Monopole vector spherical harmonics

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(Received 11 August 1993)

Eigenfunctions of total angular momentum for a charged vector field interacting with a magnetic monopole are constructed and their properties studied. In general, these eigenfunctions can be obtained by applying vector operators to the monopole spherical harmonics in a manner similar to that often used for the construction of the ordinary vector spherical harmonics. This construction fails for the harmonics with the minimum allowed angular momentum. These latter form a set of vector fields with vanishing covariant curl and covariant divergence, whose number can be determined by an index theorem.

PACS number(s): 14.80.Hv, 03.65.Ge

I. INTRODUCTION

When analyzing fields in a spherically symmetric background, it is often useful to expand the field in eigenfunctions of angular momentum. In most cases these are simply the spherical harmonics for scalar fields, while for fields of higher spin one can construct spinor, vector, or higher harmonics. The situation is somewhat more complicated when one considers charged fields in the background of a magnetic monopole. Superficially, this is because the electromagnetic vector potential is not manifestly spherically symmetric, even though the corresponding magnetic field is. At a deeper level, it is a consequence of the extra angular momentum associated with a charge-monopole pair.

The monopole analogues of the ordinary spherical harmonics were first constructed by Tamm [1] in the context of determining the wave function of an electron in the field of a magnetic monopole. The subject was revisited by Wu and Yang [2], whose conventions and notation I follow. Olsen, Oslund, and Wu [3] obtained monopole vector harmonics from the scalar harmonics by utilizing Clebsch-Gordan technology. In this paper I also study monopole vector harmonics, but from a somewhat different approach. Rather than using Clebsch-Gordan coefficients, I construct the vector harmonics (apart from an exceptional case described below) by applying vector differential operators to the scalar harmonics, in analogy with the construction often used [4] for the ordinary ($q=0$) vector harmonics. Not only are the resulting expressions simpler, but they are also more convenient for use in further calculations. In addition, with this approach the vector harmonics can be chosen to be eigenfunctions of the radial component of the spin, rather than of the magnitude of the orbital angular momentum, as was done in Ref. [3]. With this choice of basis, the expressions for the curls and divergences of the vector harmonics are easily derived and take particularly simple forms. There is also a natural separation between radial and transverse vectors, making this choice particularly useful for studying fields in spherically symmetric but curved spacetimes, such as one encounters when studying fields about magnetically charged black holes.

I consider fields with electric charge e in the present of a monopole with magnetic charge q/e; the Dirac quantization condition restricts q to integer or half-integer values.[1] The corresponding scalar monopole spherical harmonics $Y_{qLM}(\theta,\phi)$ are eigenfunctions of $\mathbf{L}^2$ and L_z, where

$$\mathbf{L}=\mathbf{r}\times(\mathbf{p}-e\,\mathbf{A})-q\hat{\mathbf{r}} \tag{1.1}$$

with $\hat{\mathbf{r}}=\mathbf{r}/r$. The first term on the right-hand side is the usual orbital angular momentum, while the second is the extra charge-monopole angular momentum. These are orthogonal, so classically $|\mathbf{L}|\geq q$. Correspondingly, although the quantum numbers L and M have their usual meaning, the minimum value of L is not zero, but q.

The monopole vector spherical harmonics are eigenfunctions of $\mathbf{J}^2$ and J_z, where the total angular momentum $\mathbf{J}$ is the sum of $\mathbf{L}$ and the spin angular momentum $\mathbf{S}$. By the usual rules for adding angular momenta, one sees that the total angular momentum quantum number J has a minimum value of $q-1$, except for the two cases $q=0$ and $q=\frac{1}{2}$ (where $J_{\min}=q$). The vector harmonics with $J\geq q$ can be constructed by applying vector operators to the scalar harmonics. The vector harmonics with the minimum allowed angular momentum, $J=q-1$, cannot be constructed in this manner (essentially, because there are no scalar harmonics with $L<q$) and so must be treated specially. However, it turns out that these latter span the space of vectors whose covariant curl and covariant divergence both vanish. An index theorem shows that this space has dimension $2q-1$, just as one would expect for a multiplet with angular momentum $q-1$.

The remainder of this paper is organized as follows. In Sec. II the scalar monopole harmonics are reviewed and some general properties of the vector harmonics are derived. The construction of the vector harmonics with $J\geq q$ is described in Sec. III, where some properties of these harmonics are derived. The exceptional case $J=q-1$ is discussed in Sec. IV. The relationship be-

[1]Throughout this paper it will be assumed that $q\geq 0$; the extension of the analysis to negative values of q is straightforward.

tween these harmonics and those of Ref. [3] is given in an Appendix.

II. GENERAL CONSIDERATIONS

In the absence of spin, the angular momentum operator in the presence of a magnetic monopole may be written as

$$\mathbf{L} = -i\mathbf{r}\times\mathbf{D} - q\hat{\mathbf{r}} , \tag{2.1}$$

where $\mathbf{D} = \nabla - ie\,\mathbf{A}$ is the gauge covariant derivative. One can readily verify the commutation relations $[L_i, r_j] = i\epsilon_{ijk} r_k$ and $[L_i, D_j] = i\epsilon_{ijk} D_k$. It follows that any vector constructed from $\mathbf{r}$ and $\mathbf{D}$ will obey $[L_i, v_j] = i\epsilon_{ijk} v_k$ and, in particular, that $\mathbf{L}$ satisfies the usual angular momentum commutation relation $[L_i, L_j] = i\epsilon_{ijk} L_k$. It also obeys the useful identity

$$\tilde{\mathbf{D}}^2 = -\frac{1}{r^2}(\mathbf{L}^2 - q^2) , \tag{2.2}$$

where

$$\tilde{\mathbf{D}} \equiv \mathbf{D} - \hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{D} \tag{2.3}$$

is the purely angular part of the covariant derivative.

The monopole spherical harmonics $Y_{qLM}(\theta,\phi)$ obey

$$\begin{aligned} \mathbf{L}^2 Y_{qLM} &= L(L+1) Y_{qLM} , \\ L_z Y_{qLM} &= M Y_{qLM} , \end{aligned} \tag{2.4}$$

where L and M can take on the values

$$\begin{aligned} L &= q, q+1, \ldots , \\ M &= -L, -L+1, \ldots, L . \end{aligned} \tag{2.5}$$

When Eq. (2.4) is solved to give an explicit expression for these harmonics, one finds that they possess singularities that coincide with the Dirac string [5] of the monopole. However, as was pointed out by Wu and Yang [2], the harmonics, are in fact nonsingular, provided that they are viewed as sections rather than as ordinary functions. In this approach, one divides the space outside of the monopole into two overlapping regions. For each region one makes a choice of the vector potential, and thus of the monopole harmonics, which is nonsingular within that region. In the overlap of the two regions the two vector potentials, and the two sets of monopole harmonics, are related by a nonsingular gauge transformation characterized by q. The explicit form of the harmonics depends on the choice of gauge and of the two regions. Expressions for a particularly convenient choice are given in Ref. [2], although we will not need these in this paper.

The Y_{qLM} are orthonormal, with

$$\begin{aligned} \int d\Omega\, Y^*_{qLM} Y_{qL'M'} &\equiv \int_0^\pi d\theta \int_0^{2\pi} d\phi\, Y^*_{qLM} Y_{qL'M'} \\ &= \delta_{LL'}\delta_{MM'} . \end{aligned} \tag{2.6}$$

Further, they form a complete set, in that any smooth section can be expanded as a linear combination of monopole harmonics.

The monopole vector spherical harmonics are eigenfunctions of $\mathbf{J}^2$ and J_z. The allowed values of the total angular momentum quantum number J are $q-1, q, \ldots,$ except in the two cases $q=0$ and $q=\frac{1}{2}$ where $J=q-1$ is absent. In general, there is more than one way to obtain a given value of J, and thus several multiplets of harmonics with the same total angular momentum. If, as was done in Ref. [3], the harmonics are chosen to be eigenfunctions of $\mathbf{L}^2$, as well as of $\mathbf{J}^2$ and J_z, the multiplet structure is[2] the following.

For $J = q-1 \ge 0$, one multiplet, with $L = J+1$.
For $J = q > 0$, two multiplets, with $L = J+1$ and $L = J$.
For $J = q = 0$, one multiplet, with $L = 1$.
For $J > q$, three multiplets, with $L = J+1$, $L = J$, and $L = J-1$.

An alternative approach, which I follow in this paper, is to classify the multiplets by the eigenvalue of $\hat{\mathbf{r}}\cdot\mathbf{S}$, which will be denoted λ. In general, λ can take on the values 1, 0, and -1. However, it is further restricted by the requirement that

$$\hat{\mathbf{r}}\cdot\mathbf{J} = \hat{\mathbf{r}}\cdot\mathbf{L} + \hat{\mathbf{r}}\cdot\mathbf{S} = -q + \lambda \tag{2.7}$$

lie in the range $-J$ to J. This gives the following.
For $J = q-1 \ge 0$, one multiplet, with $\lambda = 1$.
For $J = q > 0$, two multiplets, with $\lambda = 1$ and 0.
For $J = q = 0$, one multiplet, with $\lambda = 0$.
For $J > q$, three multiplets, with $\lambda = 1$, 0, and -1.

Thus, let us denote the vector harmonics by $\mathbf{C}^{(\lambda)}_{qJM}$, with

$$\begin{aligned} \mathbf{J}^2 \mathbf{C}^{(\lambda)}_{qJM} &= J(J+1)\mathbf{C}^{(\lambda)}_{qJM} , \\ J_z \mathbf{C}^{(\lambda)}_{qJM} &= M \mathbf{C}^{(\lambda)}_{qJM} , \\ \hat{\mathbf{r}}\cdot\mathbf{S}\, \mathbf{C}^{(\lambda)}_{qJM} &= \lambda \mathbf{C}^{(\lambda)}_{qJM} . \end{aligned} \tag{2.8}$$

Because the spin matrices $(S^k)_{ij} = -i\epsilon_{ijk}$, the last of these equations is equivalent to

$$\hat{\mathbf{r}}\times\mathbf{C}^{(\lambda)}_{qJM} = -i\lambda\mathbf{C}^{(\lambda)}_{qJM} . \tag{2.9}$$

From this we see that the $\lambda=0$ harmonics are purely radial vectors, while those with $\lambda=\pm 1$ are transverse. It also follows that

$$\begin{aligned} (\lambda'-\lambda)\mathbf{C}^{(\lambda)*}_{qJM}\cdot\mathbf{C}^{(\lambda')}_{qJ'M'} &= (i\hat{\mathbf{r}}\times\mathbf{C}^{(\lambda)*}_{qJM})\cdot\mathbf{C}^{(\lambda')}_{qJ'M'} \\ &\quad + i\mathbf{C}^{(\lambda)*}_{qJM}\cdot(\hat{\mathbf{r}}\times\mathbf{C}^{(\lambda')}_{qJ'M'}) \\ &= 0 \end{aligned} \tag{2.10}$$

so that any two vector harmonics with different values of λ are orthogonal as vectors at every point.

Furthermore, the usual methods can be used to show that any two harmonics with different values of J, M, or λ are orthogonal in the functional sense. It will be convenient to normalize them so that

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[2]Note that a $J=0$ mode occurs only for $q=0$ and $q=1$. Thus, for any other value of q it is impossible to construct a spherically symmetric configuration involving a charged vector field. In the context of a spontaneously broken SU(2) gauge theory, this explains why nonsingular spherically symmetric monopole configurations are possible only for unit magnetic charge [6].

$$\int d\Omega\, \mathbf{C}^{(\lambda)*}_{qJM}\cdot\mathbf{C}^{(\lambda')}_{qJ'M'}=\frac{\delta_{JJ'}\delta_{MM'}\delta_{\lambda\lambda'}}{r^2}\ . \tag{2.11}$$

With this normalization, the vector harmonics are homogeneous of degree -1 in the Cartesian coordinates, so that

$$(\mathbf{r}\cdot\mathbf{D})\mathbf{C}^{(\lambda)}_{qJM}=-\mathbf{C}^{(\lambda)}_{qJM}\ . \tag{2.12}$$

In addition, Eqs. (2.9) and (2.11), together with the observation that $\mathbf{r}\cdot\mathbf{C}^{(\lambda)}_{qJM}$ is a scalar, imply that

$$\mathbf{r}\cdot\mathbf{C}^{(\lambda)}_{qJM}=\delta_{\lambda 0}Y_{qJM}\ . \tag{2.13}$$

III. HARMONICS FOR $J\geq q$

The ordinary vector harmonics can be constructed by applying vector operators to the Y_{LM}. In this section I generalize this construction to obtain the monopole vector harmonics for $J\geq q$. To begin, let $\mathbf{v}$ be any vector operator constructed from $\mathbf{r}$ and $\mathbf{D}$. The commutation relation $[L_i,v_j]=i\epsilon_{ijk}v_k$ implies that

$$\begin{aligned}[\mathbf{L}^2,v_k]&=-2i\epsilon_{ijk}L_iv_i-2v_k\\ &=-2(\mathbf{L}\cdot\mathbf{S}\mathbf{v})_k-2v_k\ .\end{aligned} \tag{3.1}$$

Hence,

$$(\mathbf{L}+\mathbf{S})^2\mathbf{v}Y_{qKM}=\mathbf{v}\mathbf{L}^2Y_{qKM}=K(K+1)\mathbf{v}Y_{qKM}\ . \tag{3.2}$$

Thus if $\mathbf{v}_\lambda$ is a vector satisfying $\hat{\mathbf{r}}\times\mathbf{v}_\lambda=-i\lambda\mathbf{v}_\lambda$, then the desired $\mathbf{C}^{(\lambda)}_{qJM}$ will be given, up to a (possibly r-dependent) normalization factor, by $\mathbf{v}_\lambda Y_{qJM}$. A set of such vectors is

$$\begin{aligned}\mathbf{v}_{\pm1}&=r\mathbf{D}\pm i\mathbf{r}\times\mathbf{D}\ ,\\ \mathbf{v}_0&=\hat{\mathbf{r}}\ .\end{aligned} \tag{3.3}$$

The normalization factor for the harmonics with $\lambda=\pm1$ can be determined by using Eq. (2.2), together with the fact that $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are equivalent when acting on the Y_{qJM}, to obtain

$$\begin{aligned}\int d\Omega|\mathbf{v}_{\pm1}Y_{qJM}|^2&=\int d\Omega|(r\tilde{\mathbf{D}}\pm i\mathbf{r}\times\tilde{\mathbf{D}})Y_{qJM}|^2\\ &=\int d\Omega\, Y^*_{qJM}[-2r^2\tilde{\mathbf{D}}^2\pm 2ir\mathbf{r}\cdot\tilde{\mathbf{D}}\times\tilde{\mathbf{D}}]Y_{qJM}\\ &=\int d\Omega\, Y^*_{qJM}[2(\mathbf{L}^2-q^2)\pm 2er\epsilon_{ijk}r_iF_{jk}]Y_{qJM}\\ &=2r^2[\mathcal{J}^2\pm q]\ ,\end{aligned} \tag{3.4}$$

where

$$\mathcal{J}^2\equiv J(J+1)-q^2\ . \tag{3.5}$$

[Note that the integral in Eq. (3.4) vanishes for $J=q>0$ and $\lambda=-1$ and for $J=q=0$ and $\lambda=\pm1$, in accord with the earlier statement that the corresponding harmonics should be absent.] For $\lambda=0$, the normalization integral simply reduces to Eq. (2.6). Thus, the properly normalized vector harmonics are

$$\begin{aligned}\mathbf{C}^{(1)}_{qJM}&=[2(\mathcal{J}^2+q)]^{-1/2}[\mathbf{D}+i\hat{\mathbf{r}}\times\mathbf{D}]Y_{qJM}\ , && J\geq q>0\ ,\\ \mathbf{C}^{(0)}_{qJM}&=\frac{1}{r}\hat{\mathbf{r}}Y_{qJM}\ , && J\geq q\geq 0\ ,\\ \mathbf{C}^{(-1)}_{qJM}&=[2(\mathcal{J}^2-q)]^{-1/2}[\mathbf{D}-i\hat{\mathbf{r}}\times\mathbf{D}]Y_{qJM}\ , && J>q\geq 0\ .\end{aligned} \tag{3.6}$$

(One might think that it would have been simpler to choose two of the vector harmonics to be proportional to $\mathbf{D}Y_{qJM}$ and $\hat{\mathbf{r}}\times\mathbf{D}Y_{qJM}$, by analogy with the common practice in the $q=0$ case [4]. The problem is that these are not orthogonal if $q\neq0$; their orthogonality for $q=0$ follows from parity arguments, but parity is not a good quantum number in the presence of the monopole.)

It is useful to have formulas for the covariant curls and divergences of these vectors. For $\lambda=0$,

$$\begin{aligned}\mathbf{D}\times\mathbf{C}^{(0)}_{qJM}&=-\frac{1}{r}\hat{\mathbf{r}}\times\mathbf{D}Y_{qJM}\\ &=\frac{i}{r}\left[\left[\frac{\mathcal{J}^2+q}{2}\right]^{1/2}\mathbf{C}^{(1)}_{qJM}-\left[\frac{\mathcal{J}^2-q}{2}\right]^{1/2}\mathbf{C}^{(-1)}_{qJM}\right]\end{aligned} \tag{3.7}$$

and

$$\mathbf{D}\cdot\mathbf{C}^{(0)}_{qJM}=\mathbf{D}\cdot\left[\frac{\hat{\mathbf{r}}}{r}\right]Y_{qJM}=\frac{1}{r^2}Y_{qJM}\ . \tag{3.8}$$

For $\lambda=\pm1$ we first note that

$$\begin{aligned}\mathbf{r}\times(\mathbf{D}\times\mathbf{C}^{(\pm1)}_{qJM})&=\mathbf{D}(\mathbf{r}\cdot\mathbf{C}^{(\pm1)}_{qJM})-\mathbf{C}^{(\pm1)}_{qJM}-(\mathbf{r}\cdot\mathbf{D})\mathbf{C}^{(\pm1)}_{qJM}\\ &=0\ ,\end{aligned} \tag{3.9}$$

where the vanishing of the first term on the right-hand side follows from Eq. (2.13), while the cancellation of the last two terms is a consequence of Eq. (2.12). [This was the motivation for choosing the normalization condition (2.11).] Thus, the covariant curl of $\mathbf{C}^{(\pm1)}_{qJM}$ is a vector in the radial direction with magnitude $\hat{\mathbf{r}}\cdot\mathbf{D}\times\mathbf{C}^{(\pm1)}_{qJM}$. The latter quantity is a scalar and can therefore be expanded in scalar harmonics, with the coefficient functions determined by the integrals

$$\begin{aligned}\int d\Omega\, Y^*_{qJ'M'}\hat{\mathbf{r}}\cdot\mathbf{D}\times\mathbf{C}^{(\pm1)}_{qJM}&=\int d\Omega\, Y^*_{qJ'M'}\hat{\mathbf{r}}\cdot\tilde{\mathbf{D}}\times\mathbf{C}^{(\pm1)}_{qJM}\\ &=r\int d\Omega\, \mathbf{C}^{(0)*}_{qJ'M'}\cdot\tilde{\mathbf{D}}\times\mathbf{C}^{(\pm1)}_{qJM}\\ &=-r\int d\Omega\, \tilde{\mathbf{D}}\times\mathbf{C}^{(0)*}_{qJ'M'}\cdot\mathbf{C}^{(\pm1)}_{qJM}\\ &=-r\int d\Omega\, \mathbf{D}\times\mathbf{C}^{(0)*}_{qJ'M'}\cdot\mathbf{C}^{(\pm1)}_{qJM}\\ &=\pm\frac{i}{r^2}\left[\frac{\mathcal{J}^2\pm q}{2}\right]^{1/2}\delta_{JJ'}\delta_{MM'}\ .\end{aligned} \tag{3.10}$$

Hence

$$\mathbf{D}\times\mathbf{C}^{(\pm1)}_{qJM}=\pm\frac{i}{r}\left[\frac{\mathcal{J}^2\pm q}{2}\right]^{1/2}\mathbf{C}^{(0)}_{qJM}\ . \tag{3.11}$$

The formula for the divergence is obtained by observing that

$$\begin{aligned}\mathbf{D}\cdot\mathbf{C}^{(\pm1)}_{qJM}&=\pm i\mathbf{D}\cdot\hat{\mathbf{r}}\times\mathbf{C}^{(\pm1)}_{qJM}\\ &=\mp i\hat{\mathbf{r}}\cdot\mathbf{D}\times\mathbf{C}^{(\pm1)}_{qJM}\\ &=\frac{1}{r^2}\left[\frac{\mathcal{J}^2\pm q}{2}\right]^{1/2}Y_{qJM}\ .\end{aligned} \tag{3.12}$$

These results for the covariant divergences and curls can be compactly summarized by

$$\mathbf{D}\cdot\mathbf{C}^{(\lambda)}_{qJM}=\frac{1}{r^2}a_\lambda Y_{qJM} \tag{3.13}$$

with

$$a_0=1\ ,\quad a_{\pm1}=\left[\frac{\mathcal{J}^2\pm q}{2}\right]^{1/2}, \tag{3.14}$$

and

$$\mathbf{D}\times\mathbf{C}^{(\lambda)}_{qJM}=\frac{i}{r}\sum_{\lambda'}b_{\lambda\lambda'}\mathbf{C}^{(\lambda')}_{qJM}\ , \tag{3.15}$$

where the only nonvanishing $b_{\lambda\lambda'}$ are $b_{\pm1,0}=b_{0,\pm1}=\pm a_{\pm1}$. Furthermore, although they have only been derived thus far for $J\geq q$, we will see in the next section that the covariant curls and divergences of the $\mathbf{C}^{(1)}_{q(q-1)M}$ vanish, in agreement with the above equations, so that these expressions are in fact valid for all allowed values of J, M, and λ.

IV. VECTOR HARMONICS FOR $J=q-1$

A. Curls and divergences

For $J=q-1$ there is a single multiplet of vector harmonics, with $L=q$ and $\lambda=1$. These cannot be constructed by the methods of the previous section, since there are no scalar harmonics for $J<q$. In this section I first show that the covariant curls and divergences of these harmonics vanish. I then prove an index theorem showing that the space of such curl-free and divergenceless vectors on the unit two-sphere has dimension $2q-1$, and is thus spanned by the $J=q-1$ multiplet. Finally, the harmonics are displayed explicitly for a particular gauge choice of the vector potential. To simplify notation, let $\mathbf{C}^{(1)}_{q(q-1)M}\equiv\mathbf{U}_M$.

Consider first the divergence of $\mathbf{U}_M$. Since this is a scalar, it will vanish if

$$I_{J'M'}\equiv\int d\Omega\, Y^*_{J'M'}\mathbf{D}\cdot\mathbf{U}_M \tag{4.1}$$

vanishes for all possible values of J' and M'. To this end, note that the fact that $\lambda=1$ implies that $\hat{\mathbf{r}}\cdot U_M=0$, from which it follows that $\mathbf{D}\cdot\mathbf{U}_M=\tilde{\mathbf{D}}\cdot\mathbf{U}_M$. Hence,

$$\begin{aligned}I_{J'M'}&=-\int d\Omega(\tilde{\mathbf{D}}Y_{J'M'})^*\cdot\mathbf{U}_M\\&=-\int d\Omega\left[\left[\frac{\mathcal{J}'^2+q}{2}\right]^{1/2}\mathbf{C}^{(1)*}_{qJ'M'}\right.\\&\qquad\left.+\left[\frac{\mathcal{J}'^2-q}{2}\right]^{1/2}\mathbf{C}^{(-1)*}_{qJ'M'}\right]\cdot\mathbf{U}_M\ .\end{aligned} \tag{4.2}$$

But the last integral must vanish, since $J'\geq q$ while $\mathbf{U}_M$ is a vector harmonic with angular momentum $J=q-1$. Hence,

$$\mathbf{D}\cdot\mathbf{U}_M=0\ . \tag{4.3}$$

Proceeding as in Eq. (3.12) we see that this implies the vanishing of $\hat{\mathbf{r}}\cdot\mathbf{D}\times\mathbf{U}_M$, while, from Eq. (3.9), $\hat{\mathbf{r}}\times(\mathbf{D}\times\mathbf{U}_M)=0$. Therefore,

$$\mathbf{D}\times\mathbf{U}_M=0\ . \tag{4.4}$$

The $\mathbf{U}_M$ are thus a set of $2q-1$ linearly independent curl-free and divergenceless vector fields. Conversely, any vector field whose covariant curl and divergence both vanish is a linear combination of the $\mathbf{U}_M$. To see this, expand the field in vector harmonics and then use Eqs. (3.7), (3.8), (3.11), and (3.12) to show that the coefficients of the $\mathbf{C}^{(\lambda)}_{qJM}$ vanish for $J\geq q$.

B. An index theorem

Any vector field with vanishing curl and divergence is fixed uniquely by its values on the unit two-sphere. Since, in addition, the $\mathbf{U}_M$ are orthogonal to $\hat{\mathbf{r}}$, they are equivalent to a set of curl-free and divergenceless vector fields on this two-dimensional manifold. Let the metric on the two-sphere be denoted g_{ab}, and the coordinate plus gauge covariant derivative be $\mathcal{D}_a$. One can define a duality transformation, with the dual of vector V^a being

$$\tilde{V}^a=-\frac{i}{\sqrt{g}}\epsilon^{ab}V_b\ , \tag{4.5}$$

where $g\equiv\det g_{ab}$ and ϵ^{ab} is the antisymmetric symbol with $\epsilon^{12}=\epsilon^{\theta\phi}=1$. Three-dimensional vectors with $\lambda=1$ ($\lambda=-1$) correspond to self-dual (anti-self-dual) vectors on the two-sphere. The operators P_+ (P_-) projecting onto the space of self-dual (anti-self-dual) vectors are

$$P^{ab}_{\pm}=\frac{1}{2}\left[g^{ab}\mp\frac{i}{\sqrt{g}}\epsilon^{ab}\right]. \tag{4.6}$$

With the aid of the identity $\epsilon^{ab}\epsilon_{bc}=-g^a_c g$, one can verify that $P^a_{\pm b}P^b_{\pm c}=P^a_{\pm c}$. Furthermore, $\mathcal{D}_aP^{bc}_{\pm}=0$.

The space of curl-free self-dual (anti-self-dual) vectors may be identified as the kernel of the operator $\mathcal{O}_+$ ($\mathcal{O}_-$) mapping such vector fields onto the space of scalar fields according to

$$\mathcal{O}_\pm V=\frac{1}{\sqrt{g}}\epsilon^{ab}\mathcal{D}_aP_{\pm bc}V^c=\mp i\mathcal{D}_aP^a_{\pm b}V_b\ . \tag{4.7}$$

The second equality shows that any curl-free self-dual or anti-self-dual vector must also be divergenceless. Conversely, it is easy to see that any vector with vanishing curl and divergence must be either self-dual or anti-self-dual.

Angular momentum considerations suggest that the dimension of the kernel of $\mathcal{O}_+$ should be $2q-1$ for $q>0$, and that the kernel should vanish for $q=0$. Furthermore, since an anti-self-dual curl-free vector field would correspond to a field with the forbidden values $\lambda=-1$ and $J=q-1$, the kernel of $\mathcal{O}_-$ should vanish. These results correspond to index theorems relating the magnetic charge to the index:

$$\mathcal{I}(\mathcal{O}_\pm)\equiv\dim(\text{kernel }\mathcal{O}_\pm)-\dim(\text{kernel }\mathcal{O}^\dagger_\pm)\ . \tag{4.8}$$

Here the adjoint operators $\mathcal{O}^\dagger_\pm$ mapping scalar fields onto self-dual or anti-self-dual vector fields, are

$$(\mathcal{O}_\pm^\dagger S)^a = -i(P_\pm^{ba})^* \mathcal{D}_b S = -iP_\pm^{ab}\mathcal{D}_b S \ . \tag{4.9}$$

The first step in deriving these theorems is to note that the kernels of $\mathcal{O}_\pm$ and $\mathcal{O}_\pm^\dagger$ are the same as those of $\mathcal{O}_\pm^\dagger\mathcal{O}_\pm$ and $\mathcal{O}_\pm\mathcal{O}_\pm^\dagger$, respectively. Furthermore, if ψ is an eigenfunction of $\mathcal{O}_\pm^\dagger\mathcal{O}_\pm$ with nonzero eigenvalue, then $\mathcal{O}_\pm\psi$ is an eigenfunction of $\mathcal{O}_\pm\mathcal{O}_\pm^\dagger$ with the same eigenvalue. Assuming that these eigenfunctions form a complete basis, it follows that

$$\mathcal{I}(\mathcal{O}_\pm) = \mathrm{Tr}\left[\frac{M^2}{2\mathcal{O}_\pm^\dagger\mathcal{O}_\pm + M^2}\right] - \mathrm{Tr}\left[\frac{M^2}{2\mathcal{O}_\pm\mathcal{O}_\pm^\dagger + M^2}\right] , \tag{4.10}$$

where M^2 is an arbitrary parameter. (The somewhat unconventional factors of 2 are for later convenience.) We will find it most convenient to evaluate this expression in the limit $M^2 \to \infty$.

With the aid of the identities $P_\pm^2 = P_\pm$ and

$$P_\pm^{ab}P_\pm^{cd} = P_\pm^{ad}P_\pm^{cb} \tag{4.11}$$

one finds that

$$\begin{aligned}(\mathcal{O}_\pm^\dagger\mathcal{O}_\pm)^a{}_b &= -P_{\pm b}^a P_\pm^{cd}\mathcal{D}_d\mathcal{D}_c \\ &= -\frac{1}{2}P_{\pm b}^a\left[\mathcal{D}^c\mathcal{D}_c \pm \frac{i}{2\sqrt{g}}\epsilon^{cd}[\mathcal{D}_c,\mathcal{D}_d]\right]\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\mathcal{O}_\pm\mathcal{O}_\pm^\dagger &= -P_\pm^{cd}\mathcal{D}_c\mathcal{D}_d \\ &= -\frac{1}{2}\left[\mathcal{D}^c\mathcal{D}_c \mp \frac{i}{2\sqrt{g}}\epsilon^{cd}[\mathcal{D}_c,\mathcal{D}_d]\right] .\end{aligned} \tag{4.13}$$

The commutator of two coordinate and gauge covariant derivatives $\mathcal{D}_a$ is the sum of the commutator of the corresponding gauge covariant derivatives D_a and the commutator of the corresponding coordinate covariant derivatives ∇_a. Acting on charged fields, the former is

$$[D_a, D_b] = -ieF_{ab} = -\frac{iq}{\sqrt{g}}\epsilon_{ab} , \tag{4.14}$$

where the second equality follows from the expression for the magnetic field at unit radius from the monopole. The latter commutator vanishes when acting on scalar fields, while on self-dual or anti-self-dual vector fields

$$\begin{aligned}\epsilon^{cd}[\nabla_c,\nabla_d]V^a &= \epsilon^{cd}[\nabla_c,\nabla_d]P_{\pm b}^a V^b \\ &= -\epsilon^{cd}P_{\pm b}^a R^b{}_{ecd}V^e \\ &= \epsilon^{cd}P_{\pm b}^a(g^b{}_c g_{ed} - g^b{}_d g_{ec})V^e \\ &= \pm 2i\sqrt{g}\,P_{\pm e}^a V^e \\ &= \pm 2i\sqrt{g}\,V^a ,\end{aligned} \tag{4.15}$$

where the explicit form of the curvature tensor on the two-sphere has been used on the third line. We thus have

$$(\mathcal{O}_\pm^\dagger\mathcal{O}_\pm)^a{}_b = \tfrac{1}{2}P_{\pm b}^a(-\mathcal{D}^c\mathcal{D}_c \mp q + 1) \tag{4.16}$$

and

$$\mathcal{O}_\pm\mathcal{O}_\pm^\dagger = \tfrac{1}{2}(-\mathcal{D}^c\mathcal{D}_c \pm q) \ . \tag{4.17}$$

The factors of q enter with the opposite sign for $\mathcal{O}_+$ and $\mathcal{O}_-$ because the magnetic field is parity violating and so distinguishes between self-dual and anti-self-dual fields. There is no such asymmetry as far as the geometry of the sphere is concerned, and so the factor of 1 arising from the curvature displays no sign change.

Equation (4.16) shows that $\mathcal{O}_-^\dagger\mathcal{O}_-$ is positive definite for $q \ge 0$. Hence, the kernel of $\mathcal{O}_-$ vanishes, and there are no curl-free anti-self-dual vector fields, as expected. Furthermore, since $\mathcal{D}^c\mathcal{D}_c$ is equivalent to $-(\mathbf{L}^2 - q^2)$ when acting on scalar fields, Eq. (4.17) may be rewritten as

$$\mathcal{O}_\pm\mathcal{O}_\pm^\dagger = \tfrac{1}{2}[(\mathbf{L}^2 - q^2) \pm q] \ . \tag{4.18}$$

The eigenfunction of $\mathcal{O}_\pm\mathcal{O}_\pm^\dagger$ are thus the scalar monopole harmonics Y_{qLM}, with eigenvalues $L(L+1) - q(q \mp 1)$. For the lower signs, a zero eigenvalue is obtained only for $L = q$. Since there are $2L+1 = 2q+1$ possible values of M, the kernel of $\mathcal{O}_-^\dagger$ has dimension $2q+1$, and

$$\mathcal{I}(\mathcal{O}_-) = -2q - 1 \ . \tag{4.19}$$

For the upper signs, a zero eigenvalue is possible only for $q = 0$, $L = 0$. Thus, for $q > 0$, the kernel of $\mathcal{O}_+^\dagger$ vanishes, and the dimension of the kernel of $\mathcal{O}_+$ is equal to $\mathcal{I}(\mathcal{O}_+)$.

To calculate this last quantity, substitute Eqs. (4.16) and (4.17) into Eq. (4.10), and expand the denominators about $-\mathcal{D}^c\mathcal{D}_c + M^2$. Thus,

$$\begin{aligned}\mathcal{I}(\mathcal{O}_\pm) \\ = M^2\mathrm{Tr}&\left[\frac{P_\pm}{-\mathcal{D}^c\mathcal{D}_c + M^2} \pm \frac{(q \mp 1)P_\pm}{(-\mathcal{D}^c\mathcal{D}_c + M^2)^2} + \cdots\right] \\ -M^2\mathrm{Tr}&\left[\frac{1}{-\mathcal{D}^c\mathcal{D}_c + M^2} \mp \frac{q}{(-\mathcal{D}^c\mathcal{D}_c + M^2)^2} + \cdots\right] ,\end{aligned} \tag{4.20}$$

where the ellipsis represents terms which vanish in the limit $M^2 \to \infty$. The contributions from the first terms in the two expansions cancel.[3] To deal with the second terms in the expansions, note that replacing $\mathcal{D}^c\mathcal{D}_c$ by the flat-space two-dimensional Laplacian Δ gives an error of order M^{-2}. Therefore

[3]This is less obvious than it might seem, since the operator $-\mathcal{D}^c\mathcal{D}_c$ is understood to be acting on vectors fields in one case and scalar fields in the other, and so in curved space will take two different forms. However, one can verify the cancellation by comparing the result for $\mathcal{I}(\mathcal{O}_-)$ obtained below with that given in Eq. (4.19).

$$\begin{aligned}\mathcal{I}(\mathcal{O}_\pm)&=(\pm 2q-1)\lim_{M^2\to\infty}\int d^2x\, M^2\langle x|(-\Delta+M^2)^{-2}|x\rangle\\&=(\pm 2q-1)\lim_{M^2\to\infty}\int d^2x\int\frac{d^2k}{(2\pi)^2}\frac{M^2}{(k^2+M^2)^2}\\&=\pm 2q-1\ .\end{aligned}\tag{4.21}$$

C. Explicit expressions

To obtain explicit expressions for the $\mathbf{U}_M$, we must first fix the vector potential. Following Wu and Yang [2], let R_a be the region $0\le\theta<(\pi/2)+\delta$ and R_b be the region $(\pi/2)-\delta<\theta\le\pi$. A nonsingular choice for the vector potential which maintains the manifest rotational symmetry about the z axis is $\mathbf{A}_r=\mathbf{A}_\theta=0$ and

$$\begin{aligned}\mathbf{A}_\phi&=\frac{q}{e}(1-\cos\theta)\ \text{ in } R_a\ ,\\ \mathbf{A}_\phi&=-\frac{q}{e}(1+\cos\theta)\ \text{ in } R_b\ .\end{aligned}\tag{4.22}$$

The action of J_z on a scalar function is then

$$J_z\psi=L_z\psi=(-i\partial_\phi\mp q)\psi\ ,\tag{4.23}$$

where the upper (lower) sign refers to region R_a (R_b). Applying this to the scalar $\hat{\mathbf{z}}\cdot\mathbf{U}_M$, and using the fact that $\hat{\mathbf{z}}$ is invariant under rotations about the z axis, we find that the eigenvalue equation $J_z\mathbf{U}_M=M\mathbf{U}_M$ implies

$$(-i\partial_\phi\mp q)(\mathbf{U}_M)_\theta=M(\mathbf{U}_M)_\theta\ .\tag{4.24}$$

Hence $(\mathbf{U}_M)_\theta$ must be of the form

$$(\mathbf{U}_M)_\theta=e^{i(M\pm q)\phi}f_{qM}(\theta)\ .\tag{4.25}$$

The self-duality condition then gives

$$(\mathbf{U}_M)_\phi=ie^{i(M\pm q)\phi}\sin\theta f_{qM}(\theta)\ .\tag{4.26}$$

Substituting Eq. (4.25) into Eq. (4.3) we obtain

$$\partial_\theta(\sin\theta f_{qM})-(M+q\cos\theta)f_{qM}=0\tag{4.27}$$

whose solution is

$$\begin{aligned}f_{qM}(\theta)&=a_{qM}\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{M/2}\sin^{q-1}\theta\\&=a_{qM}(1-\cos\theta)^M\sin^{q-M-1}\theta\\&=a_{qM}(1+\cos\theta)^{-M}\sin^{q+M-1}\theta\ .\end{aligned}\tag{4.28}$$

The normalization constant a_qM is determined (up to an arbitrary phase) by Eq. (2.11) to be

$$a_{qM}=\frac{1}{2^q\sqrt{2\pi}}\left[\frac{(2q-1)!}{(q+M-1)!(q-M-1)!}\right]^{1/2}\ .\tag{4.29}$$

Thus, the $\mathbf{U}_M$ may be written as

$$\left.\begin{aligned}(\mathbf{U}_M)_\theta&=(\mathbf{C}^{(1)}_{q(q-1)M})_\theta=\frac{a_{qM}}{r}e^{i(M+q)\phi}\sin^{q+M-1}\theta(1+\cos\theta)^{-M}\\(\mathbf{U}_M)_\phi&=(\mathbf{C}^{(1)}_{q(q-1)M})_\phi=\frac{ia_{qM}}{r}e^{i(M+q)\phi}\sin^{q+M}\theta(1+\cos\theta)^{-M}\end{aligned}\right\}\ \text{in } R_a\ ,$$
$$\left.\begin{aligned}(\mathbf{U}_M)_\theta&=(\mathbf{C}^{(1)}_{q(q-1)M})_\theta=\frac{a_{qM}}{r}e^{i(M-q)\phi}\sin^{q-M-1}\theta(1-\cos\theta)^{M}\\(\mathbf{U}_M)_\phi&=(\mathbf{C}^{(1)}_{q(q-1)M})_\phi=\frac{ia_{qM}}{r}e^{i(M-q)\phi}\sin^{q-M}\theta(1-\cos\theta)^{M}\end{aligned}\right\}\ \text{in } R_b\ .\tag{4.30}$$

For these expressions to be single-valued, $q-M$ and $q+M$ must be integers. To avoid a singularity at $\theta=0$ (in region R_a), we must require $q+M-1\ge 0$, while at $\theta=\pi$ (in region R_b) we have the condition $q-M-1\ge 0$. This leaves $2q-1$ allowed values of M, thus confirming the index calculation and giving the full $J=q-1$ angular momentum multiplet.

ACKNOWLEDGMENTS

I would like to thank Kimyeong Lee, Parameswaran Nair, and Alexander Ridgway for helpful comments. I am also grateful to the Aspen Center for Physics, where part of this work was done. This work was supported in part by the U.S. Department of Energy.

APPENDIX

Olsen *et al.* [3] define monopole vector harmonics $\mathbf{Y}^{(q)}_{JLM}$ which are eigenfunctions of $\mathbf{J}^2,\mathbf{L}^2$, and J_z. In this appendix I obtain the relationship between these harmonics and the $\mathbf{C}^{(\lambda)}_{qJM}$ defined in this paper. Because the $\mathbf{Y}^{(q)}_{JLM}$ are orthonormal, while the $\mathbf{C}^{(\lambda)}_{qJM}$ are normalized according to Eq. (2.11), the two sets of harmonics are related by

$$\mathbf{C}^{(\lambda)}_{qJM}=\frac{1}{r}\sum_L M_{\lambda L}(q,J,M)\mathbf{Y}^{(q)}_{JLM}\ ,\tag{A1}$$

where the matrices $M_{\lambda L}(q,J,M)$ are unitary.

For $J=q-1$, there is only one allowed value each for λ and for L, and so $M(q,q-1,M)$ is simply a complex number of unit magnitude. Carrying out explicitly the construction of Ref. [3] and comparing with Eq. (4.30), one finds that

$$M(q,q-1,M)=(-1)^{q+M} . \tag{A2}$$

For larger values of J the $M_{\lambda L}(q,J,M)$ are either 2×2 (if $J=q$) or 3×3 (if $J>q$). The $0L$ elements of $M_{\lambda L}(q,J,M)$ can be obtained directly from Eq. (4.13) of Ref. [3], which expresses $\hat{\mathbf{r}}Y_{qLM}$ in terms of the $\mathbf{Y}^{(q)}_{JLM}$. Explicitly,

$$\begin{aligned} M_{0(J-1)}(qJM)&=\left[\frac{J^2-q^2}{(2J+1)J}\right]^{1/2} ,\\ M_{0J}(qJM)&=-\frac{q}{\sqrt{J(J+1)}} ,\\ M_{0(J+1)}(qJM)&=-\left[\frac{(J+1)^2-q^2}{(2J+1)(J+1)}\right]^{1/2} . \end{aligned} \tag{A3}$$

The first step in obtaining the remaining elements of $M_{\lambda L}(q,J,M)$ is to use Eq. (4.5) of Ref. [3]:

$$\mathbf{r}\times\mathbf{Y}^{(q)}_{JlM}=ir\sum_L A^{(q)}_{JlL}\mathbf{Y}^{(q)}_{JLM} , \tag{A4}$$

where the matrices $A^{(q)}_{JlL}$ are given explicitly in Ref. [3]. Together with Eqs. (2.9) and (A1), this leads to

$$\sum_l M_{\lambda l}A_{lL}=-\lambda M_{\lambda L} . \tag{A5}$$

This, together with the unitarity of $M_{\lambda L}$, determines the $M_{\lambda(J\pm 1)}$ in terms of the $M_{\lambda J}$, and fixes the latter up to a λ-dependent phase. Specifically,

$$\begin{aligned} M_{1J}(qJM)&=e^{i\alpha}\left[\frac{\mathcal{J}^2+q}{2J(J+1)}\right]^{1/2} ,\\ M_{-1J}(qJM)&=e^{i\beta}\left[\frac{\mathcal{J}^2-q}{2J(J+1)}\right]^{1/2} . \end{aligned} \tag{A6}$$

The next step is to use Eq. (4.14) of Ref. [3]:

$$\mathbf{L}Y_{qJM}=\sqrt{J(J+1)}\mathbf{Y}^{(q)}_{JLM} . \tag{A7}$$

Substituting Eqs. (A1), (A4), and (A6) into this and equating coefficients of $\mathbf{Y}^{(q)}_{JJM}$ yields

$$e^{i\alpha}(\mathcal{J}^2+q)-e^{i\beta}(\mathcal{J}^2-q)+2\mathcal{J}^2=0 \tag{A8}$$

whose only solution is $e^{i\alpha}=-e^{i\beta}=-1$. Hence,

$$M_{\pm 1J}(qJM)=\pm\left[\frac{\mathcal{J}^2\pm q}{2J(J+1)}\right]^{1/2} \tag{A9}$$

from which one obtains

$$\begin{aligned} M_{\pm 1(J+1)}(qJM)&\\ &=\left[\frac{\mathcal{J}^2\pm q}{4J(J+1)}\right]^{1/2}\left[\frac{J+1\pm q}{(J+1)(J+1\mp q)}\right]^{1/2} ,\\ M_{\pm 1(J-1)}(qJM)&=\left[\frac{\mathcal{J}^2\pm q}{4J(J+1)}\right]^{1/2}\left[\frac{J\mp q}{J(J\pm q)}\right]^{1/2} . \end{aligned} \tag{A10}$$

Note that if $J=q$, both M_{-1L} and $M_{\lambda(J-1)}$ vanish, so that $M_{\lambda L}$ is actually a 2×2 matrix, as required.

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