Boundary conditions in the Aharonov-Bohm scattering of Dirac particles and the effect of Coulomb interaction

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We consider the question of the physically correct choice of the boundary conditions, and therefore of the dynamics, for the scattering of spin-$\frac{1}{2}$ particles by a thread of magnetic flux. It is shown that even when the problem is approached by first considering a source of finite radius R, the resulting dynamics is a consequence of how much penetration of the wave function inside the tube one chooses to allow. (The indeterminacy in the boundary condition occurs for one value of the total angular momentum for a given value of the flux.) If a Coulomb interaction of the particle with the sources is introduced in the problem, the above-mentioned indeterminacy is enlarged. We show that for a given value of the flux there are now at least two values of the angular momentum n for which the indeterminacy occurs.

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I. INTRODUCTION

The problem of a two-dimensional Dirac particle in the presence of a thread of magnetic flux appears in a variety of problems in theoretical physics, primarily in the discussion of the Aharonov-Bohm effect [1,2] and its variant, the Aharonov-Casher effect [3] for particles with spin [4]. It also appears in the study of the interaction of matter with cosmic strings [5–7]. In the nonrelativistic limit it is present in the study of anyons in the theory of high-T_c superconductivity [8].

It was shown by de Souza Gerbert [9] that the dynamics of the case with the flux $\phi \neq 0$ displays unexpected features when compared to the free ($\phi=0$) case. Namely if $\phi \neq 0$, the determination of eigenfunctions requires specification of boundary conditions that are to be chosen from among a one-parameter family of admissible ones. Moreover, it is *impossible* to stick to the usual regularity assumption for the wave function, as opposed to the case $\phi=0$, where the only admissible boundary condition is of regularity at the origin. Subsequently, there were attempts by Alford *et al.* [6] and Hagen [10] to provide a physical motivation for the choice of the boundary conditions among the admissible ones. Both computed the eigenfunctions using the same limiting procedure of replacing the thread of flux by a fictitious flux tube of radius R (thus removing the ambiguity) and then taking the limit $R \to 0$ at the end of the computation. In this paper we show that although there is nothing wrong with the computations of Refs. [6,10], and in spite of the physical appeal of the limiting procedure, the boundary condition so obtained produces wave functions which as functions of the flux ϕ are discontinuous at all nonzero integer values of ϕ. Moreover, as observed by Alford *et al.* [6], this procedure also produces a breakdown of the Aharonov-Bohm symmetry $\phi \to \phi+1$ of the formally defined Hamiltonian. One of the purposes of this paper is to answer the question: Are there boundary conditions which preserve the apparent symmetry $\phi \to \phi+1$ and/or are continuous in ϕ? The question is answered in the affirmative and it is also shown that these boundary conditions may be obtained through the same limiting procedure of Refs. [6,10] through the introduction of a fine-tuned interaction inside the tube of radius R.

This paper is organized as follows. In Sec. II we introduce our notation and briefly review the results of [9,6,10], carefully pointing out the problems mentioned above. We also show how the problem is modified by the introduction of a Coulomb interaction with the source. In Sec. III we discuss how to obtain all admissible boundary conditions by a suitable modified limiting procedure, $R \to 0$, allowing for some interaction inside the tube. Finally, in Sec. IV we examine the effect of a Coulomb interaction with the source.

It should be mentioned that the boundary conditions of [6,10] are also compatible with conservation of an also formally defined helicity operator. This is sometimes used as an argument in favor of these boundary conditions. These issues will be examined in a separate work.

II. PRELIMINARIES

The Hamiltonian of the two-dimensional Dirac particle in the presence of a thread of magnetic flux is *formally* given by

$$H_0 = \boldsymbol{\alpha} \cdot \left[\mathbf{p} + \frac{e\,\mathbf{A}}{c} \right] + \beta m \ , \tag{1}$$

where a possible choice for the Dirac matrices is $\alpha_1 = \sigma_1$, $\alpha_2 = \sigma_2$, $\beta = s, \sigma_3$, where σ_i, $i=1,2,3$ are the 2×2 Pauli spin matrices and $s=\pm 1$.

The two choices of $s=\pm 1$ lead to inequivalent representations of the Dirac matrices. In the nonrelativistic limit, if $s=+1$ ($s=-1$) only the upper (lower) component survives, describing then a Schrödinger particle with spin $+\frac{1}{2}$ ($-\frac{1}{2}$). A more convenient choice for polar coordinates will be made later.

The vector potential $\mathbf{A}$ describes a thread of magnetic flux $(2\pi c/e)\phi$ at the origin and may be chosen as

$$\frac{e}{c}\mathbf{A}=\phi\left[\frac{-x_2\mathbf{i}+x_1\mathbf{j}}{r^2}\right], \tag{2}$$

so that

$$\nabla\times\mathbf{A}=\mathbf{H}=\left[\frac{2\pi c}{e}\right]\phi\frac{\delta(r)}{r}\mathbf{k}\ . \tag{3}$$

The Hamiltonian (1) for its complete determination requires a specification of a domain in order to make it a self-adjoint operator, thus characterizing a dynamics for the system [9]. From a more pragmatic point of view this is equivalent to choosing boundary conditions for its eigenfunctions. We shall discuss this problem by first exploring the rotation invariance of H given our choice of $\mathbf{A}$. For the eigenvalue equation

$$H_0\psi=E\psi\ , \tag{4}$$

separation of variables

$$\psi(r,\varphi)=\begin{bmatrix}\chi_1(r)\\ \chi_2(r)e^{i\varphi}\end{bmatrix}e^{in\varphi}\ , \tag{5}$$

where $n+\frac{1}{2}$ is the total (orbital+spin) angular momentum, leads to the radial eigenvalue problem

$$\mathcal{H}_n(\phi)\chi(r)=\begin{bmatrix} ms & -i\left[\partial_r+\dfrac{\nu+1}{r}\right]\\ -i\left[\partial_r-\dfrac{\nu}{r}\right] & -ms\end{bmatrix}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}=E\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}, \tag{6}$$

with $\nu=n+\phi$, where we have made use of the choice $\alpha_r\equiv(\alpha_1x_1+\alpha_2x_2)/r=\sigma_1$ and $\alpha_\varphi=(-\alpha_2x_1+\alpha_1x_2)/r=\sigma_2$, $\beta=s\sigma_3$. Notice that the Hamiltonian (1) has a symmetry $\phi\to\phi+1$ since $\mathcal{H}_{n+1}(\phi)=\mathcal{H}_n(\phi+1)$.

For the free two-dimensional Dirac equation ($\phi=0$) and for all integer values of ϕ, the only admissible boundary condition is that of regularity of the wave functions $\chi_1(r)$ and $\chi_2(r)$ at the origin (see discussion below). For $\phi\neq0$ this problem was thoroughly first discussed by de Sousa Gerbert. His finding may be summarized as follows.

For $\nu=n+\phi\geq0$ or $\nu\leq-1$, again only solutions with regular, upper, and lower components are admissible as wave functions. However, in the open interval $-1<\nu<0$ it is *impossible* to stick to the regularity condition simultaneously for both upper and lower components: an indeterminacy occurs and there is a one-parameter family of admissible boundary conditions. They are parametrized by an angle θ through

$$\chi(r)\underset{r\to0}{\sim}\begin{bmatrix} i(mr)^\nu\sin\left[\dfrac{\pi}{4}+\dfrac{\theta}{2}\right]\\ (mr)^{-(1+\nu)}\cos\left[\dfrac{\pi}{4}+\dfrac{\theta}{2}\right]\end{bmatrix},\quad -\pi\leq\theta\leq\pi\ . \tag{7}$$

Notice that for every noninteger value of the flux ϕ there is exactly one value of n, namely $n=-[\phi]-1$ (where $[x]$=largest integer$\leq x$) such that the indeterminacy occurs. Each choice of θ in (7) specifies a dynamics leading to θ-dependent spectra and cross sections.

Subsequently there were attempts by Alford *et al.* [6] and Hagen [10] to provide a physical motivation for the determination of the parameter θ. Their starting point was the replacement of the pointlike thread of flux by a magnetic field concentrated on the surface of a tube of radius R, i.e.,

$$\mathbf{H}=\left[\frac{2\pi c}{e}\right]\phi\frac{\delta(r-R)}{R}\mathbf{k}\ . \tag{8}$$

Wave functions were calculated with $R>0$ and only then the limit $R\to0$ was taken. Their result was that in this limit the wave function obeys condition (7) with $\theta=-(\mathrm{sgn}\phi)(\pi/2)$, i.e., regular upper component and singular lower component for $\phi>0$, $s=\pm1$. In particular, for $\phi>0$, the component surviving the nonrelativistic limit is regular if $s=1$, and singular if $s=-1$. In general, the component surviving the nonrelativistic limit is singular if $(\mathrm{sgn}\phi)s=-1$.

As mentioned in the Introduction and stated more carefully now, although there is nothing wrong with the computations of Refs. [6,10], the boundary condition so obtained produces, for every n, wave functions which as functions of the flux ϕ are discontinuous at $\phi=-n-1$ if $n\geq0$ and $\phi=-n$ if $n<0$. This amounts to a discontinuity in the dynamics as a function of ϕ for every nonzero integer value of ϕ. Moreover, as pointed out by Alford *et al.* [6], these boundary conditions also produce a breakdown of the symmetry $\phi\to\phi+1$ for the Hamiltonian (1). In the next section we show that there are other choices of boundary conditions which are also ϕ dependent and which are free from each or both of the above, perhaps undesirable, features. The possibility of occurrence of bound states for these boundary conditions is discussed. We then show that a more complete discussion of what happens inside or at the surface of the tube is necessary for the determination of the relevant boundary condition for the Aharonov-Bohm effect. In fact, it turns out that by conveniently suppressing the penetration of the wave function inside the tube by means of a constant repulsive potential u_R it is possible to obtain in the limit $R\to0$ all admissible boundary conditions for the problem, given by [9]. In particular, if we impose those components of the wave functions surviving the nonrelativistic limit to be zero at the surface of the tube we get the boundary conditions with the standard nonrelativistic limit and with preservation of the $\phi \to \phi+1$ symmetry. Therefore it could be argued that this boundary condition is the one of relevance to the Aharonov-Bohm effect, in spite of a discontinuity at all integer values of ϕ that, however, occurs only in those components vanishing in the nonrelativistic limit. This is the condition used by Aharonov-Bohm [1,2] in the nonrelativistic limit (see also Ref. [11]). This is shown in Sec. II.

All the above considerations, with suitable modifications, also apply to the problems of spin-1 particles [12].

The introduction of a Coulomb interaction with the source is relevant in the study of charged anyons and also in the study of the Aharonov-Bohm effect for a metallic grounded solenoid. In this case the Hamiltonian is

$$H=H_0-\frac{\xi}{r}\ , \tag{9}$$

and we show that (i)

$$\text{for } \nu>\nu_+(\xi)=\frac{-1+\sqrt{1+4\xi^2}}{2}$$

$$\text{or } \nu<\nu_-(\xi)=\frac{-1-\sqrt{1+4\xi^2}}{2}$$

the wave functions must be taken to be regular at the origin, whereas (ii) for $\nu_-(\xi)<\nu<\nu_+(\xi)$ there is again a one-parameter family of admissible boundary conditions for functions in the domain of the operator described by

$$\chi(r) \underset{r\to 0}{\sim} \begin{bmatrix} i(mr)^{\gamma_-}\sin\left[\frac{\theta}{2}+\frac{\pi}{4}\right] \\ (mr)^{\gamma_-}\cos\left[\frac{\theta}{2}+\frac{\pi}{4}\right] \end{bmatrix} \tag{10}$$

with $\gamma_\pm=-\frac{1}{2}\pm[(\nu+\frac{1}{2})^2-\xi^2]^{1/2}$. As opposed to the $\xi=0$ case, this indeterminacy for some values of the flux occurs for two or more values of the angular momentum n, as soon as $\xi\neq 0$.

III. BOUNDARY CONDITIONS IN THE LIMIT $R\to 0$

In this section we consider an especially simple model for a tube of radius R, where in addition to the tube of magnetic field [6,10] we control the penetration of the wave function inside the tube by means of a constant potential u_R. For this model the radial eigenvalue equation (6) is replaced by

$$\mathcal{H}_n(\phi)\chi(r)=\begin{bmatrix} ms & -i\left[\partial_r+\frac{\nu(r)+1}{r}\right] \\ -i\left[\partial_r-\frac{\nu(r)}{r}\right] & -ms \end{bmatrix}\chi(r)=[E-V_R(r)]\chi(r)\ , \tag{11}$$

where

$$\nu(r)=\begin{cases}\nu\ , & r\geq R\ ,\\ n\ , & r<R\ ,\end{cases}\quad V_R(r)=\begin{cases}u_R\ , & r<R\ ,\\ 0\ , & r>R\ ,\end{cases}\quad \text{and } \chi(r)=\begin{bmatrix}\chi_1(r)\\ \chi_2(r)\end{bmatrix}.$$

For $r<R$ we take the regular solution (this is actually not a matter of choice since the singular solution is not square integrable in a neighborhood of zero), and so:

$$\chi(r)=\begin{cases}\dfrac{1}{N}\begin{bmatrix} J_n(k_0r) \\ \dfrac{ik_0}{ms+E-u_R}J_{n+1}(k_0r)\end{bmatrix} & \text{for } r<R\ ,\\[2ex] \dfrac{1}{N}\begin{bmatrix} FJ_\nu(kr)+GJ_{-\nu}(kr) \\ \dfrac{-i}{ms+E}\left[F\left[\partial_r-\frac{\nu}{r}\right]J_\nu(kr)+G\left[\partial_r-\frac{\nu}{r}\right]J_{-\nu}(kr)\right]\end{bmatrix} & \text{for } r>R\ ,\end{cases} \tag{12}$$

where $k_0^2=(E-u_R)^2-m^2$, $k=(E^2-m^2)^{1/2}$, and N is an overall normalization factor. We then impose continuity of both upper and lower components at R. (Incidentally this is equivalent to imposing (i) continuity of the upper component and (ii) a jump proportional to ϕ of its derivative accounting for the δ function contribution emphasized by Hagen [10], see also the discussion by Jackiw [13].) We get

$$\left[\frac{G}{F}\right]_R=\frac{[k/(ms+E)]J_{-(1+\nu)}(kR)J_n(k_0R)+[k_0/(ms+E-u_R)]J_{-\nu}(kR)J_{n+1}(k_0R)}{[k/(ms+E)]J_{1+\nu}(kR)J_n(k_0R)-[k_0/(ms+E-u_R)]J_\nu(kR)J_{n+1}(k_0R)}\ . \tag{13}$$

If we take the limit $R \to 0$ with $u_R = u$ constant then $\lim_{R\to 0}(F/G)_R = 0$, that is, we obtain the boundary condition $\theta = -\mathrm{sgn}(\phi)(\pi/2)$, in Eq. (7). Here we used the asymptotic behavior:

$$J_\nu(x) \underset{x\to 0}{\sim} \frac{1}{\Gamma(1+\nu)} \left[\frac{x}{2} \right]^\nu .$$

Let us now discuss some peculiarities of this boundary condition. We first remark that for $\nu \geq 0$ the only admissible boundary condition is given by $\theta = \pi/2$ whereas if $\nu \leq -1$, $\theta = -\pi/2$, otherwise the wave functions are not square integrable in the neighborhood of the origin. For a given value of the orbital angular momentum n we see that in order to ensure continuity of the wave functions as a function of ϕ one has to interpolate the function

$$\theta(\phi) = \frac{\pi}{2} \quad \text{for } \phi \geq -n ,$$

$$\theta(\phi) = -\frac{\pi}{2} \quad \text{for } \phi \leq -n-1$$

in the open interval $-n-1 < \phi < -n$. If we then choose

$$\theta(\phi) = f(\phi + n + 1) , \quad -n-1 < \phi < -n ,$$

where f is a fixed *n-independent* continuous function defined in the interval (0,1), the symmetry $\phi \to \phi + 1$ is preserved. If, moreover, we require $f(\theta) = -\pi/2$ and $f(1) = \pi/2$, continuity is also ensured.

Thus the boundary conditions obtained in Ref. [6], that is, $\theta(\phi) = -(\mathrm{sgn}\phi)\pi/2$, lead to a discontinuity at $\phi = -n$, if $n < 0$ and at $\phi = -n-1$ if $n > 0$. Moreover, this choice leads also to a breakdown of the $\phi \to \phi + 1$ symmetry.

Figure 1 illustrates the above features. In Fig. 1(a) the dashed line shows the boundary condition $\theta = -(\mathrm{sgn}\phi)(\pi/2)$ for a negative n. In Fig. 1(b) the dashed line shows $\theta = -(\mathrm{sgn}\phi)(\pi/2)$ for positive n. In both figures the dashed-dotted line shows how one could interpolate between $\theta(\phi) = \pi/2$ and $\theta(\phi) = -\pi/2$ obtaining continuity and preserving the symmetry $\phi \to \phi + 1$. The asymmetry between positive n and negative n (dashed lines) explains why $\theta = -(\mathrm{sgn}\phi)(\pi/2)$ breaks the symmetry $\phi \to \phi + 1$. The discontinuity of the dynamics at nonzero integer values of ϕ, with the choice $\theta = -(\mathrm{sgn}\phi)(\pi/2)$ can be seen as follows: For *each* value of the angular momentum n a discontinuity will take place at $\phi = -n$ if $n < 0$ and $\phi = -n-1$ if $n \geq 0$.

(i) Consider first the case $n < 0$. Then the critical region $-1 < \nu = \phi + n < 0$ occurs for positive value of ϕ. If $\phi \downarrow -n$, $\nu = n + \phi$ goes to zero, $\nu \downarrow 0$, i.e., ν is outside the critical region, and therefore $\theta \downarrow \pi/2$ since $\theta = \pi/2$ for all $\nu \geq 0$ (see discussion above). Therefore the wave function behaves as

$$\chi \sim \begin{bmatrix} i(mr)^\nu \\ 0 \end{bmatrix} .$$

Consider now $\phi \uparrow -n$. Then $\nu = n + \phi$ goes to zero, $\nu \uparrow 0$, so that ν is in the critical region. Since ϕ is positive, the boundary condition θ implies $\theta = -\pi/2$. Therefore the wave function behaves as

$$\chi \sim \begin{bmatrix} 0 \\ (mr)^{-(1+\nu)} \end{bmatrix} ,$$

which is not square integrable in the limit $\nu \uparrow 0$. Therefore the limit from the left has no meaning. This means that the operator $\lim_{\phi \uparrow n} \mathcal{H}_n(\phi)$ has no limit in any of the usual senses [14]. Notice that $\mathcal{H}_n(\phi)$ is continuous at $\phi = -n-1$.

(ii) Consider now the case $n \geq 0$. Then the critical region $-1 < \nu < 0$ occurs for negative values of ϕ. If $\phi \uparrow -n-1$, then $\nu = n + \phi \leq -1$, i.e., outside the critical region with $\theta = -\pi/2$, which implies, in the limit $\nu \uparrow -1$,

$$\chi \sim \begin{bmatrix} 0 \\ (mr)^{-(1+\nu)} \end{bmatrix} .$$

However if $\phi \downarrow -n-1$, $\nu = n + \phi$ goes to $\nu \downarrow -1$ and therefore is in the critical region. The value of $\theta = -(\mathrm{sgn}\phi)\pi/2$ implies $\theta = \pi/2$ since ϕ is negative. Therefore the wave function behaves as

$$\chi \sim \begin{bmatrix} i(mr)^\nu \\ 0 \end{bmatrix} ,$$

which again is not integrable when $\nu \downarrow -1$. Again we

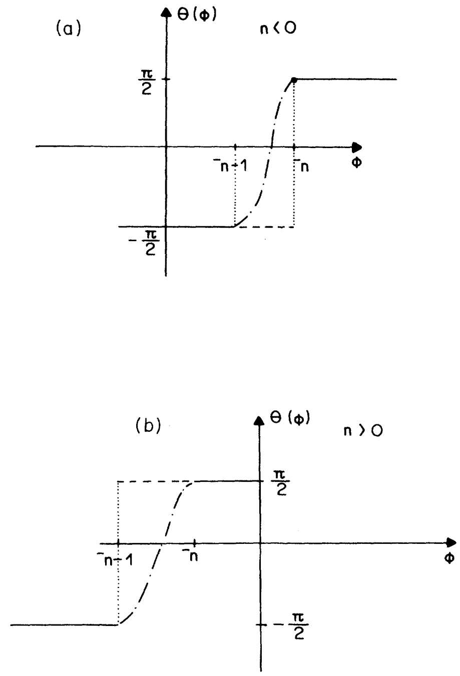

FIG. 1. The dashed line shows the boundary condition $\theta = -s(\pi/2)$ for (a) negative and (b) positive n. In both figures the dashed-dotted line shows how one could interpolate between $\theta(\phi) = \pi/2$ and $\theta(\phi) = -\pi/2$.

have an ill-defined limit $\lim_{\phi \downarrow -n-1} \mathcal{H}_n(\phi(\phi)$. Notice that there is continuity at $\phi=-n$. Notice also that the discontinuities never occur at $\phi=0$.

As shown by de Sousa Gerbert [9], bound states for these systems occur if $-\pi \leq \theta < -\pi/2$ or $\pi/2 < \theta \leq \pi$. From the above discussion it follows that if we choose the interpolating function to be monotonically increasing (as in the dashed line in Fig. 1), then θ remains outside these intervals, and so, no bound states occur.

Let us now fine-tune u_R by letting it go to $+\infty$ as $R \rightarrow 0$ at the rates:

$$Ru_R = \begin{cases} x_{n+1} + \dfrac{1}{\beta}(mR)^{-(1+2\nu)} & \text{for } -1<\nu<-\dfrac{1}{2}, \\ \bar{x} \quad \text{for } \nu=-\dfrac{1}{2}, \\ x_n - \beta(mR)^{1+2\nu} & \text{for} -\dfrac{1}{2}<\nu<0, \end{cases} \tag{14}$$

where $x_n \neq 0$ denotes the first zero of $J_n(x)$, and $\bar{x}$ is any number such that $J_n(\bar{x}) \neq 0$ and $J_{n+1}(\bar{x}) \neq 0$. Under these conditions we obtain

$$\lim_{R\rightarrow 0}\left[\frac{F}{G}\right]_R = \begin{cases} \beta\dfrac{\Gamma(1+\nu)}{\Gamma(-\nu)}\left[\dfrac{k}{2m}\right]^{-(1+2\nu)}\left[\dfrac{E-ms}{E+ms}\right]^{1/2}, & \nu \neq -\dfrac{1}{2}, \\ \dfrac{J_n(\bar{x})}{J_{n+1}(\bar{x})}\left[\dfrac{E-ms}{E+ms}\right]^{1/2}, & \nu=-\dfrac{1}{2}, \end{cases} \tag{15}$$

that is,

$$\tan\left[\frac{\theta}{2}+\frac{\pi}{4}\right] = \begin{cases} \beta, & \nu \neq -\dfrac{1}{2}, \\ \dfrac{J_n(\bar{x})}{J_{n+1}(\bar{x})}, & \nu=-\dfrac{1}{2}. \end{cases} \tag{16}$$

In conclusion, the $\theta=-\operatorname{sgn}(\phi)(\pi/2)$ boundary condition is obtained with a weak suppression of the wave function as $R \rightarrow 0$. On the other hand, the boundary condition of Refs. [1,2], $\theta=-s\pi/2$ independent of ϕ, may also be obtained. For instance, if $s=1$ take $\beta \rightarrow 0$ for $\nu \neq -\frac{1}{2}$ or $\bar{x} \rightarrow x_n$ for $\nu=-\frac{1}{2}$.

We can now see that this boundary condition also leads to discontinuity in the wave function as a function of the flux for all integer values of the flux, including $\phi=0$. However, the discontinuity occurs only in those components that vanish in the nonrelativistic limit. Figure 2 shows the discontinuity for the case $n=0$.

Let us now discuss the situation of a completely impenetrable tube, i.e., $\chi(r)=0$ for $r<R$. This situation still requires a specification of a boundary condition at R. The most natural one is the requirement of $\chi_1(R)=0$ and so:

$$\chi(r)=\frac{1}{N}\begin{bmatrix} J_{-\nu}(kR)J_\nu(kR)-J_\nu(kR)J_{-\nu}(kr) \\ \dfrac{-ik}{ms+E}[-J_{-\nu}(kR)J_{\nu+1}(kR)-J_\nu(kR)J_{-(\nu+1)}(kr)] \end{bmatrix}. \tag{17}$$

Notice that $\chi_2(R) \neq 0$, and so it is discontinuous across the boundary. In this case we have

$$\left[\frac{F}{G}\right]_R = \frac{J_{-\nu}(kR)}{J_\nu(kR)}, \tag{18}$$

and so

$$\lim_{R\rightarrow\infty}\left[\frac{F}{G}\right]_R = 0,$$

i.e., the boundary conditions of Refs. [1,2]. The condition $\chi_1(R)=0$ is the standard boundary condition in the nonrelativistic limit with fixed R[11] (see [14] for nonstandard boundary conditions for the nonrelativistic problem). Of course, all other admissible boundary conditions are obtained if we simply impose

$$\left[\frac{F}{G}\right]_R = \left[\frac{E-ms}{E+ms}\right]^{1/2}\left[\frac{k}{2m}\right]^{-(1+2\nu)}\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right] \tag{19}$$

and take the limit $R \rightarrow 0$ with θ fixed. In conclusion, if we choose to have inpenetrability of the tube, the "correct" boundary conditions as $R \rightarrow 0$ depend on the physics at the surface of the tube for $R>0$.

IV. THE EFFECT OF A COULOMB INTERACTION WITH THE SOURCE

We can again separate variables in the study of the eigenvalue problem for the Hamiltonian (9), and the radial eigenvalue problem then reads (for simplicity we consider only the case $s=1$)

$$\mathcal{H}_n(\phi,\xi)\chi(r)=\begin{bmatrix} m-\frac{\xi}{r} & -i\left[\partial_r+\frac{\nu+1}{r}\right] \\ -i\left[\partial_r-\frac{\nu}{r}\right] & -\left[m+\frac{\xi}{r}\right]\end{bmatrix}\chi(r)=E\chi(r)\ . \tag{20}$$

Introducing for further symmetrization of

$$u(r)=\begin{bmatrix} u_1(r) \\ u_2(r)\end{bmatrix}\ ,\quad \chi(r)=\frac{u(r)}{r^{1/2}}$$

one obtains

$$\begin{bmatrix} m & -i\left[\partial_r+\frac{\nu+\frac{1}{2}}{r}\right] \\ -i\left[\partial_r-\frac{\nu+\frac{1}{2}}{r}\right] & -m\end{bmatrix}u(r)=\left[E-\frac{\xi}{r}\right]u(r)\ . \tag{21}$$

Let us then analyze the asymptotic behavior of the solution when $r\to 0$. Assuming then $u_1(r)=ar^{\gamma}$, $u_2=br^{\gamma}$, $|a|+|b|>0$, and inserting it in Eq. (21), we get

$$\begin{bmatrix} \xi & -i(\gamma+\nu+\frac{1}{2}) \\ -i(\gamma-\nu-\frac{1}{2}) & \xi\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix}=0 \tag{22}$$

and this implies

$$\gamma=\pm\sigma,\quad \sigma=\sqrt{(\nu+\tfrac{1}{2})^2-\xi^2}\ . \tag{23}$$

Let us now analyze the square integrability of the singular solution

$$\chi(r)=\begin{bmatrix} a \\ b\end{bmatrix}r^{-\sigma-1/2}\ .$$

Computing

$$\int_0^1 r\,dr\,\chi(r)^{\dagger}\chi(r)\cong(|a|^2+|b|^2)\int_0^1 r^{-2\sigma}dr \tag{24}$$

we see that if $2\sigma\geq 1$, the irregular solution is not admissible, and therefore the dynamics is uniquely defined. Technically this means that when restricted to a function

$$\psi(r)=\begin{bmatrix}\psi_1(r,\theta) \\ \psi_2(r,\theta)\end{bmatrix}\quad \text{with } \chi_1,\chi_2\in C_0^{\infty}(\mathbb{R}^2/\{0\})$$

(infinitely differentiable functions with support which is compact and does not contain the origin), the operator (9) is essentially self-adjoint [15].

If, however, $2\sigma<1$, we have an indeterminacy of the same type discussed before, and a specification of a boundary condition is required. Notice that the condition $2\sigma<1$ translates into

$$\nu_-(\xi)=-\tfrac{1}{2}-\sqrt{\tfrac{1}{4}+\xi^2}<\nu<-\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+\xi^2}=\nu_+(\xi)\ . \tag{25}$$

Therefore the range of values of ν for which there is an indeterminacy is enlarged by the introduction of the Coulomb interaction. It has a length

$$\nu_+(\xi)-\nu_-(\xi)=\sqrt{1+4\xi^2}>1$$

if $\xi^2>0$. A remarkable consequence of this fact is that for some values of ϕ it is not possible to have two values of the angular momentum for which ν falls in the critical region given by (25), as soon as $\xi^2>0$. If we take ξ^2 sufficiently large, we may have an arbitrarily given number of values of n so that ν falls in the region given by (25). On the other hand, this is in agreement with the fact that even in the free case, $\phi=0$, the introduction of a Coulomb interaction produces an indeterminacy for all values of $\xi\neq 0$, namely for all values of angular momentum $s=n+\frac{1}{2}$ in the range

$$-\sqrt{\tfrac{1}{4}+\xi^2}<j<+\sqrt{\tfrac{1}{4}+\xi^2} \tag{26}$$

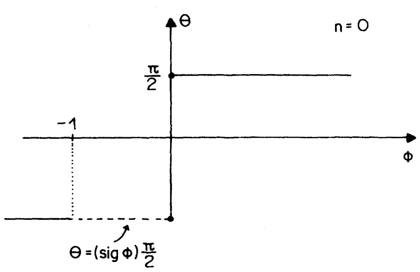

FIG. 2. The dashed line shows the boundary condition $\theta=-s(\pi/2)$ for $n=0$.

as follows from (25) with $\nu=n$. This result should be compared with the results for the Dirac equation in three space dimensions with a Coulomb interaction [15,16], where essential self-adjointness (only regular solutions) exists if $|\xi|<\frac{3}{4}$.

Using standard methods (see, for instance, the paper by de Sousa Gerbert [9] and references therein), we obtain the admissible boundary conditions given in Eq. (10).

Alternatively, one could have reached the same conclusions by writing down the general solution of Eq. (21) in terms of confluent hypergeometric functions as in p. 89 of Ref. [17].

We may now repeat the discussion of Sec. II, with the introduction of a tube of radius R, taking the magnetic field as given by Eq. (8) and introducing an external potential $W_R(r)$ used in Eq. (11) which now reads:

$$\begin{bmatrix} m & -i\left[\partial_r+\dfrac{\nu(r)+1}{r}\right] \\ -i\left[\partial_r-\dfrac{\nu(r)}{r}\right] & -m \end{bmatrix}\chi(r)=[E-W(r)]\chi(r)\ , \tag{27}$$

where

$$W(r)=\begin{cases} u_R\ , & r<R\ , \\ \dfrac{\xi}{r}\ , & r>R\ . \end{cases} \tag{28}$$

We are switching off the Coulomb potential inside the tube, in order to make only the regular solution at the origin acceptable as long as R is positive. The general solution is now

$$\chi(r)=\begin{cases} \dfrac{1}{N}\begin{bmatrix} J_n(k_0r) \\ \dfrac{ik_0}{m+E-u_R}J_{n+1}(k_0r) \end{bmatrix}\ , & r<R\ , \\ \dfrac{1}{N}\begin{bmatrix} F\,R_{\gamma,\nu}(kr)+G\,S_{\gamma,\nu}(kr) \\ \dfrac{-i}{m+\left[E-\dfrac{\xi}{r}\right]}\left[F\left[\partial_r-\dfrac{\nu}{r}\right]R_{\gamma,\nu}(kr)+G\left[\partial_r-\dfrac{\nu}{r}\right]S_{\gamma,\nu}(kr)\right] \end{bmatrix}\ , & r>R\ , \end{cases}$$

where $R_{\gamma,\nu}(kr)$ and $S_{\gamma,\nu}(kr)$ are respectively the regular and singular solutions of Eq. (27) for $r>R$. So

$$\frac{F}{G}=\frac{J_n(k_0R)(\partial_r-\nu/R)S_{\gamma,\nu}(kR)-[ik_0/(m+E-u_R)]J_{n+1}(k_0R)S_{\gamma,\nu}(kR)}{R_{\gamma,\nu}[ik_0/(m+E-u_R)]J_{n+1}(k_0R)+J_n(k_0R)\{i/[m+(E-\xi/R)]\}\left[\partial_r-\dfrac{\nu}{R}\right]R_{\gamma,\nu}(kR)}\ ,$$

and it is now clear how to fine-tune u_R, as in Eq. (14) with the changes $\nu\to\gamma_-$ in order to obtain all admissible boundary conditions.

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