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Hamiltonian embedding of a second-class system with a Chern-Simons term

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Using the generalized canonical (Hamiltonian) formalism, the second-class system comprising complex scalars coupled to a Chern-Simons term is converted into first class. It leads to a *new* Wess-Zumino-type embedding which cannot be obtained by conventional Lagrangian methods.

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It is customary to think of second-class dynamical systems as the outcome of explicit gauge-breaking terms occurring in the classical Lagrangian or as a consequence of quantum anomalies. Thus the massive Maxwell theory (Proca theory)

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+m^2A_\mu A^\mu \tag{1}$$

is a clean example of the first kind of a second-class theory. Alternatively, anomalous (chiral) gauge theories provide examples of the second category. In both of these cases, moreover, there is a standard Lagrangian prescription [1–3] which converts these second-class systems into first-class (i.e., true) gauge systems by extending the configuration space. In (1) this is the Stückelberg [1] mechanism which corresponds to performing a gauge transformation $A_\mu \to A_\mu+\partial_\mu\lambda$ and identifying the new field λ as the Stückelberg scalar. Similarly, for anomalous gauge theories, the Wess-Zumino method [2] of introducing scalars to cancel the anomalies yields the desired first-class theory [3]. Both these procedures lead to manifestly invariant (embedded) actions which reduce to the original action when the new (extra) fields are set to zero [1–3].

We shall show in the present paper that dynamical systems with a Chern-Simons (CS) term provide a very remarkable departure from the above (conventional) Lagrangian description. Let us consider the familiar model of complex scalars coupled to a CS term [4],

$$\mathcal{L}=(D_\mu\phi)^*(D^\mu\phi)+\frac{\theta}{4\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \ , \tag{2}$$

where $D_\mu=\partial_\mu+iA_\mu$ and the metric is chosen as

$$g_{\mu\nu}=(+1,-1,-1), \quad \varepsilon_{012}=\varepsilon^{012}=1 \ .$$

It is known [5] that this model has two first-class constraints,

$$\pi_0\approx 0, \quad P=\partial^i\pi_i+J_0+\frac{\theta}{4\pi^2}\varepsilon^{ij}\partial_i A_j\approx 0, \quad i=1,2 \tag{3}$$

along with a pair of second-class constraints,

$$P_i=\pi_i-\frac{\theta}{4\pi^2}\varepsilon_{ij}A^j\approx 0 \ , \tag{4}$$

where π_μ is the momentum conjugate to A^μ and J_0 is the zeroth component of the conserved current:

$$J_\mu=i[(D_\mu\phi)^*\phi-\phi^*(D_\mu\phi)] \ .$$

The Lagrangian (2) is invariant (up to a total divergence) under the gauge transformations

$$\phi(x)\to e^{i\alpha(x)}\phi(x) \ ,$$
$$A_\mu(x)\to A_\mu(x)-\partial_\mu\alpha(x) \ .$$

Moreover, as is well known [6], odd dimensional theories such as (2) are not afflicted with quantum anomalies. Consequently the occurrence of the second-class constraints (4) cannot be attributed to either of the two categories mentioned at the outset. Indeed it is a manifestation of the symplectic structure of the CS term. Naturally the conventional Lagrangian procedure for embedding [1–3] a second-glass system into a first-class system is inadequate. We therefore take recourse to the Hamiltonian formalism developed recently [7,8]. We shall initially discuss the operatorial conversion of the second-class system into first class by explicitly constructing the first-class constraints and the involutive Hamiltonian in an extended phase space. We next define the partition function in this phase space and make contact with the Lagrangian formulation by explicitly evaluating it in the (i) unitary [7] and (ii) Faddeev-Popov [9] gauges. While (i) reproduces the original theory, (ii) yields the generalized Wess-Zumino (WZ) [2] functional. The latter result is new and differs from conventional Lagrangian (Faddeev-Popov [9]) embeddings [1–3] because (a) it does *not* reduce to the original theory when the new (Wess-Zumino) field is set to zero, (b) it is not manifestly Lorentz invariant, and (c) it has a nonpolynomial term.

The first step is to convert the second-class constraints (4) whose Poisson algebra is

$$\Delta_{ij}(x,y)=\{P_i(x),P_j(y)\}=-\frac{\theta}{2\pi^2}\varepsilon_{ij}\delta(x-y); \quad \varepsilon_{12}=1 \ , \tag{5}$$

into first class. This is easily done by following the prescription of Ref. [8] by introducing new dynamical

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variables $\Phi^i(x)$ in the extended phase space,

$$(\phi,\pi,\phi^*,\pi^*,A_\mu,\pi^\mu)\oplus(\Phi^i)\,,$$

obeying the Poisson algebra

$$\{\Phi^i(x),\Phi^j(y)\}=W^{ij}(x,y)\,, \tag{6}$$

where W is an invertible antisymmetric matrix. Then the first-class constraints P_i' are given by [8]

$$P_i'(\pi_j,A^k;\Phi^l)=\sum_{n=0}^{\infty}P_i'^{(n)},\quad P_i'^{(n)}\sim(\Phi)^n \tag{7}$$

subject to the boundary condition

$$P_i'^{(0)}=P_i'(\pi_j,A^k;0)=P_i\,. \tag{8}$$

After (8), the next term in the series (7) is given by [8]

$$P_i'^{(1)}(x)=\int dy\,X_{ij}(x,y)\Phi^j(y), \tag{9}$$

where

$$\int dz\,dz'[X_{ij}(x,z)W^{jk}(z,z')X_{kl}(z',y)]=-\Delta_{il}(x,y) \tag{10}$$

with $\Delta_{il}(x,y)$ defined in (5). The other terms ($n>1$) in (7) are obtained by a recursion relation [8]. As we shall see, these are redundant in our example. A possible choice for W^{ij} and X_{ij} compatible with (10) is

$$W^{ij}(x,y)=2\begin{bmatrix}0&1\\-1&0\end{bmatrix}\delta(x-y)\,,$$

$$X_{ij}(x,y)=\begin{bmatrix}1&0\\0&\dfrac{\theta}{4\pi^2}\end{bmatrix}\delta(x-y)\,. \tag{11}$$

There is a "natural arbitrariness" [7,8] in this choice which just corresponds to canonical transformations in the extended phase space. The above choice considerably simplifies the algebra. Using (4), (7)–(9), and (11) the new set of constraints are found to be

$$P_1'=P_1+\Phi^1\,,$$
$$P_2'=P_2+\frac{\theta}{4\pi^2}\Phi^2\,, \tag{12}$$

which are strongly involutive, i.e.,

$$\{P_i',P_j'\}=0\,,$$

illustrating the redundancy of the other ($n>1$) terms in the series (7). The algebra of the original pair of first-class constraints (3) is unmodified since the additional terms in (12) just involve the new fields. Consequently the complete set of constraints π_0,P,P_i' is strongly involutive. Hence the theory is of rank zero [10].

The next step is to construct the first-class (involutive) Hamiltonian, which, following Ref. [8], can be expressed as a series,

$$H'(\phi,\pi,\phi^*,\pi^*,A_\mu,\pi^\mu;\Phi^i)=\sum_{n=0}^{\infty}H'^{(n)};\quad H'^{(n)}\sim(\Phi)^n\,, \tag{13}$$

subject to the initial condition

$$H'^{(0)}=H'(\phi,\pi,\phi^*,\pi^*,A_\mu,\pi^\mu;0)=H_c\,, \tag{14}$$

where H_c is the canonical Hamiltonian [5] obtained by a Legendre transform of (2),

$$H_c=\int\left[|\pi|^2-A_0\left[J_0+\frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_iA_j\right]-(D_i\phi)^*(D^i\phi)\right]\,, \tag{15}$$

and $\pi(\pi^*)$ is the momentum conjugate to ϕ (ϕ^*). The general expression for $H'^{(n)}$ in (13) is [8]

$$H'^{(n+1)}=-\frac{1}{n+1}\int dx\,dy\,dz[\Phi^i(x)W_{ij}(x,y)X^{jk}(y,z)G_k^{(n)}(z)]\ (n\geq 0)\,, \tag{16}$$

where W_{ij} and X^{jk} are the inverse of the matrices W^{ij}, X_{jk} respectively, defined in (11). The generating functional $G_k^{(n)}$ has the simple form

$$G_k^{(0)}(x)=\{P_k(x),H_c\}\,,$$
$$G_k^{(n)}(x)=\{P_k'^{(1)}(x),H'^{(n-1)}\}_{\mathcal{O}}+\{P_k(x),H'^{(n)}\}_{\mathcal{O}};\quad(n\geq 1)\,, \tag{17}$$

the genesis of which is contained in the intelligent choice (11). Indeed a look at the general structure for $G_k^{(n)}$ given in Eq. (2.54) of [8] will convince the reader of the remarkable algebraic simplification achieved in (17). The symbol $\mathcal{O}$ appearing in (17) indicates that the Poisson brackets are computed among the original (old) variables. Using (4) and (13) to (17), we find that the series (13) truncates after $n=2$ and the final expression for the first-class Hamiltonian is

$$H'=H_c+\int\left[\frac{2\pi^2}{\theta}\Phi^1\left[J_2+\frac{\theta}{2\pi^2}\partial_1A_0+\frac{2\pi^2}{\theta}\Phi^1|\phi|^2\right]+\frac{\Phi^2}{2}\left[-J_1+\frac{\theta}{2\pi^2}\partial_2A_0+\frac{\Phi^2}{2}|\phi|^2\right]\right]\,, \tag{18}$$

which has the following involution relations with the first-class constraints (3), (12):

$$\{P(x),H'\}=\{P_i'(x),H'\}=0\ ,\qquad \{\pi_0(x),H'\}=\partial_i P_i'+P\quad (i=1,2)\ . \tag{19}$$

This completes the operatorial conversion of the original second-class system (with constraints π_0, P, P_i and the canonical Hamiltonian H_c) into first class (with constraints π_0, P, P_i' and the Hamiltonian H').

In order to make contact with the Lagrangian formulation, we first identify the new variables Φ^1 and Φ^2 occurring in the extended phase space, as a canonically conjugate pair (λ,π_λ), where

$$\Phi^1\rightarrow 2\lambda,\quad \Phi^2\rightarrow\pi_\lambda\ ,$$

as may be easily verified from (6) and (11). In our subsequent analysis we always refer to (Φ^1,Φ^2) as $(2\lambda,\pi_\lambda)$. Moreover, we denote the set of first-class constraints (3), (12) collectively by F_α ($\alpha=0,1,2,3$):

$$F_0=\pi_0,\quad F_i=P_i',\quad F_3=P\ . \tag{20}$$

Then the phase space partition function is given by

$$Z=\int(\mathcal{D}\phi\mathcal{D}\pi\mathcal{D}\phi^*\mathcal{D}\pi^*\mathcal{D}A_\mu\mathcal{D}\pi^\mu\mathcal{D}\lambda\mathcal{D}\pi_\lambda)\times\prod_{\alpha,\beta}[\delta(F_\alpha)\delta(G_\beta)]\det|\{F_\alpha,G_\beta\}|e^{iS}\ , \tag{21a}$$

where

$$S=\int[\pi\dot\phi+\pi^*\dot\phi^*+\pi_\mu\dot A^\mu+\pi_\lambda\dot\lambda-\mathcal{H}']\ , \tag{21b}$$

where $\mathcal{H}'$ is the Hamiltonian density corresponding to H' (18) and G_β are the gauge-fixing conditions chosen so that the determinant in (21) is nonvanishing. Expression (21) is the familiar form first given by Faddeev [11,10]. Now three of the constraints F_0,F_i are algebraic and the integrals over π^0,π^1,π^2 are trivially done by exploiting the delta functions $\delta(F_0)$, $\delta(F_i)$. The δ function involving the Gauss constraint $\delta(F_3)$ is expressed by its corresponding Fourier transform (with the Fourier variable ξ). Making a change of variables

$$A_0\rightarrow A_0+\xi$$

and performing the Gaussian integration over π,π^*, we obtain the following form for the quantum action:

$$S=\int\left[S_c+2\lambda\left[F_{01}-\frac{2\pi^2}{\theta}J_2-\frac{8\pi^4}{\theta^2}\lambda|\phi|^2\right]+\pi_\lambda\left[-\frac{1}{4}|\phi|^2\pi_\lambda+\dot\lambda+\frac{\theta}{4\pi^2}F_{02}+\frac{1}{2}J_1\right]\right]\ , \tag{22}$$

where S_c is the classical action corresponding to the Lagrangian (2), and the measure is given by

$$[d\mu]=(\mathcal{D}\xi\mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu\mathcal{D}\lambda\mathcal{D}\pi_\lambda)\times\prod_\beta\{\delta[G_\beta(A_0+\xi,A_i,\lambda,\pi_\lambda)]\}\det|\{F_\alpha,G_\beta\}|\ .$$

The original theory (2) is reproduced in one line by choosing the unitary gauge [7]

$$G_1=\lambda,\quad G_2=\pi_\lambda\ . \tag{23}$$

To realize the WZ [2] (-type) functional, one chooses the Faddeev-Popov-like gauges [9] which do not involve the momenta. Then the Gaussian integration over π_λ [see (22)] can be performed and we obtain

$$S=S_c+S_{\rm WZ}\ , \tag{24a}$$

where the WZ [2] term is

$$S_{\rm WZ}=\int\left[\frac{1}{|\phi|^2}\left[\dot\lambda+\frac{\theta}{4\pi^2}F_{02}+\frac{J_1}{2}\right]^2+2\lambda\left[F_{01}-\frac{2\pi^2}{\theta}J_2-\frac{8\pi^4}{\theta^2}\lambda|\phi|^2\right]\right] \tag{24b}$$

and the corresponding Liouville measure just comprises the configuration space variables:

$$[d\mu]=(\mathcal{D}\xi\mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu\mathcal{D}\lambda)\times\prod_\beta\{\delta(G_\beta)\}\det|\{F_\alpha,G_\beta\}|(\det|\phi|^2)^{-1/2}\ ,$$

where the last factor is a consequence of the π_λ integration. It is straightforward to verify that, starting from the Lagrangian corresponding to (24), one reproduces the set of first-class constraints F_α (20) and the Hamiltonian obtained from this Lagrangian by a Legendre transform is identical to the involutive Hamiltonian H' (18). This serves as a consistency check on our analysis.

Equation (24) is the *new* type of WZ [2] embedding obtained by this Hamiltonian procedure. Contrary to conventional examples [1–3,12–14] it does not reduce to the original action when the WZ scalar λ is set to zero. This clearly illustrates the inadequacy of usual Lagrangian [1–3] (Faddeev-Popov [9]) embeddings for systems whose second-class nature is a consequence of the CS term, and not the result of a genuine gauge violation, either at the classical or at the quantum level. We may mention that recently [12–14] within the Hamiltonian formalism [7,8], embeddings of familiar second-class systems have been done which merely reproduce the standard results obtained by usual Lagrangian procedures [1–3]. While such investigations [12,13] may have a pedagogic value, they do not illustrate the complete power of the Hamiltonian embedding, which has been illuminated in this analysis.

Another noteworthy feature of (24), contrasted with usual WZ [2] terms, is its lack of *manifest* Lorentz invariance. This need not be alarming since such examples, in other contexts [15,16], have appeared. The important thing is that the *actual* invariance is preserved. This is true in our case due to the theorem [10,11,17] proving the gauge independence of (21). We had earlier shown how, in the unitary gauge (23), the original manifestly invariant theory was reproduced. Consequently the quantum physics following from (24), which is an outcome of a different (Faddeev-Popov) gauge, is compatible with relativistic invariance. The reason for the lack of manifest invariance is simple. It consists in breaking the manifestly invariant structure of the second-class constraints P_i (4), when converted to first class P_i' (12). Finally, note the presence of a nonpolynomial term in (24).

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