Negative energy densities in extended sources generating closed timelike curves in general relativity with and without torsion

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Near a spinning point particle in (2+1)-dimensional gravity (or near an infinitely thin, straight, spinning string in 3+1 dimensions) there is a region of space-time with closed timelike curves. Exact solutions for extended sources with apparently physically acceptable energy-momentum tensors have produced the same exterior space-time structure. Here it is pointed out that, in the case with torsion, closed timelike curves appear only for spin densities so high that the spin energy density is higher than the net effective energy density. In models without torsion, the presence of closed timelike curves is related to a heat flow of unphysical magnitude. This corroborates earlier arguments against the possibility of closed timelike curves in space-time geometries generated by physical sources.

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It has been conjectured [1] that physically acceptable sources together with reasonable boundary conditions produce space-times without causality violation in (2+1)-dimensional gravity, and, in the case of spinless point particles, it has been proved that closed timelike curves (CTC's) cannot be realized by physical timelike sources [2–5]. The appearance of CTC's near a *spinning* point particle in (2+1)-dimensional gravity [1,6] (or a spinning string in 3+1 dimensions [7]) could be attributed to a torsion singularity at the source [8]. Hence, in this case one could argue that the causality violation is due to a point (or line) singularity which by itself seems unphysical.

The singularity argument against these time machines could be removed if one could find an extended source for this space-time without losing the CTC's. Within the Einstein-Cartan theory [9], such a model has been constructed [10]. It represents an infinite, straight (3+1)-dimensional string with spin polarization along its axis. According to the Einstein-Cartan theory, the gravitational effect of spin is torsion. Although torsion does not propagate in vacuum in this theory, the matching conditions at the surface of a medium with torsion lead to distant gravitational effects. Thus, spin is known to produce exterior gravitational fields similar to those induced by orbital angular momentum [11]. The spinning string problem has been studied within general relativity without torsion, in which case spin polarization is replaced with orbital angular momentum [12] (see also the related system in Ref. [13]), and it was claimed that these constructions were examples of time machines that respect the energy conditions, but according to a recent theorem by Menotti and Seminara [14] a torsionless, stationary, and rotational symmetric extended source would have to violate the weak energy condition in order to produce CTC's. In this Brief Report this contradiction is resolved by a careful reexamination of the models in Refs. [10,12,13], and it will be shown that the sources are indeed unphysical when CTC's exist. Thus, the present study corroborates the earlier results of Refs. [2–5,14] for Einstein's theory and indicates an extension to the Einstein-Cartan theory.

The geometry of the model is specified by the orthonormal one-forms (tetrad frame)

$$\begin{aligned} \omega^0 &= dt + M d\phi \, , \\ \omega^1 &= dr \, , \\ \omega^2 &= \rho d\phi \, , \\ \omega^3 &= dz. \end{aligned} \tag{1}$$

M and ρ are functions of r. The metric in the coordinate frame is found by using that $ds^2 = \eta_{\mu\nu}\omega^\mu\omega^\nu$. When torsion is polarized along the z axis, we get only one nonvanishing component of the torsion tensor: namely,

$$T^0{}_{12} = \sigma. \tag{2}$$

Defining

$$\Omega \equiv -\frac{1}{2}\sigma + \frac{M'}{2\rho}, \tag{3}$$

the Einstein tensor takes the form [10]

$$\begin{aligned} G^0{}_0 &= -3\Omega^2 - \sigma\Omega + \rho''/\rho \, , \\ G^1{}_1 &= G^2{}_2 = \Omega^2 \, , \\ G^3{}_3 &= -\Omega^2 - \sigma\Omega + \rho''/\rho \, , \\ G^0{}_2 &= -\Omega'. \end{aligned} \tag{4}$$

If

$$\rho = \frac{\sin(\sqrt{\lambda} r)}{\sqrt{\lambda}}, \tag{5}$$

and torsion is nonzero and constant throughout the source, one can integrate the field equations assuming that both the heat flow $G^0{}_2$ and the radial pressure $G^1{}_1$ vanish. Then, using units such that $8\pi G = 1$, the energy-momentum tensor takes the form $T_{\mu\nu} = \mathrm{diag}[\lambda, 0, 0, -\lambda]$ which satisfies the weak energy condition by construction.

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This model can be matched to a torsionless vacuum characterized by the orthonormal frame

$$\begin{aligned} \theta^0 &= dt + \tfrac{j}{2\pi} d\phi \,, \\ \theta^1 &= dr \,, \\ \theta^2 &= \left(1 - \tfrac{\mu}{2\pi}\right)(r + r_0)\, d\phi \,, \\ \theta^3 &= dz. \end{aligned} \tag{6}$$

The parameters in the exterior metric are determined by the Arkuszewski-Kopczyński-Ponomariev junction conditions [15] appropriate for the Einstein-Cartan theory. The result is (see Ref. [10] for details)

$$\begin{aligned} j &= M(R) \,, \\ \mu &= 2\pi\left(1 - \cos(\sqrt{\lambda}R)\right) \,, \\ \sqrt{\lambda}r_0 &= \tan(\sqrt{\lambda}R) - \sqrt{\lambda}R. \end{aligned} \tag{7}$$

Here R is the surface radius of the spinning string. μ is the angle deficit in the exterior geometry, and, in the present units, it is also the mass per length of the string. j is the angular momentum per length. For realistic strings, with $\mu \ll 2\pi$, CTC's can exist only for unrealistic spin densities [10]. In general, closed timelike curves appear at the surface of the string provided $M(R)^2 > \rho(R)^2$ because ϕ is then a timelike coordinate. By use of Eqs. (5) and (7), this condition can be written

$$\left(1 - \frac{\mu}{2\pi}\right)^2 > 1 - \left(\frac{j}{2\pi}\right)^2 \lambda, \tag{8}$$

which, due to the relation $j = \sigma/\lambda\mu$, takes the simple form

$$\mu > 4\pi\left(1 + \sigma^2/\lambda\right)^{-1}. \tag{9}$$

But $\mu < 2\pi$, in a conical universe. This leads to the requirement that

$$\sigma^2 > \lambda. \tag{10}$$

Thus, in order that the spinning string in the Einstein-Cartan model induce a causality-violating space-time, the spin energy density σ^2 has to be larger than the net effective energy density λ. If the effective energy density λ is a sum of the spin energy and a "bare" energy, it follows that this "bare" energy density has to be negative.

In the case of vanishing torsion, it is impossible to avoid a heat flow. To generate CTC's, the source has to produce a nonvanishing $M(r)$, and to avoid a torsion singularity at the origin, $M(0) = 0$. It follows that $M(r)$ is not a constant, and by the definition (3), and that $M'(r)$ and $\Omega(r) \neq 0$. Now, to match smoothly to the exterior vacuum, the radial pressure at the surface has to vanish. Since by the form of the Einstein tensor (4), $p_r = G^1{}_1 = \Omega^2$, vanishing surface pressure implies that $\Omega(R) = 0$. Because Ω must be zero at one point and nonzero elsewhere, $\Omega'(r) \neq 0$. In Refs. [12,13], it was argued that by taking $\rho(r)$ as the same function as in the torsion case, cf. Eq. (5), the energy density becomes positive definite by construction. Then Ω' was taken to be a constant of magnitude equal or less than the minimal energy density of the model. It then followed from Einstein's field equations that

$$T^{00} \geq |T^{\mu i}|, \tag{11}$$

and it was assumed that the dominant energy condition was satisfied. This is wrong. Indeed, consider an energy-momentum tensor of the form

$$T = \begin{bmatrix} \rho & q \\ q & p \end{bmatrix}, \tag{12}$$

where ρ is the energy density, q is the heat flow, and p is the pressure, and where the spatial dimensions orthogonal to the heat flow have been suppressed. Boost it by use of the Lorentz transformation

$$L = \begin{bmatrix} \gamma & -u\gamma \\ -u\gamma & \gamma \end{bmatrix}. \tag{13}$$

Then one finds (in the limit $u = 1$) that the energy density in the new frame is given as

$$\rho' = (\rho - 2q + p)\gamma^2. \tag{14}$$

Hence, the weak energy condition is violated unless the magnitude of the heat flow, q, is everywhere equal or less than half the sum of energy density and pressure. Thus, the time machines of Refs. [12,13] require an energy-momentum tensor that corresponds to a negative energy density in some frames, and these solutions are therefore not in conflict with the general result that the energy conditions protect against CTC's [14].

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