}$$

$$+\delta(z)\frac{2}{y^2}\{K_b(-4\hat{q}^6+2\hat{q}^4 y +3\hat{q}^4 y^2-\hat{q}^2 y^3-\hat{q}^2 y^4+y^5)\}$$

$$\left.-K_b\delta'(z)\frac{2\hat{q}^2}{y^3}(y^2-\hat{q}^2)^2(y-\hat{q}^2)\right]\cos\theta\,,\tag{5.12}$$

$$\frac{1}{\Gamma_b}\frac{d\Gamma}{dy\,d\cos\theta} = \cdots + \left[(1+\epsilon_b)\left\{(1-2y)y^2-3y^2\rho+\frac{3y^2\rho^2}{(1-y)^2}-\frac{(1+y)y^2\rho^3}{(1-y)^3}\right\}\right.$$

$$\left.+K_b\left\{-\frac{10y^3}{3}+\frac{2y^3(5-2y)\rho^2}{(1-y)^4}-\frac{4y^3(5+2y-y^2)\rho^3}{3(1-y)^5}\right\}\right]\cos\theta\,,\tag{5.13}$$

$$\frac{1}{\Gamma_b}\frac{d\Gamma}{d\cos\theta} = \cdots + (1+\epsilon_b-K_b)[\{-\tfrac{1}{6}+2\rho+6\rho^2-\tfrac{22}{3}\rho^3-\tfrac{1}{2}\rho^4+6\rho^2\ln\rho+4\rho^3\ln\rho\}\,]\cos\theta\,.\tag{5.14}$$

The $\rho\to 0$ limits of the polarized Λ_b distributions are

$$\frac{1}{\Gamma_b}\frac{d\Gamma}{dy\,d\hat{q}^2 d\cos\theta} = \cdots + \left[\frac{\theta(z)}{y}\{6(1+\epsilon_b)(y-\hat{q}^2)(-2\hat{q}^2+y+\hat{q}^2 y -y^2)+2K_b(-6\hat{q}^4+12\hat{q}^2 y +4\hat{q}^4 y -3y^2-6\hat{q}^2 y^2)\}\right.$$

$$+\delta(z)\frac{2}{y^2}\{K_b(-4\hat{q}^6+2\hat{q}^4 y +3\hat{q}^4 y^2-\hat{q}^2 y^3-\hat{q}^2 y^4+y^5)\}$$

$$\left.-K_b\delta'(z)\frac{2\hat{q}^2}{y^3}(y^2-\hat{q}^2)^2(y-\hat{q}^2)\right]\cos\theta\,,\tag{5.15}$$

$$\frac{1}{\Gamma_b}\frac{d\Gamma}{dy\,d\cos\theta} = \cdots + \left[(1+\epsilon_b)\{(1-2y)y^2\}+K_b\left\{-\frac{10y^3}{3}+\delta(1-y)-\frac{1}{3}\delta'(1-y)\right\}\right]\cos\theta\,,\tag{5.16}$$

$$\frac{1}{\Gamma_b}\frac{d\Gamma}{d\cos\theta} = \cdots -\tfrac{1}{6}(1+\epsilon_b-K_b)\cos\theta\,.\tag{5.17}$$

The formulas obtained here will be discussed in detail in the following sections.

VI. PHYSICAL INTERPRETATION OF THE $1/m_b^2$ CORRECTIONS

There is a way to obtain most of the $1/m_b^2$ corrections which also provides a physical picture of these corrections. One can obtain a class of $1/m_b^2$ corrections by taking the lowest-order expression, Eq. (3.6), for $T^{\mu\nu}$ (or $W^{\mu\nu}$) and smearing it over a distribution of b-quark momenta in the H_b hadron. This will give the terms proportional to E_b and K_b in Secs. IV and V, but not those proportional to G_b. This is why we did not simplify the results using $E_b=G_b+K_b$.

We begin with the spin-independent terms. The b-quark momentum in $|H_b(v,s)\rangle$ can be written as $p=m_b v+k$. The lowest-order (parton model) expressions $T_0^{\mu\nu}(q,v,m_b,m_j)$ were obtained by considering the decay of a quark of mass m_b and velocity v in the rest frame $v=(1,0,0,0)$. A quark with momentum $m_b v+k$

can be considered to be an on-shell quark with mass m_b' and velocity v', where $v'^2=1$, and $m_b'v'=m_b v+k$. The decay rate of such a quark can be obtained by using $T_0^{\mu\nu}(q,v',m_b',m_j)/v'^0$, where v'^0 is the Lorentz time dilation factor for a moving particle. The value of $T^{\mu\nu}$ is obtained by averaging $T_0^{\mu\nu}(q,v',m_b',m_j)/v'^0$ over a distribution of b-quark momenta. This average is most easily done by writing $v'^0=v\cdot m_b'v'/m_b'$. $m_b'T_0^{\mu\nu}(q,v',m_b',m_j)/v\cdot m_b'v'$ can be written as a function of q, m_j, and the product $m_b'v'$. This makes the averaging simple because one can use the substitution $m_b'v'=m_b v+k$ to rewrite the expression in terms of $m_b v$ and k, and then average over k. Since q is unaffected by the averaging, terms proportional to q^μ or q^ν will remain proportional to q^μ and q^ν. Thus, averaging T_4 or T_5 will not produce T_{1-3}. Similarly, averaging T_1 or T_3 will produce terms only proportional to T_1 or T_3, but averaging T_2 produces terms proportional to both T_2 and T_1.

As a simple example, we will consider the average of the piece of $T^{\mu\nu}$ containing T_3 explicitly. The average we need is

$$\left\langle \frac{m_b'}{v \cdot m_b' v'} [-i\epsilon^{\mu\nu\alpha\beta} v_\alpha' q_\beta T_3(q, q \cdot v', m_b')] \right\rangle = \frac{1}{2} \left\langle \frac{1}{v \cdot m_b' v'} \left[-i\epsilon^{\mu\nu\alpha\beta} m_b' v_\alpha' q_\beta \frac{1}{(m_b' v' - q)^2 - m_j^2} \right] \right\rangle$$

$$= \frac{1}{2} \left\langle \frac{1}{v \cdot (m_b v + k)} \left[-i\epsilon^{\mu\nu\alpha\beta}(m_b v + k)_\alpha q_\beta \frac{1}{(m_b v + k - q)^2 - m_j^2} \right] \right\rangle . \quad (6.1)$$

The averages over k can be written in terms of

$$\langle k^\alpha \rangle = E_b m_b v^\alpha ,$$

$$\langle k^\alpha k^\beta \rangle = \tfrac{1}{3} \langle k^2 \rangle (g^{\alpha\beta} - v^\alpha v^\beta) \quad (6.2)$$

$$= -\tfrac{2}{3} K_b m_b^2 (g^{\alpha\beta} - v^\alpha v^\beta) ,$$

where E_b and K_b are the mean total energy and mean kinetic energy in units of m_b^2. This gives the terms in Eq. (3.44) with the exception of the G_b term. A similar computation also reproduces T_1 and T_2 in Eq. (3.44).

The above averaging procedure computes the $1/m_b^2$ corrections using a distribution of quark momenta k in a hadron H_b. Here k is considered to be a number, not an operator. The operator product expansion gives a similar result, with k replaced by the covariant derivative iD. The commutator of two covariant derivatives is proportional to the gluon field strength, $[D^\alpha, D^\beta] = igG^{\alpha\beta}$. The covariant derivatives commute if we neglect all terms involving $G^{\alpha\beta}$, i.e., all terms involving G_b. Thus, the $1/m_b$ terms obtained by the averaging method are identical to those obtained using the operator product expansion neglecting $G^{\alpha\beta}$. This result holds to all orders in the $1/m_b$ expansion due to reparametrization invariance [13].

Our averaging procedure provides a useful check on the total decay rate. The total decay rate (neglecting $1/m_b$ corrections) can be written as $\Gamma_0 = m_b^5 f(m^2/m_b^2)$.

The total decay rate for a quark distribution is given by averaging

$$m_b'^5 f(m^2/m_b'^2) / v'^0 = m_b'^6 f(m^2/m_b'^2) / v \cdot m_b' v' .$$

Now

$$m_b'^2 = m_b'^2 v'^2 = (m_b v + k)^2 = m_b^2(1 + 2E_b - 2K_b) ,$$

and $v \cdot m_b' v' = m_b(1 + E_b)$. This gives

$$\Gamma = m_b^5(1 + E_b)^{-1}(1 + 2E_b - 2K_b)^3$$

$$\times f(\rho(1 + 2E_b - 2K_b)^{-1}) , \quad (6.3)$$

where $\rho = m^2/m_b^2$. Expanding this and retaining corrections to order $1/m_b^2$ gives

$$\Gamma = \Gamma_0 + E_b \left[5\Gamma_0 - 2\rho \frac{d\Gamma_0}{d\rho} \right]$$

$$+ K_b \left[-6\Gamma_0 + 2\rho \frac{d\Gamma_0}{d\rho} \right] . \quad (6.4)$$

The result, Eq. (5.7), agrees with this check.

One can also use the averaging procedure to determine the leptonic spectrum. Recall that we needed to compute the average of $m_b' T_0^{\mu\nu} / v \cdot m_b' v'$, where $m_b' T_0^{\mu\nu}$ is a function only of $m_b' v'$ and q, and has mass dimension zero. This can be written as

$$\left\langle \frac{m_b' T_0^{\mu\nu}}{v \cdot m_b' v'} \right\rangle = \frac{1}{m_b(1 + E_b)} \left\langle m_b T_0^{\mu\nu} + k^\alpha \frac{\partial}{\partial m_b v^\alpha} m_b T_0^{\mu\nu} + \frac{1}{2} k^\alpha k^\beta \frac{\partial^2}{\partial m_b v^\alpha \partial m_b v^\beta} m_b T_0^{\mu\nu} \right\rangle$$

$$= \left[T_0^{\mu\nu} - E_b T_0^{\mu\nu} + E_b v^\alpha \frac{\partial}{\partial m_b v^\alpha} m_b T_0^{\mu\nu} - \frac{1}{3} K_b m_b (g^{\alpha\beta} - v^\alpha v^\beta) \frac{\partial^2}{\partial m_b v^\alpha \partial m_b v^\beta} m_b T_0^{\mu\nu} \right] , \quad (6.5)$$

using Eq. (6.2). The decay rate depends on $L_{\mu\nu} T^{\mu\nu} \equiv F$ which has mass dimension two, and is a function only of k_e, k_ν, $m_b v$, m_j, and q. One can rewrite F in terms of the dimensionless variables y, $\hat{q}$, ρ, and an overall factor of m_b^2. The variables y, $\hat{q}$, and ρ were defined in Eq. (5.4), and x is defined by

$$x = \frac{2E_\nu}{m_b} = \frac{2m_b v \cdot k_\nu}{(m_b v)^2} . \quad (6.6)$$

The averaging formula, Eq. (6.5), for $T^{\mu\nu}$ implies that F can be written in terms of the lowest-order expression $F_0 = L_{\mu\nu} T_0^{\mu\nu}$, by differentiating with respect to $m_b v$. In performing the differentiation, it is important to remember that $T_0^{\mu\nu}$ and F_0 are to be considered as functions only of the product $m_b v$, not of m_b and v separately. Thus, m_b is an implicit function of v, with

$$\frac{\partial}{\partial m_b v^\alpha} m_b = \frac{\partial}{\partial m_b v^\alpha} [(m_b v) \cdot (m_b v)]^{1/2} = v_\alpha . \quad (6.7)$$

Using the partial derivatives

$$\frac{\partial x}{\partial(m_b v^\alpha)} = \left[\frac{2k_{v\alpha}}{m_b^2} - \frac{2xv_\alpha}{m_b}\right] , \quad \frac{\partial \hat{q}^2}{\partial(m_b v^\alpha)} = -\frac{2\hat{q}^2}{m_b}v_\alpha ,$$

$$\frac{\partial y}{\partial(m_b v^\alpha)} = \left[\frac{2k_{e\alpha}}{m_b^2} - \frac{2yv_\alpha}{m_b}\right] , \quad \frac{\partial \rho}{\partial(m_b v^\alpha)} = -\frac{2\rho}{m_b}v_\alpha ,$$

(6.8)

and Eq. (6.5) our averaging procedure implies that

$$F = \left\{ 1 + E_b \left[1 - 2\rho\frac{\partial}{\partial\rho} - 2\hat{q}^2\frac{\partial}{\partial\hat{q}^2} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} \right] \right.$$
$$\left. + K_b \left[-2 + 2\rho\frac{\partial}{\partial\rho} + 2\hat{q}^2\frac{\partial}{\partial\hat{q}^2} + 2y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x} + \frac{1}{3}y^2\frac{\partial^2}{\partial y^2} + \frac{1}{3}x^2\frac{\partial^2}{\partial x^2} + \frac{2}{3}(xy - 2q^2)\frac{\partial^2}{\partial x\partial y} \right] \right\} F_0 .$$

(6.9)

The differential decay rate is given by integrating F over the phase space, which is proportional to $dy\, dx\, d\hat{q}^2$. Integrating Eq. (6.9) with respect to x gives the formula

$$\frac{d\Gamma}{dy\, d\hat{q}^2} = \left\{ 1 + E_b \left[2 - 2\rho\frac{\partial}{\partial\rho} - 2\hat{q}^2\frac{\partial}{\partial\hat{q}^2} - y\frac{\partial}{\partial y} \right] + K_b \left[-\frac{10}{3} + 2\rho\frac{\partial}{\partial\rho} + 2\hat{q}^2\frac{\partial}{\partial\hat{q}^2} + \frac{4}{3}y\frac{\partial}{\partial y} + \frac{1}{3}y^2\frac{\partial^2}{\partial y^2} \right] \right\} \frac{d\Gamma_0}{dy\, d\hat{q}^2} ,$$

(6.10)

where $d\Gamma_0/dy\, d\hat{q}^2$ is the parton model decay rate obtained by setting $E_b, K_b, G_b \to 0$ in Eq. (5.2). Integrating Eq. (6.10) with respect to $\hat{q}^2$ gives

$$\frac{d\Gamma}{dy} = \left\{ 1 + E_b \left[4 - 2\rho\frac{\partial}{\partial\rho} - y\frac{\partial}{\partial y} \right] \right.$$
$$+ K_b \left[-\frac{16}{3} + 2\rho\frac{\partial}{\partial\rho} + \frac{4}{3}y\frac{\partial}{\partial y} \right.$$
$$\left. \left. + \frac{1}{3}y^2\frac{\partial^2}{\partial y^2} \right] \right\} \frac{d\Gamma_0}{dy} .$$

(6.11)

Integrating Eq. (6.11) with respect to y gives

$$\Gamma = \left\{ 1 + E_b \left[5 - 2\rho\frac{\partial}{\partial\rho} \right] + K_b \left[-6 + 2\rho\frac{\partial}{\partial\rho} \right] \right\} \Gamma_0 ,$$

(6.12)

which is the same result we obtained in Eq. (6.4). Equations (6.10)–(6.12) agree with Eqs. (5.2)–(5.7) on setting $G_b \to 0$. One can also understand the origin of the δ-function terms in Sec. V. Since the decay distribution does not vanish at the end point, the derivatives in Eq. (6.11) produce δ functions and derivatives of δ functions.

One can also apply the averaging method to obtain the spin-dependent form factors. One considers the b quark in the hadron to have a distribution of spin and momentum, with

$$\langle S^\mu \rangle = (1 + \epsilon_b + K_b)s^\mu ,$$

$$\langle S^\mu k^\nu \rangle = m_b K_b s^\mu v^\nu + \tfrac{2}{3}m_b K_b s^\nu v^\mu ,$$

(6.13)

$$\langle S^\mu k^\alpha k^\beta \rangle = -\tfrac{2}{3}m_b^2 K_b(g^{\alpha\beta} - v^\alpha v^\beta)s^\mu ,$$

where S^μ is the quark spin, and s^μ is the hadron spin.

VII. DECAY DISTRIBUTIONS FOR HADRONS CONTAINING A c QUARK

The decay distributions for semileptonic c-quark decay can be readily obtained from the calculations in the previous section for b-quark decay. The charged lepton distribution in c decay is equal to the neutrino distribution in b decay, and vice versa. This is equivalent to changing the signs of W_3, G_3, G_8, and G_9. The lepton energy spectra are less singular than for b decay because the free quark decay rate vanishes at the end point. The double differential c-decay distribution for an unpolarized hadron H_c containing a c quark is

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy\, d\hat{q}^2} = \theta(z)\{12y(1-\rho-y) + 12yE_c(1+\rho) - 24yK_c + 8yG_c(\rho-2)\}$$

$$+ \delta(z)\frac{1}{y}\{ 12E_c\hat{q}^2(y-1)(\hat{q}^2 - 2y + y^2) + 4K_c(3\hat{q}^4 - 6\hat{q}^2 y - 4\hat{q}^4 y + 6\hat{q}^2 y^2 + y^4)$$

$$+ 8\hat{q}^2 G_c(1-y)(\hat{q}^2 + y - y^2)\} + \delta'(z)K_c\frac{4\hat{q}^2}{y^2}(y^2 - \hat{q}^2)^2(1-y) ,$$

(7.1)

where

$$\Gamma_c = |V_{jc}|^2 G_F^2 \frac{m_c^5}{192\pi^3} \; . \tag{7.2}$$

Integrating this with respect to $\hat{q}^2$ and then y gives

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy} = \left\{ 12(1-y)y^2 - 24y^2\rho + \frac{12y^2\rho^2}{(1-y)} \right\} + E_c \left\{ 12(2-y)y^2 + \frac{12y^2(y-2)\rho^2}{(1-y)^2} \right\}$$
$$+ K_c \left\{ -8y^2(3+y) + \frac{8y^2(3-2y)\rho^2}{(1-y)^3} \right\} + G_c \left\{ -8y^3 + \frac{8y^3\rho^2}{(1-y)^2} \right\} , \tag{7.3}$$

and

$$\frac{1}{\Gamma_c}\Gamma = \{1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2\ln\rho\} + E_c\{5 - 24\rho + 24\rho^2 - 8\rho^3 + 3\rho^4 - 12\rho^2\ln\rho\}$$
$$+ K_c\{-6 + 32\rho - 24\rho^2 - 2\rho^4 + 24\rho^2\ln\rho\} + G_c\{-2 + 16\rho - 16\rho^3 + 2\rho^4 + 24\rho^2\ln\rho\} \; . \tag{7.4}$$

The $\rho\to0$ limits of Eqs. (7.1)–(7.4) are

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy\,d\hat{q}^2} = \theta(z)\{12(y-y^2) + 12yE_c - 24yK_c - 16yG_c\}$$
$$+ \delta(z)\frac{1}{y}\{ 12E_c\hat{q}^2(y-1)(\hat{q}^2 - 2y + y^2) - 4K_c(-3\hat{q}^4 + 6\hat{q}^2y + 4\hat{q}^4y - 6\hat{q}^2y^2 - y^4)$$
$$- 8\hat{q}^2 G_c(y-1)(\hat{q}^2 + y - y^2)\} + \delta'(z)K_c\frac{4\hat{q}^2}{y^3}(y^2 - \hat{q}^2)^2(1-y) , \tag{7.5}$$

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy} = \{12(1-y)y^2\} + E_c\{12(2-y)y^2\} + K_c\{-8y^2(3+y) + 4\delta(1-y)\} + G_c\{-8y^3\} , \tag{7.6}$$

$$\Gamma = \Gamma_c\{1 + 5E_c - 6K_c - 2G_c\} \; . \tag{7.7}$$

The $\cos\theta$ terms in the decay distributions for a polarized Λ_c are

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy\,d\hat{q}^2 d\cos\theta} = \cdots + \left[\theta(z)(1+\epsilon_c)\{6y(1-\rho-y) - 6y(1-\rho)K_c\} \right.$$
$$\left. + \delta(z)\{2K_c(-\hat{q}^4 - 3\hat{q}^2y + 3\hat{q}^2y^2 + y^3)\} + \delta'(z)K_c\frac{2\hat{q}^2}{y^2}(y^2 - \hat{q}^2)^2(1-y) \right]\cos\theta , \tag{7.8}$$

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy\,d\cos\theta} = \cdots + \left[(1+\epsilon_c)\left\{ 6(1-y)y^2 - 12y^2\rho + \frac{6y^2\rho^2}{(1-y)} \right\} + K_c\left\{ -10y^3 + \frac{2y^3(5-3y)\rho^2}{(1-y)^3} \right\} \right]\cos\theta , \tag{7.9}$$

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{d\cos\theta} = \cdots + (1+\epsilon_c - K_c)[\{\tfrac{1}{2} - 4\rho + 4\rho^3 - \tfrac{1}{2}\rho^4 - 6\rho^2\ln\rho\}]\cos\theta \; . \tag{7.10}$$

The $\rho\to0$ limits of these distributions are

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy\,d\hat{q}^2 d\cos\theta} = \cdots + \left[\theta(z)(1+\epsilon_c)\{6y(1-y) - 6yK_c\} + \delta(z)\{2K_c(-\hat{q}^4 - 3\hat{q}^2y + 3\hat{q}^2y^2 + y^3)\} \right.$$
$$\left. + \delta'(z)K_c\frac{2\hat{q}^2}{y^2}(y^2 - \hat{q}^2)^2(1-y) \right]\cos\theta , \tag{7.11}$$

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{dy\,d\cos\theta} = \cdots + [(1+\epsilon_c)\{6(1-y)y^2\} + K_c\{-10y^3 + 2\delta(1-y)\}]\cos\theta , \tag{7.12}$$

$$\frac{1}{\Gamma_c}\frac{d\Gamma}{d\cos\theta} = \cdots + \tfrac{1}{2}(1+\epsilon_c - K_c)\cos\theta \; . \tag{7.13}$$

VIII. INCLUSIVE $H_b \to X_u e \bar{\nu}_e$ NEAR THE BOUNDARY OF PHASE SPACE

For an exclusive decay $H_b \to X e \bar{\nu}_e$ the kinematically allowed region of phase space is

$$0 < q^2 < 2E_e M_{H_b} + \frac{2M_X^2 E_e}{2E_e - M_{H_b}} . \tag{8.1}$$

The maximum value of the electron energy $E_e^{(\max)}$ occurs when the right-hand side of Eq. (8.1) is zero;

$$E_e^{(\max)} = \frac{M_{H_b}^2 - M_X^2}{2M_{H_b}} . \tag{8.2}$$

The results of Sec. V show that in QCD, the inclusive decay kinematics are governed by the quark mass m_b, rather than the hadron mass M_{H_b}. The kinematically allowed region in the Dalitz plot is

$$0 < q^2 < 2E_e m_b + \frac{2m_j^2 E_e}{2E_e - m_b} , \tag{8.3}$$

where m_j is the charm quark mass for $b \to c$ transitions and zero for $b \to u$ transitions (neglecting light quark masses). The difference between these two kinematic regions is shown in Fig. 4 for $b \to u$ decay.

For $b \to u$ transitions the region (8.3) of the q^2, E_e plane becomes the interior of a right triangle with sides $q^2 = 0$, $q^2 = 2E_e m_b$, and $E_e = m_b/2$. Since m_b is less than M_{H_b} a comparison of Eqs. (8.3) and (8.1) reveals that a part of the kinematically allowed phase space is not populated. This part corresponds to the production of states with mass squared:

$$M_{X_u}^2 \lesssim (M_{H_b} - m_b) M_{H_b} \left[1 - \frac{q^2}{M_{H_b}^2} \right] . \tag{8.4}$$
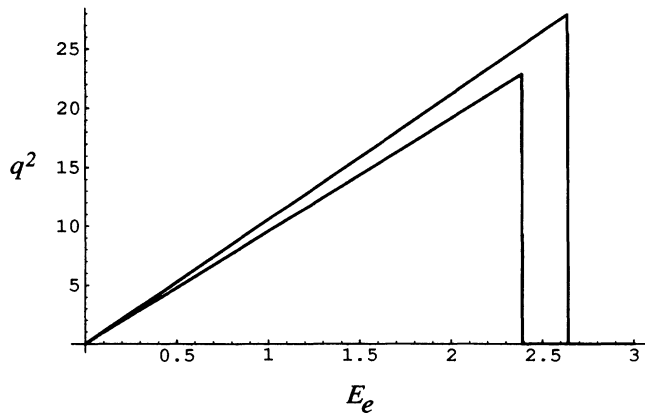

FIG. 4. The allowed region of the Dalitz plot for $B \to \bar{X}_u e \bar{\nu}_e$ decay. E_e is in GeV, and q^2 is in GeV^2. The outer curve is the region allowed using physical hadron masses. The inner curve is obtained using quark-quark masses, with $\bar{\Lambda}$ for the B meson chosen to be 500 MeV. The free-quark decay distribution is nonzero only inside the inner triangle.

The physical origin of this discrepancy has been discussed by Isgur, Scora, Grinstein, and Wise [16] and by Isgur [17]. For simplicity, consider QCD in the large-N_c limit where the final state X must be a $\bar{q}q$ bound state (nonresonant final states are produced with an amplitude suppressed by a factor of $1/\sqrt{N_c}$). Provided $q^2/M_{H_b}^2$ is not very close to unity, the production of states with masses much less than in Eq. (8.4) is strongly suppressed by hadronic form factors. For very large m_b this form-factor suppression arises from the transfer of a large momentum to the spectator antiquark by a single hard gluon. Clearly such effects are subdominant to those given by Eq. (5.8). However, when the mass of the final $\bar{q}q$ resonance becomes of order $\sqrt{m_b \Lambda_{\text{QCD}}}$ [see Eq. (8.4)], the antiquark in the X meson has such a broad distribution of momentum that no form-factor suppression is required. Thus, the differential decay rate in Eq. (5.8) corresponds to a sum over exclusive final states with masses greater than $\sim \sqrt{m_b \Lambda_{\text{QCD}}}$. Note that for large m_b these hadronic masses are much greater than the QCD scale.

There are theoretical limitations on the extent to which our prediction for the differential $H_b \to X_u e \bar{\nu}_e$ decay rate $d\Gamma/dq^2 dE_e$ can be compared with experiment. In the region of the phase space very near $q^2 = M_{H_b}^2$, which corresponds to low-mass final hadronic states recoiling at low momentum, our expression for the differential decay rate is not valid (the operator product expansion cannot be justified for low mass states). Also, the appearance in Eq. (5.8) of a δ function and its first derivative indicates that along the boundaries $q^2 = 2m_b E_e$ and $E_e = m_b/2$ the differential decay rate must be smeared over a region of electron energies. The amount of smearing necessary is determined by demanding that corrections proportional to K_b, E_b, and G_b give a contribution to the smeared differential rate that is small compared with the leading "free-quark decay" contribution.

For definiteness consider the boundary $q^2 = 2m_b E_e$. In the free $b \to u e \bar{\nu}_e$ decay it corresponds to a configuration (in the b rest frame) where the electron and u quark are moving in the same direction and the antineutrino goes in the opposite direction. Since the weak current is left handed the free-quark decay amplitude vanishes here by angular momentum conservation. Define (recall $y = 2E_e/m_b$, $\hat{q}^2 = q^2/m_b^2$)

$$S(q^2) = \int dy \frac{1}{\Gamma_b} \frac{d^2\Gamma}{dy d\hat{q}^2} W(y) , \tag{8.5}$$

where

$$W = \frac{1}{\sqrt{\pi}\epsilon} e^{-[y - (\hat{q}^2 + \epsilon)]^2/\epsilon^2} . \tag{8.6}$$

$S(q^2)$ corresponds to smearing the differential cross section at $y = \hat{q}^2 + \epsilon$ over a region of y of order ϵ. (The averaging is at $y = \hat{q}^2 + \epsilon$ instead of $y = \hat{q}^2$ so that the term containing a derivative of a δ function is not made artificially small.) Demanding that the leading free-quark decay contribution to $S(q^2)$ be large compared to that of the corrections proportional to K_b, E_b, and G_b gives the condition

$$\epsilon^2 > (\Lambda_{\rm QCD}/m_b)^2 \ . \qquad (8.7)$$

This corresponds to smearing over a range of electron energies $\Delta E_e > \Lambda_{\rm QCD}$. In deriving Eq. (8.7) the matrix elements K_b, E_b, and G_b were estimated to be of order $(\Lambda_{\rm QCD}/m_b)^2$. A similar result holds for the region near the boundary of phase space at $E_e = m_b/2$. The differential cross section in Eq. (5.8) must be smeared over a region of electron energies $\Delta E_e > \Lambda_{\rm QCD}$ to be physically meaningful. Away from the boundaries $E_e = m_b/2$ and $q^2 = 2m_b E_e$ of phase space some smearing may also be required. As one varies E_e (at fixed q^2) new thresholds are encountered corresponding to different M_X in Eq. (8.1). However, here the situation is likely to be similar to

$$R(s) = \sigma(e^+ e^- \to {\rm hadrons})/\sigma(e^+ e^- \to \mu^+ \mu^-) \ ,$$

where at large s the density of hadronic states is so large that the size of the region s that must be averaged over becomes negligibly small.

In semileptonic B decay the end-point region of electron energies, where $B \to X_c e \bar{\nu}_e$ decays are forbidden, is of great interest because of its potential use to determine $|V_{ub}|$. However, in this end-point region $2.2 < E_e < 2.6$ GeV, our predictions are not very useful. Very near $q^2 = M_B^2$ our results do not apply and away from this value of q^2 the averaging of electron energies over a region $\Delta E_e > \Lambda_{\rm QCD}$ makes it impossible to isolate the end-point part of the electron spectrum. This is discussed more quantitatively in the next section.

IX. NUMERICAL RESULTS

The $1/m_b^2$ corrections to the decay distribution for semileptonic H_b decay have been computed in terms of the parameters G_b and K_b. These parameters depend on the hadron H_b, and will be denoted by $G_b(H_b)$ and $K_b(H_b)$. The parameter G_b is the leading operator that breaks the heavy quark spin symmetry, so its matrix element can be determined in terms of the hyperfine spin splittings within heavy quark multiplets. For example, $G_b(\Lambda_b) = 0$ and

$$m_b G_b(B) = -\tfrac{3}{4}[M(B^*) - M(B)] \ ,$$
$$m_b G_b(B^*) = \tfrac{1}{4}[M(B^*) - M(B)] \ . \qquad (9.1)$$

The experimental value of 46 MeV for the $B^* - B$ mass difference determines $G_b(B)$ to be -0.0065 (where m_b can be equated with the hadron mass to this order). The parameter $K_b(B)$ cannot be simply determined. Quark model estimates suggest that $K_b(B)$ is approximately 0.01. The size of the $1/m_b^2$ correction to the lepton spectrum $d\Gamma/dy$ for $B \to X_c e \bar{\nu}_e$ decay is plotted in Fig. 5 for $G_b(B) = -0.0065$ and $K_b(B) = 0.01$. The plot shows the ratio of the distribution Eq. (5.6) to the free-quark decay spectrum without any smearing. Figure 6 shows the same result as a percentage correction, so that one can see how the corrections become large only near the end point. The results must be averaged in y over a region sufficiently large that the correction terms are small. The
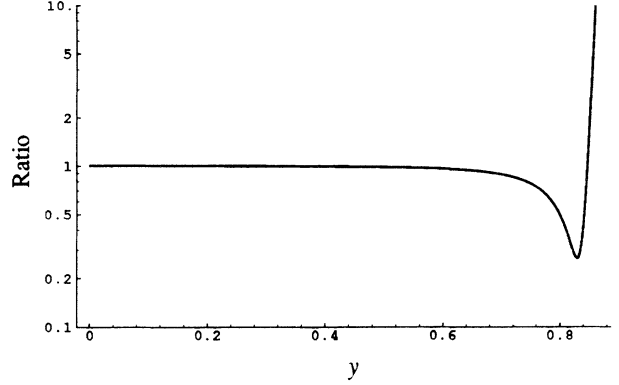

FIG. 5. The ratio of the lepton spectrum for $B \to X_c e \bar{\nu}_e$ decay including $1/m_b^2$ corrections with $G_b = -0.0065$ and $K_b = 0.01$ to the free quark $b \to c$ decay spectrum. Note the logarithmic scale.

$1/m_c^2$ corrections to charm decay are a factor of $(m_b/m_c)^2$ larger than the corrections for b decay.

To better understand the size of the $1/m_b^2$ corrections, it is interesting to plot the $B \to X_c e \bar{\nu}_e$ lepton spectrum including $1/m_b^2$ corrections without smearing. This is illustrated in Fig. 7, where the free-quark $b \to c$ decay and $b \to u$ decay electron spectra are compared to the $B \to X_c e \bar{\nu}_e$ decay spectrum including $1/m_b^2$ corrections. The peak near the end point for $B \to X_c e \bar{\nu}_e$ decay gives an indication of the minimum size of the smearing region that must be used before the QCD calculation is valid. The peak must be smeared over a large enough region that it produces a small correction to the decay spectrum. This indicates that smearing region should be at least $\Delta y = 0.2$, which corresponds to a lepton energy spread of around 500 MeV.

The dominant uncertainty in the extraction of V_{ub} is the shape of the end point of the $B \to X_u e \bar{\nu}_e$ lepton spectrum. It is difficult to estimate the size of the smearing necessary in QCD from the unsmeared spectrum, Eq. (5.10), since it contains δ-function singularities at the end point. Figure 8 shows the smeared lepton decay distribution using Gaussian smearing in y with different widths. The smearing extends the spectrum beyond the parton model end point $y = 1$, but the curves are only plotted for $0 \le y \le 1$. For simplicity, only the δ functions in Eq. (5.10) have been smeared, and then added to the remain-
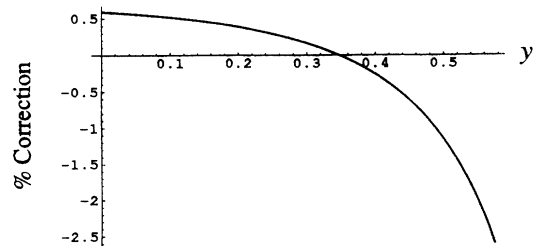

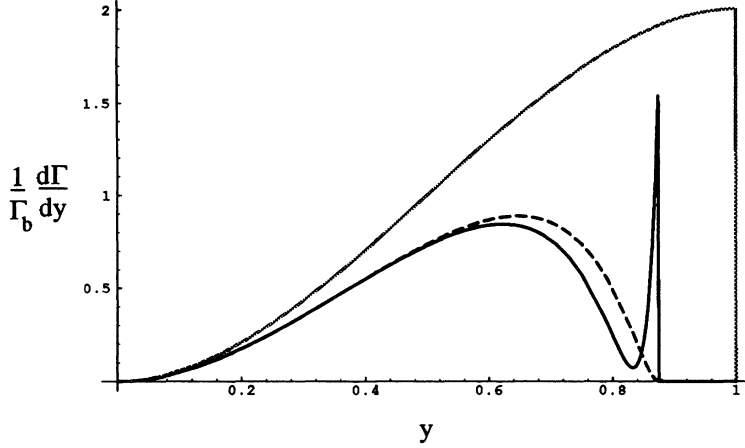FIG. 6. The same spectrum as Fig. 5, plotted as a percentage correction on a linear scale.

FIG. 7. The electron spectrum for free-quark $b \rightarrow c$ decay (dashed line), free-quark $b \rightarrow u$ decay (grey line), and $B \rightarrow X_c e \bar{\nu}_e$ decay including $1/m_b^2$ corrections (solid line) with $G_b = -0.0065$ and $K_b = 0.01$. The QCD calculation of the $1/m_b^2$ corrections has not been smeared.

ing terms. Clearly, the dip in the spectrum for a smearing width of $\Delta y = 0.1$ is unphysical. The curves in Fig. 8 indicate that the minimum smearing width in Δy is around 0.2, which corresponds to a lepton energy width of 500 MeV.

Even though the parameters $K_b(H_b)$ are not known except by model calculations, one can determine the difference of K_b between two hadrons. For example, the masses of the Λ_b baryon and the B and B^* mesons are

$$M(\Lambda_b) = m_b + \bar{\Lambda}(\Lambda_b) + m_b K_b(\Lambda_b) \ ,$$

$$M(B) = m_b + \bar{\Lambda}(B) + m_b K_b(B) + m_b G_b(B) \ , \qquad (9.2)$$

$$M(B^*) = m_b + \bar{\Lambda}(B) + m_b K_b(B) - \tfrac{1}{3} m_b G_b(B) \ ,$$

including all corrections to order $1/m_b$ in the mass. We have used the heavy quark spin-symmetry relations $G_b(B) = -3 G_b(B^*)$ and $K_b(B) = K_b(B^*)$. The difference between the Λ_b mass and the average meson mass

$$M(B)_{\text{avg}} = [M(B) + 3M(B^*)]/4$$

is

$$M(\Lambda_b) - M(B)_{\text{avg}}$$

$$= \bar{\Lambda}(\Lambda_b) - \bar{\Lambda}(B) + m_b [K_b(\Lambda_b) - K_b(B)] \ . \qquad (9.3)$$
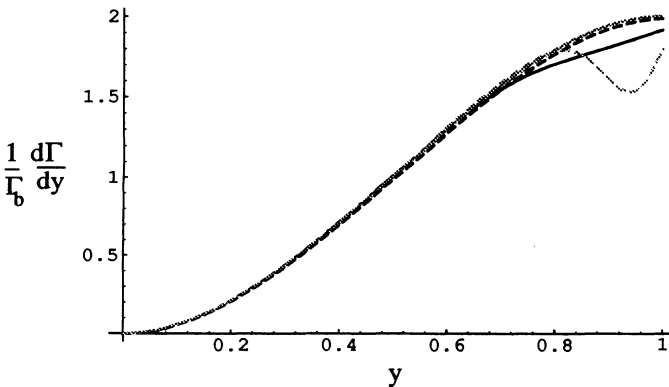
A similar expression holds for the D mesons and the Λ_c baryon. Heavy quark symmetry implies that $\bar{\Lambda}(H_c) = \bar{\Lambda}(H_b)$ and $m_b^2 K_b(H_b) = m_c^2 K_c(H_c)$, so that

$$[M(\Lambda_c) - M(D)_{\text{avg}}] - [M(\Lambda_b) - M(B)_{\text{avg}}]$$

$$= m_b \left[\frac{m_b}{m_c} - 1 \right] [K_b(\Lambda_b) - K_b(B)] \ . \qquad (9.4)$$

Equating the quark masses with the meson masses to this order gives $K_b(\Lambda_b) - K_b(B) = -0.002 \pm 0.006$, using the present value of 5641 ± 50 MeV for the Λ_b mass.

The difference between G_b and K_b for different hadrons gives predictions for the differences in the semileptonic decay distributions. The order α_s corrections due to gluon radiation that have been computed [2,3,5] are the same for all hadrons, since they correct the free-quark decay formula, and cancel in the difference. Thus, the difference is known to corrections of order $\alpha_s(m_b)/m_b$ and $1/m_b^3$. For example, one can compute the differences in the total semileptonic decay widths for $H_b \rightarrow X_u e \bar{\nu}_e$ decay:

$$\frac{\Gamma(H_b) - \Gamma(H_b')}{\Gamma(H_b) + \Gamma(H_b')} = \tfrac{3}{2} [G_b(H_b) - G_b(H_b')]$$

$$- \tfrac{1}{2} [K_b(H_b) - K_b(H_b')] \ . \qquad (9.5)$$

FIG. 8. Plot of the smeared corrections for $B \rightarrow X_u e \bar{\nu}_e$ decay for $G_b = -0.0065$ and $K_b = 0.01$. The plots are shown only in the region $0 \le y \le 1$. The curves are the free-quark spectrum (solid grey), the $1/m_b^2$ corrected spectrum smeared over $\Delta y = 0.1$ (dashed grey), 0.2 (solid black), and 0.5 (dashed black).

This gives

$$\frac{\Gamma(\Lambda_b) - \Gamma(B)}{\Gamma(B)} = \frac{9}{4} \frac{M(B^*) - M(B)}{M(B)}$$

$$- [K_b(\Lambda_b) - K_b(B)]$$

$$= 0.021 \pm 0.006 \ . \qquad (9.6)$$

Similarly, the decay width differences for $H_c \rightarrow X_d e^+ \nu_e$ decay are

$$\frac{\Gamma(\Lambda_c) - \Gamma(D)}{\Gamma(D)} = \frac{9}{4} \frac{M(D^*) - M(D)}{M(D)}$$

$$- [K_c(\Lambda_c) - K_c(D)]$$

$$= 0.19 \pm 0.04 \ . \qquad (9.7)$$

The uncertainties in Eqs. (9.6) and (9.7) are due to the ± 50 MeV uncertainty in the Λ_b mass. The differences of the decay widths for the decay modes $H_b \rightarrow X_c e \bar{\nu}_e$ and $H_c \rightarrow X_s e^+ \nu_e$ can be computed similarly using Eqs. (5.11) and (7.4). In the $b \rightarrow c$ case, $[\Gamma(\Lambda_b) - \Gamma(B)]/\Gamma(B) = 0.029 \pm 0.004$.

X. CONCLUSIONS

One of the important points of the analysis in this paper is that the decay distribution is determined by the quark mass m_b, rather than the hadron mass M_{H_b}. Thus, the free-quark decay rate Γ_b depends on m_b^5, not $M_{H_b}^5$. The difference between the two masses is the $\bar{\Lambda}$ parameter of the heavy quark theory. There should have been $1/m_b$ corrections proportional to $\bar{\Lambda}$ if a free-quark decay model with the decay rate given by M_{H_b} was appropriate. Corrections of this form are absent in QCD. For $b \rightarrow c e \bar{\nu}_e$ decays, the charm quark mass m_c also appears. To the order we are working, this is the pole mass in the heavy c-quark propagator, or equivalently, the mass in the phase factor that relates the c-quark field in QCD to the c-quark field in the heavy quark effective theory. Up to corrections of order $1/m_{b,c}$, $m_b - m_c = M_B - M_D$. This connection is important because it reduces the uncertainty in the differential decay rate.

The end-point region of the inclusive lepton spectrum in semileptonic b decay is important for the extraction of V_{ub}. The QCD calculation near the end point must be smeared over a large region (around 500 MeV) before it can be compared to experiment. This region is larger than the difference in the end points of the $b \rightarrow u$ and $b \rightarrow c$ decays. This means that the extraction of V_{ub} still requires modeling the end point. The QCD computation does provide some constraints on the model. Any model which has $1/m_b$ corrections, or has a decay distribution given by hadron kinematics instead of quark kinematics, is in contradiction with QCD.

We have presented results for decays of hadrons containing a b or c quark. Formally, perturbative corrections are of the form $\alpha_s(m_b)$ or $\alpha_s(m_c)$. However, since the final state in free-quark decay is three-body, perturbative corrections to the total semileptonic decay rate may be better represented by $\alpha_s(m_b/3)$ or $\alpha_s(m_c/3)$. Only a higher-order perturbative calculation can resolve this issue. Because of this, we do not have confidence in applying the results of Sec. VII to the decays of hadrons containing a c quark.

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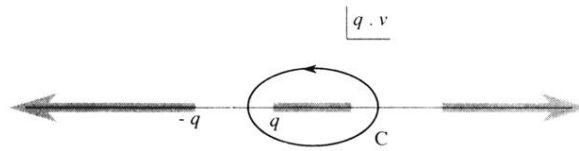[19] T. Mannel, Darmstadt Report No. IKDA-93/26 (unpublished).

FIG. 1. The analytic structure of $T^{\mu\nu}$ in the complex $q \cdot v$ plane for fixed timelike q^2. The contour C encircles the cut relevant for semileptonic b decay. The other cuts extend to infinity, and correspond to other physical processes. Here $q \equiv \sqrt{q^2}$.