BRIEF REPORTS

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Topologically massive gravity with a two-fluid source

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A solution is presented for the Einstein-Cotton field equations of topologically massive gravity where two separate fluids act as the source. Vorticity and heat flow are present, and a discussion of the thermodynamics demonstrates that the temperature and coefficient of thermal conductivity are well defined and positive and that the Gibbs relation is satisfied.

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In $(2+1)$-dimensional general relativity, the gravitational field is locally determined by the matter sources, and so gravitational excitations are absent. Because topologically massive gravity (TMG) [1,2] does *not* suffer from this shortcoming, it represents a more realistic $2+1$ analogue of the usual higher-dimensional general relativity and has consequently attracted considerable attention [3,4]. In the case of TMG, an SO(1,2) Chern-Simons term is added to the action so that variation with respect to the metric yields the Einstein-Cotton field equations

$$G^\mu_\nu + \kappa^{-1} C^\mu_\nu = 2\pi G T^\mu_\nu \ ,$$

where the constant κ has the dimension of mass and where

$$C^\mu_\nu := \epsilon^{\mu\lambda\sigma} D_\lambda (R_{\sigma\nu} - \tfrac{1}{4} g_{\sigma\nu} R)$$

is known as the *Cotton* (or *Bach*) tensor [5] and is analogous to the Weyl tensor of $3+1$ dimensions: $C_{\mu\nu}$ is symmetric, traceless, identically zero if and only if the space-time is conformally flat, and has zero covariant divergence. Gravity in $2+1$ dimensions has no Newtonian analogue, and so there are no restrictions on the constant on the right side of the Einstein-Cotton equations. It has been written as $2\pi G$ to agree with the convention of Jackiw [2].

The intent of this Brief Report is to seek solutions of the above field equations with the stress-energy tensor taking the (imperfect) fluid form

$$T^\mu_\nu = (\rho + p) u^\mu u_\nu + p \delta^\mu_\nu + q^\mu u_\nu + u^\mu q_\nu - 2\eta \sigma^\mu_\nu \ .$$

Since rotation is present in many TMG solutions [3], the starting point will be the combed hedgehog metric [6]

$$ds^2 = -\cos 2\alpha dt^2 - 2r \sin 2\alpha dt d\theta + dr^2 + r^2 \cos 2\alpha d\theta^2 ,$$

where $\alpha = \alpha(r)$ is to be chosen so that the field equations are satisfied. The following ansatz for the velocity vector is convenient [6] and consistent with the requirement $u^\mu u_\mu = -1$:

$$u^t = -u_t = \cos\alpha, \quad u^r = u_r = 0, \quad u^\theta = -r^{-2} u_\theta = r^{-1} \sin\alpha \ .$$

It follows that the heat flux vector is given by

$$q^t = q_t = -Q \sin\alpha, \quad q^\theta = r^{-2} q_\theta = r^{-1} Q \cos\alpha \ ,$$

with q^r and Q still to be determined. The only mixed components of the shear tensor not identically zero are

$$\sigma^t_r = \sigma^r_t = -\tfrac{1}{2} \sin\alpha [\partial_r \alpha - (2r)^{-1} \sin 2\alpha] \ ,$$

$$\sigma^r_\theta = r^2 \sigma^\theta_r = \tfrac{1}{2} r \cos\alpha [\partial_r \alpha - (2r)^{-1} \sin 2\alpha] \ ,$$

and the scalar vorticity and scalar curvature are

$$\omega = \tfrac{1}{2} [\partial_r \alpha + (2r)^{-1} \sin 2\alpha] \ , \ R = 2[(\partial_r \alpha)^2 + (2r)^{-2} \sin^2 2\alpha] \ .$$

The scalar expansion $\theta := u^\mu_{;\mu}$ is zero and so the isotropic and thermodynamic pressures become identical. Both will be denoted by p. Using the abbreviations

$$\Delta := \partial_r^2 \alpha + r^{-1} \partial_r \alpha - (2r^2)^{-1} \sin 2\alpha \cos 2\alpha \ ,$$

$$\Phi := 2r^{-2} \sin^2 2\alpha + R \ ,$$

the only nonzero mixed components of the Cotton tensor are

$$C^t_t = \cos 2\alpha [r^{-1} \partial_r (r\Delta) + \Phi \partial_r \alpha] + 2r^{-1} \sin 2\alpha (\partial_r \alpha)^2 \ ,$$

$$C^t_\theta + r^2 C^\theta_t = 2 \sin 2\alpha [r(\partial_r \Delta + \Phi \partial_r \alpha) + \tfrac{1}{2} \Delta] \ ,$$

$$C^t_\theta - r^2 C^\theta_t = 3[2 \sin 2\alpha (\Delta - r^{-1} \partial_r \alpha) + R \cos 2\alpha] \ ,$$

$$C^r_r = -r^{-1} [\cos 2\alpha \Delta + 4 \sin 2\alpha (\partial_r \alpha)^2] \ ,$$

$$C^\theta_\theta = -\cos 2\alpha [\partial_r \Delta + \Phi \partial_r \alpha] + 2r^{-1} \sin 2\alpha (\partial_r \alpha)^2 \ .$$

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If the $G_{\mu\nu}$ Einstein term were absent from the field equations, then the Cotton tensor being traceless would imply a traceless source term on the right-hand side of the Cotton equations—a property that is characteristic of the stress-energy tensor of a *radiative* (null) fluid. With this in mind, consider a two-fluid model [7,8] with

$$T_{\mu\nu}=T^{(m)}_{\mu\nu}+T^{(r)}_{\mu\nu} ,$$
$$\rho=\rho_m+\rho_r ,$$
$$p=p_m+p_r ,$$

where $T^{(m)}_{\mu\nu}$ and $T^{(r)}_{\mu\nu}$ refer, respectively, to the matter content and the radiation content of the spacetime. It will be assumed that the Einstein-Cotton equations can be split into two separate parts: namely,

$$\kappa^{-1}C^{\mu}_{\nu}= 2\pi G[(\rho_r+p_r)u^{\mu}u_{\nu}+p_r\delta^{\mu}_{\nu}+q^{\mu}u_{\nu}+u^{\mu}q_{\nu}-2\eta\sigma^{\mu}_{\nu}]$$

and a similar equation for G^{μ}_{ν} but with r subscripts replaced by m subscripts. Since C^{μ}_{ν} and σ^{μ}_{ν} are traceless and $q^{\mu}u_{\mu}=0$, it follows, as anticipated, that $\rho_r=2p_r$, which is the equation of state for a radiative fluid in $2+1$ dimensions [9]. The equation for $\kappa^{-1}C^{r}_{r}$ gives

$$p_r=(2\pi G\kappa r)^{-1}[\cos 2\alpha\Delta+4\sin 2\alpha(\partial_r\alpha)^2]$$

which, combined with the equation for $\kappa^{-1}(C^{t}_{t}-C^{\theta}_{\theta})$, leads to

$$r^{-1}(1-3\cos 2\alpha)\Delta+2\partial_r\Delta+4(\partial_r\alpha-\tfrac{5}{2}r^{-1}\sin 2\alpha)\times[\partial_r\alpha-(2r)^{-1}\sin 2\alpha]\partial_r\alpha=0 .$$

Although it would be difficult to find a *general* solution for this equation, a special solution can be found by noting that the equation is solved by $\partial_r\alpha=\frac{1}{2}r^{-1}\sin 2\alpha$ (whence $\Delta=0$) which implies $\alpha=\tan^{-1}(r/a)$, where a is an arbitrary constant. The metric is now completely determined,

$$ds^2=-\left[\frac{a^2-r^2}{a^2+r^2}\right]dt^2-\left[\frac{4ar^2}{a^2+r^2}\right]dt\,d\theta+dr^2+r^2\left[\frac{a^2-r^2}{a^2+r^2}\right]d\theta^2 ,$$

and can be shown to satisfy the remaining equations for C^{μ}_{ν} provided ρ_r, p_r, Q and q^{μ} are chosen as

$$\rho_r=2p_r=\frac{-8a^3}{\pi G\kappa(a^2+r^2)^3},\quad Q=\frac{-6a^2r}{\pi G\kappa(a^2+r^2)^3} ,$$
$$(q^t,q^r,q^\theta)=Q(-\sin\alpha,0,r^{-1}\cos\alpha) .$$

(The shear tensor σ^{μ}_{ν} is zero.) Jackiw [2] has pointed out that the constant κ can take either sign. The energy density ρ_r and the pressure p_r must be positive in order to be physically reasonable. Thus the constants a and κ are required to take opposite signs: $a\kappa<0$ (with the assumption $G>0$). To be specific, a will be taken as positive and κ will be taken as negative.

The G^{μ}_{ν} equations must now be checked. However, the above metric is precisely the *combed hedgehog* metric [6] which is known to be a perfect fluid solution satisfying the equations

$$G^{\mu}_{\nu}=2\pi G[(\rho_m+p_m)u^{\mu}u_{\nu}+p_m\delta^{\mu}_{\nu}] ,$$

with

$$\rho_m=3p_m=\frac{3a^2}{\pi G(a^2+r^2)^2} .$$

The mass density ρ_m and the pressure p_m are positive everywhere. It follows that the metric satisfies the full Einstein-Cotton equations

$$G_{\mu\nu}+\kappa^{-1}C_{\mu\nu}=2\pi G(T^{(m)}_{\mu\nu}+T^{(r)}_{\mu\nu}) ,$$

where $T^{(m)}_{\mu\nu}$ refers to a perfect fluid and $T^{(r)}_{\mu\nu}$ to a radiative fluid with heat flow, and where $G>0$, $\kappa<0$. The scalar vorticity and scalar curvature are

$$\omega=a(a^2+r^2)^{-1} ,$$
$$R=\tfrac{1}{3}\Phi=4a^2(a^2+r^2)^{-2} .$$

In addition to the role it plays in the field equations, the Cotton tensor also acts as a $(2+1)$-dimensional "Weyl tensor" for the purpose of determining Petrov classifications. The matrix $\|C^{\mu}_{\nu}\|$ has three distinct eigenvalues, $1\pm 3(1-r^2/a^2)^{1/2}, -2$. In the terminology of Barrow, Burd, and Lancaster [10], the metric is of Petrov class A.

The above two-fluid solution clearly satisfies the weak energy condition, $\rho>0$, and the strong energy condition, which reduces to $p>0$ in $2+1$ dimensions (Ref. [10], p. 560). For an imperfect fluid solution, it is also necessary to check that the various thermodynamic quantities, such as temperature T and coefficient of thermal conductivity χ, are positive and that the Gibbs relation leads to a well-defined entropy [7,11]. The temperature gradient law [11]

$$q^{\mu}=-\chi h^{\mu\nu}(T_{,\nu}+T\dot{u}_{\nu})$$

does *not* lead to a well-defined temperature if the temperature T is assumed to be a function only of r. However, a stationary metric can give rise to a time-dependent T, as demonstrated by Rebouças and Tiomno [12] for inhomogeneous Gödel-like spacetimes. With $T=T(r,t)$, the temperature gradient law implies

$$T=f(t)a(a^2+r^2)^{-1/2} ,$$
$$\chi=-6a[\pi G\kappa(a^2+r^2)\partial_t f]^{-1}$$

for any continuous choice of $f=f(t)$ with $\partial_t f\neq 0$. Positive T and χ are ensured by requiring $f(t)>0$ and $\partial_t f>0$ (since $\kappa<0$). Letting n denote the particle density (of the matter), it can be verified that the baryon conservation law, $(nu^{\mu})_{;\mu}=0$, is satisfied for any $n=n(r)$. The interest in TMG stems partly from its use as a toy model for $3+1$ gravity. Thus it is reasonable to consider the Gibbs relation in a form that is analogous to the Gibbs relation for two-fluid matter-radiation models in $3+1$ dimensions [7]:

$$d\left[\frac{S}{n}\right]=\frac{1}{T}\left[d\left[\frac{\rho_m}{n}\right]+p_m d\left[\frac{1}{n}\right]\right] ,$$

where the density and pressure are for the *matter* only. Since T is a function of t, the right side of the Gibbs relation is time dependent, but has no dt differential term. It follows that both sides of the equation must be zero. Thus $S \propto n$ and $d(\rho_m/n)+p_m d(1/n)=0$. The latter equation leads to

$$n = A(a^2/\pi G)^{3/4}(a^2+r^2)^{-3/2} ,$$

where A is a constant. Defining $S^\mu := Su^\mu + T^{-1}q^\mu$, the condition $\chi \geq 0$ guarantees that entropy production $S^\mu_{;\mu}$ is non-negative [11]. This is also easily checked by direct computation:

$$S^\mu_{;\mu} = \frac{-6ar^2\partial_t f}{\pi G\kappa(a^2+r^2)^3 f^2} \geq 0 .$$

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