

Multiloop stringlike formulas for QED

C. S. Lam*

Physics Department, McGill University, Montreal, Québec, Canada H3A 2T8

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Multiloop gauge-theory amplitudes written in the Feynman-parameter representation are poised to take advantage of two important developments of the past decade: the spinor-helicity technique and the superstring reorganization. The former has been considered in a previous paper; the latter will be elaborated in this paper. We show here how to write multiloop stringlike formulas in the Feynman-parameter representation for any diagram in QED, including those involving other non-electromagnetic interactions, provided the internal photon lines are not adjacent to any external photon line. The general connection between the Feynman-parameter approach and the superstring and/or first-quantized approach is discussed. In the special case of a one-loop multiphoton amplitude, these formulas reduce to the ones obtained by the superstring and the first-quantized methods. The stringlike formulas exhibit a simple gauge structure which makes the Ward-Takahashi identity apparent, and enables the integration-by-parts technique of Bern and Kosower to be applied, so that gauge-invariant parts can be extracted diagram by diagram with the seagull vertex neglected.

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I. INTRODUCTION

Feynman diagram calculations for gauge theories are always complicated because of the presence of spin and gauge dependence. Spin brings in a derivative coupling or Dirac algebra; gauge dependence introduces many gauge noninvariant terms which will be canceled at the end. For non-Abelian theories such as QCD the situation is even worse, for there is the additional complication of color algebra, as well as the presence of three- and four-gluon vertices, and ghosts.

Two techniques have been developed in the past decade to simplify gauge theory calculations for tree and one-loop diagrams. The first is the *spinor-helicity technique* [1–21], which makes use of the fact that most fermions can be treated as massless at high energies. Being massless, chirality is conserved, a fact which can be exploited effectively to reduce the number of terms in a significant manner. Even photons and gluons benefit from this conservation because a spin-one index can be written as a pair of dotted and undotted spinor indices. Furthermore, one may make use of the gauge freedom to choose the polarization vectors wisely to render many more terms zero. Polarized amplitudes benefit enormously from this technique but even unpolarized cross sections for which this technique was first invented are greatly simplified. This technique, however, is applicable only when all the momenta in the problem are linear combinations of massless momenta. This is so for tree diagrams, but not true for loop diagrams where off-shell loop momenta are present to ruin chirality conservation. Consequently most of the

applications of this technique has been confined to tree amplitudes [1–21].

This problem can be circumvented if the off-shell loop momenta are integrated out before the spinor-helicity technique is applied. One way of doing so is to make use of the connection between a gauge theory and a superstring theory in the infinite-tension limit [22–24]. There is a known formula for a one-loop superstring amplitude in which integrations are over the Koba-Nielsen variables with internal momenta absent. This formula for a one-loop multigluon (or multiphoton) amplitude can also be derived from a first-quantized particle theory interacting with a background gauge field [25], where the particle in the theory is taken to be the one running around the loop. The formula obtained in the first-quantized theory is identical to the superstring formula presumably because the particle theory used in this approach is also reparametrization invariant.

Unfortunately, the generalization of either of these two approaches beyond one loop, and to all possible scattering amplitudes, have not yet been worked out.

A different method to avoid the off-shell momenta is to introduce Feynman parameters to integrate them out [26]. The spinor-helicity technique is then applicable in the resulting Feynman-parameter representation, in any process and to any number of loops [27].

The second technique developed in the last decade is independent of the spinor-helicity technique, but works very well in conjunction with it. It can perhaps be labeled as *string reorganization* [23]. For tree and certain one-loop amplitudes when string formulas are available, one finds that the individual terms of the string formula in the infinite-tension limit are not identical to the terms obtained from the standard Feynman rules, although the sum of all the terms are necessarily the same for both. Moreover, as a rule, the terms in the string formula are organized in a much neater and a much simpler way, and

*Electronic address: Lam@physics.mcgill.ca

this beautiful reorganization can greatly simplify calculations. Why the string is so clever is not completely clear; it may have to do with the fact that it treats spacetime and color on an equal footing. I shall refer to field-theoretic formulas reorganized in the string way as *stringlike formulas*.

Because of its neatness and simplicity, and potential simplifications achieved in actual calculations, it would be desirable to develop stringlike formulas for all processes in any number of loops. It is not *a priori* clear that this can be done, and if so in what form the stringlike formulas would take, because we only know them for some amplitudes and only to one-loop order. Fortunately it turns out that in many cases the stringlike formula in one loop is sufficient of a guide for obtaining string reorganization in many loops. For example, in tree and one-loop QCD, the string [4,15–17] reorganizes the usual color factors into the Chan-Paton factors [28], which have the advantage of having the corresponding subamplitudes gauge invariant and cyclic-permutation invariant. This reorganization can be generalized to all loops and all amplitudes [27].

The background gauge emerges naturally from the one-loop string calculations of QCD [23] and the first quantized approach [25]. Its use simplifies the calculations considerably in any number of loops as well, especially when a large number of external gluon or photon lines are present [27]. This again illustrates the superiority of string reorganization.

There is yet another important feature in the one-loop stringlike formula which is the main subject of this paper. I shall refer to this feature as the *gauge characteristic*, because it makes explicit the gauge-transformation property and the Ward-Takahashi identity in the Feynman-parameter space. To see its significance take a *momentum-space* diagram in spinor QED with some external photon lines. A gauge transformation of the polarization vector produces two additional terms given by shrinking one of the two spinor propagators to which this external photon is attached. It is this local feature on which the Ward-Takahashi identity depends. In *Feynman-parameter space*, this local feature is lost, because rules governing Feynman-parameter amplitudes tend to involve the diagram globally as a whole [26]. Nevertheless, when the amplitude is reorganized into a stringlike formula, this local feature reappears once again. The details will be discussed in Sec. III, but what happens is that in this stringlike form, the additional terms arising from the gauge transformation are proportional to derivatives of the integrand with respect to some Feynman parameters α_r . The surface terms obtained by integrating these derivatives correspond to diagrams with α_r set equal to zero, which is equivalent to shrinking and short circuiting the corresponding propagators. In this way the essential local feature of the momentum space leading to the Ward-Takahashi identity is restored. This is the gauge characteristic alluded to before.

This gauge characteristic of the stringlike formula has been put to good use in the known one-loop stringlike formulas [22–25]. To explain it let us define a diagram to be a *standard diagram* when none of the *external gauge-*

particle lines are connected to the diagram either through a seagull vertex (in scalar electrodynamics) or through a four-gluon vertex (in QCD). All other diagrams will be called *seagull diagrams*. Because of the presence of this gauge characteristic in the *standard diagrams*, it is possible to perform integration by parts (IBP) in the Feynman parameters. At least in simple cases, the gauge-dependent surface terms together add up to cancel contributions from the seagull diagrams [22–25]. If this is true in general then we never have to worry about the surface terms nor the seagull diagrams. This greatly simplifies the calculation because there are many many seagull diagrams, and because in this way the gauge-invariant parts can be extracted *diagram by diagram* after discarding the surface terms from suitable IBP's.

This gauge clarity of the stringlike formula and the resulting simplification from the IBP technique makes it important to ask whether stringlike formulas can be written for all loops and all processes. We shall leave non-Abelian gauge theories aside for future considerations. The purpose of this paper is to show that stringlike formulas with the gauge characteristic can be written for standard diagrams in QED, in the Feynman-parameter representation to any number of loops and for all processes, provided a technical restriction to be discussed in Sec. III is obeyed.

In order to derive such multiloop formulas, considerable knowledge of the details of the Feynman-parameter representation and the electric circuit analogy [26,29–32] is needed. We therefore start out in Sec. II with a discussion of these topics. Some but not all of the formulas appearing there are already contained in [26], but the important and subtle roles of *external vertices* and of *level dependence* were not sufficiently discussed before. Currents in an electric circuit are the important objects to be considered in momentum-space amplitudes, and voltages are the primary quantities to be dealt with in the configuration-space amplitudes. The voltage at a vertex corresponds to the position of that point in the configuration space. Translational invariance in configuration space corresponds to the impossibility of determining the absolute (rather than the relative) voltage levels, which in turn is the cause of the level-dependence problem. Although there is no direct relationship, it is however interesting to note that in many ways level dependence in an electric circuit is analogous to gauge dependence in a gauge theory.

Together with the electric circuit analogy of a Feynman diagram it is also known that there is a particle interpretation of the same, which as we shall see comes very close to being a multiloop generalization of the string approach of Bern and Kosower [22] and the first quantized approach of Strassler [25]. This will also be discussed in Sec. II.

One feature of the stringlike formula is that it treats external vertices somewhat differently than the internal vertices. This can be seen, for example, in the appearance of the background gauge in QCD [23,25]. We shall see in Sec. II that to some extent this is a general feature, in that the topological structure of Feynman diagrams is such that certain relations which hold for external ver-

tices are no longer true for internal vertices.

The multiloop stringlike formula for scalar QED with the gauge characteristic will be derived in Sec. III. A similar formula for spinor QED will be discussed in Sec. IV. An explicit two-loop example for a photon-meson Compton scattering diagram is given in Sec. V to illustrate some of these results.

II. FEYNMAN-PARAMETER REPRESENTATION

Any ℓ -loop scattering amplitude with N internal and n external lines is given by a momentum-space integral

$$M(p) = \left[\frac{-i}{(2\pi)^4} \right]^l \int \prod_{\alpha=1}^{\ell} (d^4 k_{\alpha}) \frac{S_0(q)}{\prod_{r=1}^N (-q_r^2 + m_r^2 - i\epsilon)}, \quad (1)$$

where the internal momenta q_r are understood to be linear combinations of the external *outgoing* momenta p_i and the loop momenta k_{α} . An external momentum p_i will be assigned to *every* vertex i ; this allows certain algebraic manipulations that would otherwise be impossible. At the end of the calculation all artificially added p_i will be set equal to zero.

Vertex factors as well as numerators of spinning propagators are incorporated into the “*primitive spin factor*” $S_0(q)$. We shall work in four dimensions but it is just as easy to develop formulas for an arbitrary dimension. By introducing the Feynman parameters α_r , the loop momenta can be integrated out and a Feynman-parameter representation of the amplitude can be obtained [26]:

$$M(p) = \frac{i^N}{(-16\pi^2)^l} \int_0^{\infty} \left(\prod_{r=1}^N d\alpha_r \right) \Delta(\alpha)^{-2} \times \exp[-iD(\alpha, p)] S(J), \quad (2)$$

$$D(\alpha, p) = \sum_{r=1}^N \alpha_r m_r^2 - P(\alpha, p), \quad (3)$$

$$P(\alpha, p) = \sum_{r=1}^N \alpha_r J_r^2. \quad (4)$$

Equation (2) is written in the Schwinger proper-time formalism, or the “Nambu representation.” One can recover from it the “Chisholm representation” quoted in [27] by making the substitution $\alpha_r = \bar{\alpha}_r \lambda$, with $\sum_r \bar{\alpha}_r = 1$, and carrying out the integration over λ .

To understand and to describe the quantities J, P, Δ , and S appearing in (2), it is useful to know that a Feynman diagram can be thought of as a passive electric circuit [26,29–32], in which α_r takes on the role of resistance and p_i become the currents flowing out of the circuit. The quantity J_r is then the current flowing through the r th internal line, and P is the power consumed by the circuit. Explicit formulas for these quantities are available [26],

$$\Delta(\alpha) = \sum_{T_1} \left(\prod_{\alpha}^{\ell} \alpha \right), \quad (5)$$

$$P(\alpha, p) = \Delta(\alpha)^{-1} \sum_{T_2} \left(\prod_{\alpha}^{\ell+1} \alpha \right) (\sum p)^2, \quad (6)$$

$$J_r(\alpha, p) = \pm \Delta(\alpha)^{-1} \sum_{T_2(r)} \alpha_r^{-1} \left(\prod_{\alpha}^{\ell+1} \alpha \right) (\sum p), \quad (7)$$

and they mean the following. An ℓ -loop diagram can be made into a connected tree diagram (a “1 tree”) by cutting ℓ lines, and into a diagram with two disjoint trees (a “2 tree”) by cutting $\ell + 1$ lines. $\Delta(\alpha)$ is given by the sum over the set T_1 of all 1 trees so obtained, with the summand consisting of the product of the α ’s of the cut lines. $P(\alpha, p)$ is given by the sum over the set T_2 of all 2 trees so obtained, with the summand being the product of the α ’s of the cut lines, times the square of the sum of all the external momenta p_i attached to one of these two trees. Finally, let $T_2(r)$ be the set of all 2 trees in which line r is cut, and such that when the line r is inserted back a 1 tree results. Then $J_r(\alpha, p)$ is given by the sum of all 2 trees $T_2(r)$ with the summand equal to the product α ’s of all the cut lines except the r th, times the sum of all the external momenta p_i attached to one of these two trees. The sign \pm in (7) is determined by the orientation of J_r .

We proceed now to describe the “*modified spin factor*” $S(J)$ in (2). It is made up of the sum of several terms,

$$S(J) = \sum_{k \geq 0} S_k(J), \quad (8)$$

of which the first $S_0(J)$ is just the numerator factor $S_0(q)$ in (1) with q replaced by J . The other terms S_k are obtained from S_0 by contracting k pairs of J ’s in all possible ways according to the rule

$$J_r^{\mu} J_s^{\nu} \rightarrow -\frac{i}{2} g^{\mu\nu} H_{rs}(\alpha), \quad (9)$$

and summing over all the contracted results. The formula for the *contraction function* H_{rs} is

$$H_{rr}(\alpha) = -\Delta(\alpha)^{-1} \partial \Delta(\alpha) / \partial \alpha_r, \quad (10)$$

$$H_{rs}(\alpha) = \pm \Delta(\alpha)^{-1} \sum_{T_2(rs)} (\alpha_r \alpha_s)^{-1} \left(\prod_{\ell+1} \alpha \right) \quad (r \neq s),$$

where $T_2(rs)$ is the set of all 2 trees with both lines r and s cut, and such that when either line r or line s is inserted back a 1 tree results. The product of α ’s in (10) are over all the cut lines except the r th and the s th. If the lines r and s both point to the same tree, then the sign in (10) is $+1$. If they point to different trees, then the sign is -1 .

For practical calculations, Eq. (2) may not be the best to use because the contractions leading up to $S_k(J)$ are relatively complicated to compute. It may often be simpler to use the alternate formula [26]

$$M(p) = \frac{i^N}{(-16\pi^2)^l} \int_0^\infty \left(\prod_{r=1}^N d\alpha_r \right) \frac{1}{\Delta(\alpha)^2} \times T \left(-\frac{i}{2} \frac{\partial}{\partial p} A\beta \right) \exp[-iD(\alpha, p)] , \tag{11}$$

where the function $T(q)$ is obtained from the function $S_0(q)$ by the formula

$$T(q) = \sum_{k \geq 0} T_k(q) , \tag{12}$$

with $T_0(q) = S_0(q)$, and other $T_k(q)$ obtained from $T_0(q)$ by contracting k pairs of q 's according to the rule

$$q_r^\mu q_s^\nu \rightarrow \frac{i}{2} \beta_r \delta_{rs} g^{\mu\nu} , \tag{13}$$

and summing over all possible contractions. The quantity β_r is the conductance of the line r :

$$\beta_r = (\alpha_r)^{-1} . \tag{14}$$

The contraction rule (13) is far simpler than the contraction rule (9). For example, if each q_r does not appear more than once in $S_0(q)$, then no contraction according to (13) is possible because of the δ_{rs} factor, so $T(q) = S_0(q)$, whereas this is not the case according to (9) and $S(J) \neq S_0(J)$.

In (11), the argument q_r in $T(q)$ is replaced by the operator

$$d_r \equiv -\frac{i}{2} \sum_i \frac{\partial}{\partial p_i} A_{ir} \beta_r , \tag{15}$$

where A_{ir} is the *incidence matrix* of the graph, defined to be +1 if line r points into the vertex i , -1 if the line points out of the vertex, and 0 if line r is not connected to the vertex i . Momentum conservation at each vertex can be expressed with the help of this matrix to be

$$p_i = \sum_r A_{ir} J_r . \tag{16}$$

If line r points from vertex i to vertex j , then we shall also write it as $r = (ij)$. In that case (15) simply says that each $q_{(ij)}$ should be replaced by $(-i/2)\beta_{(ij)}[\partial/\partial p_i - \partial/\partial p_j]$.

Let us now look deeper into the various circuit quantities and their relationships. Some of the relations described below already appeared in [26], but the external-vertex relations and the level-dependent relations have not. It is important to understand these formulas, especially the subtle role of level dependence, for they will be needed in deriving the multiloop stringlike formulas in Sec. III.

Let V_i be the voltage at vertex i , and

$$v_r = v_{(ij)} = V_i - V_j = - \sum_i A_{ir} V_i \tag{17}$$

be the voltage drop across the resistor r . The current flowing through that line is

$$J_r = \beta_r v_r . \tag{18}$$

Combining (16)-(18) one gets

$$p = -(A\beta A^t)V \equiv -YV , \tag{19}$$

where p, V are n -dimensional vectors with components p_i and V_i , respectively, A is the $n \times N$ matrix with matrix elements A_{ir} , and $\beta = \alpha^{-1}$ is a diagonal $N \times N$ matrix with diagonal matrix elements $\beta_r = \alpha_r^{-1}$. The power consumed by the network is then

$$P = -Vp = VYV . \tag{20}$$

The absolute level of the voltages V_i are of course never determined; they can all be shifted by a common constant without changing any physical attribute. This is reflected in (19) by the fact that

$$\sum_j Y_{ij} = 0 , \tag{21}$$

which follows algebraically from $\sum_i A_{ir} = 0$. Being a symmetric matrix we must also have $\sum_i Y_{ij} = 0$, and in (19) this simply expresses conservation of the external currents. The matrix Y is singular, so (19) cannot be inverted to obtain V as a function of p , in agreement with the fact that the absolute level of V cannot be determined. This *level dependence* is analogous to a gauge dependence. It is unphysical, it complicates matter, but often we have no choice but to fix a *level scheme* (analogous to fixing a gauge) to carry out explicit calculations. For example, we must fix a level scheme before the inversion of (19) can be carried out.

In order to invert (19) let us fix a level scheme by choosing $V_n = 0$. We shall use a subscript 0 to denote the remaining $(n - 1)$ -dimensional quantities. Then (19) can be inverted to give

$$V_0 = -Y_0^{-1} p_0 \equiv -Z_0 p_0 . \tag{22}$$

Incorporating $V_n = 0$ we can enlarge this $(n - 1)$ -dimensional relationship into the n -dimensional relationship

$$V = -Z' p , \tag{23}$$

$$Z' = \begin{pmatrix} Z_0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{24}$$

We shall refer to this level scheme as the *primitive level scheme*. The chief advantage of this scheme is that an explicit formula for Z' is available, via Eqs. (24), (22), and (19).

The impedance matrix Z' is a level-dependent quantity. Because of p conservation, a change

$$Z'_{ij} \rightarrow Z_{ij} = Z'_{ij} + a_i + a_j \tag{25}$$

for any a_i simply changes the overall level of V_i by an amount $-\sum_j a_j p_j$ but it does not change the physical attributes of the network. This level change is analogous to a gauge transformation. Physically measurable quantities such as the current J_r and the power P must be level independent. In fact, from (17)–(25),

$$J = -\beta A^t V = \beta A^t Z' p = \beta A^t Z p, \quad (26)$$

$$P = -V p = p Z' p = p Z p = J \alpha J, \quad (27)$$

and their level independence follows easily from

$$\sum_i p_i = \sum_i A_{ir} = 0. \quad (28)$$

The variation of these level-independent quantities with respect to a change of α is most easily calculated in the $V_n = 0$ level scheme. Using (19) and (22)–(24) one gets

$$\frac{\partial P}{\partial \alpha_s} = p \frac{\partial Z'}{\partial \alpha_s} p = (p Z' A \beta)_s (\beta A^t Z' p)_s = J_s^2, \quad (29)$$

$$\begin{aligned} \frac{\partial J_r}{\partial \alpha_s} &= -\beta_s \delta_{rs} J_s + [\beta A^t Z' A \beta]_{rs} (\beta A^t Z' p)_s \\ &= H_{rs} J_s, \end{aligned} \quad (30)$$

in which we have used the definition of $H_{rs}(\alpha)$:

$$H(\alpha) = G(\alpha) - \beta, \quad (31)$$

$$\begin{aligned} G_{rs} &= [\beta A^t Z A \beta]_{rs} \\ &= \beta_r \beta_s \{Z_{ik} - Z_{jk} - Z_{il} + Z_{jl}\}, \end{aligned} \quad (32)$$

for $r = (ij)$ and $s = (kl)$. The level independence of (29)–(32) can be easily checked using (28). Note that it is this same quantity H_{rs} that appeared in (9) and (10) [26].

Equations (26), (27), (31), and (32) allow us to see directly why (2) and (11) are equivalent. To that end, first notice that the only quantity in the scalar integral in (11) depending on p is $P(\alpha, p)$. A single operator (15) operating on $\exp(iP)$ then brings down $(p Z A \beta)_r = J_r$, which corresponds to the replacement $q_r \rightarrow J_r$ used in (2). If $S_0(q) = q_r^\mu q_s^\nu$, then $T(d) = d_r^\mu d_s^\nu + (i\beta_r/2)\delta_{rs}g^{\mu\nu}$, so operating on $\exp(iP)$, $T(d)$ brings down

$$\begin{aligned} J_r^\mu J_s^\nu - \frac{i}{2} \{[\beta A^t Z A \beta]_{rs} g^{\mu\nu} - \beta_r \delta_{rs} g^{\mu\nu}\} \\ = J_r^\mu J_s^\nu - \frac{i}{2} H_{rs} g^{\mu\nu}, \end{aligned} \quad (33)$$

in agreement with (8) and (9). The equivalence for a more complicated $S_0(q)$ can be seen similarly.

One might think that only level-independent quantities need be considered in the Feynman-parameter representation. Indeed both (2) and (11) contain only level-independent quantities. This of course does *not* mean that certain calculations cannot be carried out more easily in one level scheme than another. In fact, as we shall see later, the gauge characteristic mentioned in the Intro-

duction is borne out only in the particular level scheme described below.

By using the level freedom (25) one can choose a_i so that

$$Z_{ii} = 0 \quad \forall i. \quad (34)$$

It is in this *zero-diagonal level scheme* that the gauge characteristic emerges. It is also in this scheme that the impedance matrix element Z_{ij} can be computed most easily using a graphical rule. This rule can be derived from the rule (6) for P , using (27) and (34). If we replace $(\sum p)^2$ in (6) by $-(\sum p)(\sum' p)$, where the two sums are, respectively, sums of outgoing momenta attached to the two disjoint trees, then clearly there are no p_i^2 terms contributing to (6), and the corresponding level scheme is therefore given by (34). The graphical rule in the zero-diagonal scheme is therefore

$$Z_{ij} = -\frac{1}{2} \sum_{T_2^{ij}} \prod_{\alpha} \alpha \quad (i \neq j), \quad (35)$$

where the sum is over the set of all 2 trees T_2^{ij} in which the vertices i and j lie in two different trees.

The zero-diagonal level scheme of (34) treats all indices symmetrically, whereas the primitive level scheme of (24) does not, though the latter has the simplicity of eliminating an irrelevant degree of freedom in a simple manner. In these respects the two level schemes are, respectively, analogous to the covariant and physical gauges in electrodynamics, in that the former is manifestly covariant and the latter contains no longitudinal photons.

We will now derive three *level-dependent relations* [Eqs. (43), (47), and (48) below], which will be the basis of the stringlike formulas of Secs. III and IV. In this regard it is important to note that due to external momentum conservation, neither J_r nor P is a unique function of the n variables p_i , so the derivatives $\partial P/\partial p_i$ and $\partial J_r/\partial p_i$, treating each of the n variables p_i as independent, is not uniquely defined. However, as can be seen in (26) and (27), this ambiguity is related to the level dependence of Z . Once a level scheme is defined, such derivatives become meaningful, though the results are obviously level dependent. Nevertheless, the combination of the derivatives in (15) is still level independent.

We shall call a vertex with *two* internal lines (and any number of external lines) an *external vertex*, and any vertex with more than two internal lines an *internal vertex*. This terminology is unfortunately somewhat misleading because internal vertices can contain external lines as well. At an external vertex a we will make the convention that one of the two internal lines points into the vertex ($A_{ar} = +1$), and the other points out of the vertex ($A_{ar} = -1$). The line upstream of the vertex (with $A_{ar} = +1$) will also be labeled as line a'' , and the line downstream of the vertex will be labeled as line a' . Current conservation at that vertex then takes the form

$$p_a = J_{a''} - J_{a'}. \quad (36)$$

We shall see that external vertices possess special properties not shared by the internal vertices.

For each external vertex a , define the operator

$$\partial_a = - \sum_r A_{ar} \frac{\partial}{\partial \alpha_r} = - \frac{\partial}{\partial \alpha_{a''}} + \frac{\partial}{\partial \alpha_{a'}} . \quad (37)$$

An important property of an external vertex a is that $\Delta(\alpha)$, Z_{ij} , and H_{rs} depend on $\alpha_{a'}$ only through the combination $\alpha_{a'} + \alpha_{a''}$, provided $i \neq a \neq j$ and $A_{ar} = A_{as} = 0$, as can be seen from the graphical rules (5), (10), and (35). As a consequence,

$$\partial_a \Delta(\alpha) = \partial_a H_{rs} = \partial_a Z_{ij} = 0 \quad (38)$$

provided $i \neq a \neq j$ and $A_{ar} = A_{as} = 0$.

Suppose a is an external vertex and s any internal line. Then it follows from (10) that

$$H_{a's} = H_{a''s} . \quad (39)$$

To see that suppose first that lines a' , a'' , s are all different. Then the claim follows because there is a one-to-one correspondence between 2 trees $t_2(a's)$ and $t_2(a''s)$: instead of cutting line a' to form a 2 tree, just cut line a'' . Note that lines a' and a'' belong to the same loop so they cannot be cut simultaneously either in $t_2(a's)$ or in $t_2(a''s)$. With this one-to-one correspondence, the equality in (39) follows because $\Delta H_{a's}$ does not contain $\alpha_{a'}$ and $\Delta H_{a''s}$ does not contain $\alpha_{a''}$. Now suppose lines a' and s are the same. Then we must show that $H_{a'a'} = H_{a''a'}$. Since $\Delta H_{a'a'} = -\partial \Delta / \partial \alpha_{a'}$, the terms in Δ linear in $\alpha_{a'}$ is $-\alpha_{a'} H_{a'a'}$. We shall now show that it is also given by $-\alpha_{a'} H_{a''a'}$, thereby showing that $H_{a'a'} = H_{a''a'}$. To do so, consider any 2 tree $t_2(a''a')$ used to compute $H_{a''a'}$. Since lines a' and a'' are now adjacent to each other, one of the two trees in $t_2(a''a')$ simply consists of the vertex a , and the other is a tree obtained from the original diagram with the vertex a removed. There is a one-to-one correspondence between such a tree and a 1 tree of the original diagram when line a' is cut. Hence the term of Δ linear in $\alpha_{a'}$ is $-\alpha_{a'} H_{a''a'}$. This completes the proof of (39).

Equations (38) and (39) are special features of the external vertices not generally shared by internal vertices. We shall refer to them as *external-vertex relations*.

If a, b are external vertices, then $\partial_a P$ can be computed from (29) and (36) to be

$$\partial_a P = - \sum_r A_{ar} J_r^2 = -J_{a''}^2 + J_{a'}^2 = -p_a (J_{a''} + J_{a'}) . \quad (40)$$

Alternatively, using (28) and (38) one gets

$$\partial_a P = \partial_a \left(\sum_{i,j} p_i Z_{ij} p_j \right) = 2p_a \partial_a \left(\sum_j Z_{aj} p_j \right) . \quad (41)$$

Equating (40) and (41) one concludes that

$$p_a \left[J_{a''} + J_{a'} + 2\partial_a \left(\sum_j Z_{aj} p_j \right) \right] = 0 \quad (42)$$

for every possible p , subject of course only to the restriction $\sum_i p_i = 0$. This means that

$$\begin{aligned} J_{a''} + J_{a'} &= -2\partial_a \left(\sum_j Z_{aj} p_j \right) \\ &= 2\partial_a V_a \\ &\equiv -2 \sum_j \dot{Z}_{aj} p_j \equiv 2\dot{V}_a . \end{aligned} \quad (43)$$

For an external vertex a , it is also useful to define the operator

$$D_a^\mu = \partial_a \frac{\partial}{\partial (p_a)_\mu} . \quad (44)$$

Then it is clear from (43) that we can also write

$$J_{a''} + J_{a'} = -D_a P . \quad (45)$$

Next, employ (29) and (30) to compute $\partial_a \partial_b P$:

$$\partial_a \partial_b P = 2 \sum_{r,s} A_{ar} A_{bs} J_r H_{rs} J_s . \quad (46)$$

Since a, b are external vertices, (40) is true, so using (36) and (39) we get

$$\partial_a \partial_b P = 2p_a p_b H_{a'b'} .$$

Suppose $a \neq b$. Then an alternative calculation is to use (38):

$$\partial_a \partial_b P = \partial_a \partial_b \left(\sum_{i,j} p_i Z_{ij} p_j \right) = 2p_a p_b \partial_a \partial_b (Z_{ab}) .$$

Again these two expressions must be equal for all momentum configurations, so

$$H_{a'b'} g^{\mu\nu} = \partial_a \partial_b (Z_{ab}) g^{\mu\nu} = \frac{1}{2} D_a^\mu D_b^\nu P \equiv \ddot{Z}_{ab} g^{\mu\nu} , \quad (a \neq b) , \quad (47)$$

where (38) has again been used to obtain the last equality. For $a = b$, (47) is no longer valid. Instead, (39) can be used to relate $H_{a'a'}$ to $H_{a'a''}$, which can then be calculated using (47).

There is one more relation for an external vertex a which we need,

$$D_a^\mu J_s^\nu = -H_{rs} g^{\mu\nu} \quad (A_{as} = 0) , \quad (48)$$

which is true for $r = a'$ or $r = a''$, and for $a' \neq s \neq a''$. To prove it, consider a situation where the primitive spin factor is $S_0(J) = -(J_{a'} + J_{a''})^\mu J_s^\nu$. The modified spin factor $S(J)$ is, according to (9) and (39), given by $S(J) = S_0(J) + iH_{rs} g^{\mu\nu}$. On the other hand, using (11)–(15), and the assumption that $a' \neq s \neq a''$, $S(J)$ is also given by $\exp(-iP) S_0(d) \exp(iP)$. Using (45), this becomes

$$\begin{aligned}
& -i \exp(-iP) d_s^\nu D_a^\mu \exp(iP) \\
& = -i \exp(-iP) D_a^\mu J_s^\nu \exp(iP),
\end{aligned}$$

which is equal to $-i(D_a^\mu J_s^\nu) - (J_{a'} + J_{a''})^\mu J_s^\nu$. Comparing these two ways of obtaining $S(J)$, (48) follows.

As it stands, (48) is not true when $s = a'$ or a'' . For example, using the explicit example to be considered in Sec. V, one can show explicitly that when $a = 1$ and $s = 2$, only half of the right-hand side (RHS) of (48) is obtained from the left-hand side (LHS).

It is this lack of universal validity of (48) that restricts the validity of the stringlike formula, as we shall see in the next section.

Because of the importance of this restriction it should also be pointed out that there is a false derivation of (48) which makes it appear to be valid for *all* a and s . The false argument follows from (30), (36), (37), (39), and (44). It is false because strangely enough, (36) is not a valid equation for the present application, where D_a and hence derivative of external momenta are involved. Equation (36) is of course valid in the physical case when external momentum conservation (28) is used. On the other hand, since the operation involving D_a is level dependent, we must not use this conservation law before the momentum differentiation. In that case the RHS of (36) could be equal to, say, $-\sum_{b \neq a} p_b$, which is not at all the same thing as the LHS of (36) as far as momentum differentiation is concerned. The correct way to check the validity of (36) is to compute the RHS of (36) using (26), sticking to whatever level scheme one is using and refraining from ever using external momentum conservation before the momentum differentiation. Without using momentum conservation, the currents J is no longer level independent; hence, the outcome of the right-hand side (RHS) of (36) depends on the level scheme one uses. In

this way one can show by explicit examples that (36) is generally invalid in the zero-diagonal level scheme. This observation once again shows the subtlety of the level dependence problem.

Let us turn to amplitudes in the configuration space. Since field theories are local in x , interactions should look simpler in the x space than in the p space. For conceptual reasons it is therefore worthwhile to look at the expressions in the x space, although in practical calculations we must return to the momentum space. Fourier-transforming the momentum-space amplitude (1), we get

$$\begin{aligned}
& (2\pi)^4 i^l M(p) \delta^4 \left(\sum_{i=1}^n p_i \right) \\
& = \int \left(\prod_{i=1}^n d^4 x_i \right) \exp \left[i \sum_{i=1}^n p_i x_i \right] M'(x), \quad (49)
\end{aligned}$$

$$M'(x) = S_0(-i\partial/\partial y) \prod_{r=1}^N \Delta_+(y_r), \quad (50)$$

where the scalar propagator is given by

$$\begin{aligned}
\Delta_+(y_r) &= \frac{1}{(2\pi)^4} \int d^4 q_r \frac{\exp[iq_r y_r]}{m_r^2 - q_r^2 - i\epsilon} \\
&= \frac{1}{16\pi^2} \int_0^\infty d\beta_r \exp \left[-i \frac{m_r^2}{\beta_r} + \frac{i}{4} \beta_r y_r^2 \right] \\
&= \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha_r}{\alpha_r^2} \exp \left[-i\alpha_r m_r^2 + i \frac{y_r^2}{4\alpha_r} \right]. \quad (51)
\end{aligned}$$

Hence

$$M'(x) = \left(\frac{1}{16\pi^2} \right)^N \int_0^\infty \left(\prod_{r=1}^N d\beta_r \right) S_0 \left(\frac{1}{2} \beta_r y_r \right) \exp \left[-i \sum_r \alpha_r m_r^2 + iP \right], \quad (52)$$

$$P = \frac{1}{4} \sum_{r=1}^N \beta_r y_r^2.$$

For a concrete example see Eq. (70) below. In (50) and (52) we must substitute $y_r = x_i - x_j$ for an $r = (ij)$ and as usual, $\beta_r = 1/\alpha_r$. In the electric circuit analogy, $x_i/2$ is really the potential V_i at vertex i , so $y_r/2$ is the potential drop v_r across resistance r [26,29–32]. P is again equal to the power consumed by the network. One can also obtain similar formulas and interpretations when some of the x_i 's are integrated over. For details, see [26].

A Feynman diagram can also be given a different interpretation [26,29–32] as a maze in which a particle moves in. The off-shell four-momentum of the particle along an internal line r is J_r^μ ; the four-distance it travels along r is $y_r^\mu = x_i^\mu - x_j^\mu$, and this is accomplished in an amount of

“proper time” equal to $2m\alpha_r$. Incidentally, it is perhaps more appropriate to think of the object traveling around the maze as a quantum mechanical wave rather than a classical particle because it is off shell, and because it is easier to think of a wave splitting and recombining at vertex junctions than a classical particle. In this interpretation, the integrand in the last expression of (51) is essentially the overlap matrix element $\langle x_i \tau_i | x_j \tau_j \rangle$ under the Hamiltonian $m(\dot{x}(\tau)^2 + 1)/2$ where $\tau = \tau_i - \tau_j = 2m\alpha_r$ is the proper time elapsed.

This interpretation exists for all diagrams and it can perhaps be thought of as the basis for a multiloop generalization of the string approach [22–24] and the first

quantized approach [25]. There is however a fundamental difference between a string propagation on a multi-genus world sheet and this particle propagating in the Feynman-diagram maze. The former satisfies the wave equation everywhere on the world sheet, whereas the latter can be described by free-particle equations of motion only between vertices. Vertices are singularities where interactions take place, where free-particle equations of motion break down. No such singularities are present on the world sheet on account of the string's conformal invariance [33], but this conformal invariance is lost in the infinite-tension limit when the world sheet collapses into a network of world lines because the σ variable along the string disappears. Reparametrization invariance in the τ variable can however still be kept as is done in [25], thereby preserving the string characters and the string-like formulas. As a result of the difference, although a string amplitude can be written as a path integral of a free string over multigenus world sheets, a free-particle path integral no longer exists for Feynman diagrams with internal vertices, unless their associated singularities and interactions can somehow be incorporated, as is done in the Feynman-parameter representation.

III. SCALAR QED

There are two kinds of electromagnetic vertices in scalar electrodynamics: the cubic vertex of Fig. 1,

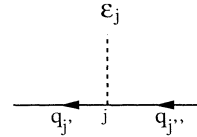


FIG. 1. The cubic electromagnetic vertex in scalar QED.

$$C_j = e\epsilon_j(q_{j''} + q_{j'}) \tag{53}$$

and the quartic (seagull) vertex of Fig. 2,

$$Q_j = 2e^2\epsilon_j\epsilon_{j+1} , \tag{54}$$

with ϵ_j being the polarization vector for the j th photon. We are interested in finding a stringlike formula for *standard diagrams*, which by definition are diagrams without seagull vertices.

Figure 3 represents a one-loop n -photon standard diagram. All other one-loop standard diagrams are obtained from it by permuting the external photon lines. The amplitude for Fig. 3 can be derived from the first quantized approach [23,25] to be

$$M(p) = -\frac{e^n}{16\pi^2} \int \left(\prod_{i=1}^n dt_i \right) t_n^{-2} \left\langle \exp \{ -im^2 t_n \} \exp \left\{ +i \sum_{i < j} [p_i p_j G_B^{ij} - (\epsilon_i p_j - \epsilon_j p_i) \dot{G}_B^{ij} + \epsilon_i \epsilon_j \ddot{G}_B^{ij}] \right\} \right\rangle . \tag{55}$$

This formula resembles a formula first obtained by Bern and Kosower [22] from string theory for QCD so it shall be referred to as a *stringlike formula*. The integration region in (55) is

$$0 \leq t_1 \leq t_2 \leq \dots \leq t_n < \infty ; \tag{56}$$

the notation $\langle f(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \rangle$ means that only terms multilinear in all the ϵ_i 's in f should be kept. In other words, if θ_i are Grassmann variables,

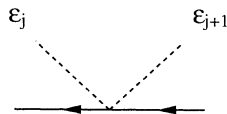


FIG. 2. The seagull electromagnetic vertex in scalar QED.

$$\langle f(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \rangle$$

$$= \pm \int d\theta_1 d\theta_2 \dots d\theta_n f(\theta_1 \epsilon_1, \theta_2 \epsilon_2, \dots, \theta_n \epsilon_n) . \tag{57}$$

The function G_B^{ij} and its derivatives are functions of $t_{ij} =$

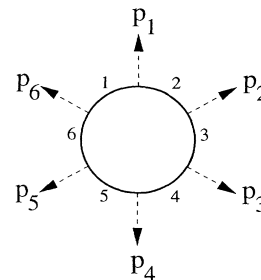


FIG. 3. A one-loop n -photon amplitude with outgoing external momenta p_i . $n = 6$ is shown in the diagram. The numbers around the loop label the internal lines.

$t_i - t_j$ whose explicit expressions for $i < j$ are

$$G_B^{ij} = t_{ij}(t_n + t_{ij})/t_n, \quad (58)$$

$$\dot{G}_B^{ij} = \frac{\partial G_B^{ij}}{\partial t_i} = (t_n + 2t_{ij})/t_n, \quad (59)$$

$$\ddot{G}_B^{ij} = \frac{\partial^2 G_B^{ij}}{\partial t_i^2} = 2/t_n. \quad (60)$$

One can also add in δ functions to simulate the seagull vertex contributions [25] but we shall not do that here.

What is crucial in this formula is that the various terms appearing in the exponent between square brackets are all given by the functions G_B^{ij} and their derivatives. It is this *gauge characteristic* that allows the integration by parts (IBP) technique discussed in the Introduction to be carried out. We shall come back to a discussion of the gauge characteristic after we derive the multiloop stringlike formula in scalar QED that exhibits this feature.

As we shall see below, a stringlike formula with the gauge characteristic exists for any process to any number of loops, provided the following technical restriction is obeyed. The formula turns out to look very similar to (55) when no derivative couplings are present, i.e., when there are no internal photon lines and when the “strong interaction” between scalar particles contains no derivative couplings. In the more general situation, a stringlike formula with the gauge characteristic still exists, but it looks a bit more complicated. See Eqs. (62) and (76).

Derivative couplings are present in electromagnetic cubic vertices (53), and perhaps in the strong interactions between the scalar particles. All of them contribute to the primitive spin factor $S_0(q)$ in Eq. (1). The technical restriction mentioned above is imposed so that Eq. (48) can be used. As we shall see below, this means that if a is an *external* vertex, then $S_0(q)$ should *not* contain any $q_{a'}$ or $q_{a''}$ other than those in C_a of (53). The notation is that used in Sec. II: a'' and a' are, respectively, the lines pointing into and out of the external vertex a . This restriction means, for example, that if there are internal photon propagators, then these internal vertices should not be adjacent to an external vertex for the following reason.

Equation (48) is used to replace the contraction function H_{rs} in (9) by differentiation D_a^μ , thereby reducing the modified spin factor $S(J)$ of (8) into factorized and simplified forms shown in Eqs. (55), (62), and (76). The restriction $A_{as} = 0$ in (48) means that when an external photon line is adjacent to an internal photon line, the current contractions associated with these external and internal photon vertices are not given completely by D_a^μ . An additional term will be necessary in (76) for correc-

tion. The precise form of the primitive spin factor $S_0(q)$, which is determined by the structure of the internal photon vertices, will play an important role in understanding how to deal with graphs that do not obey the technical restriction. Before discovering how this is done, (76), subject to the technical restriction mentioned above, is the best formula one can hope to obtain. Note however that this restriction does not wipe off all the diagrams with internal photon lines. For example, a four-loop photon self-energy diagram obtained by inserting an H into the simple one-loop diagram, where the horizontal bar of the H is taken to be an internal photon line and the rest of letter H are charged meson lines, is a diagram with an internal photon line which satisfies this technical restriction.

To obtain a stringlike formula for multiloops, it is necessary to find out first what is the generalization of G_B^{ij} . By comparing the terms quadratic in p in the exponents of (55) with (2) and (27), it is clear that if anything works G_B^{ij} has to be $Z_{ij}/2$. However, this alone does not tell us in what level scheme should Z_{ij} be expressed so as to capture the gauge characteristic. Before we solve this problem there is however a second problem that needs to be considered.

In one loop, G_B^{ij} is a function of the “time” difference $t_{ij} = t_i - t_j$. Time is connected to the Feynman parameters of Fig. 3 by

$$t_i = \sum_{k=1}^i \alpha_k. \quad (61)$$

For multiloop diagrams, Z_{ij} is a rather complicated function of α_i , and it is impossible in general to define a time parameter t_i so that Z_{ij} is a function only of $t_i - t_j$. This is related to the fact that the particle in the more general diagram has to split and recombine. Given that, the next important question to solve for multiloops is to determine what should replace the time derivatives in (59) and (60). It turns out that the time derivative $\partial/\partial t_a$ should in the general case be replaced by ∂_a defined in (37), and in so doing the gauge characteristic will be preserved *provided* that Z_{aj} is expressed in the zero-diagonal level scheme of (34).

We shall use the indices a, b to denote cubic electromagnetic vertices (53) with an external photon line, and the indices i, j to label *all* the vertices. Note that a, b are external vertices in the sense of Sec. II, viz., two internal lines are connected to each of them. In the simpler case when internal photon lines and derivative couplings between the scalar particles are absent, the multiloop stringlike formula for any multiloop standard diagram is

$$M(p) = \frac{i^{N-n} e^n}{(-16\pi^2)^l} \int_0^\infty \left(\prod_{r=1}^N d\alpha_r \right) \Delta(\alpha)^{-2} S_0'' \times \left\langle \exp \left\{ -i \sum_{r=1}^N \alpha_r m_r^2 + i \sum_{i,j} p_i p_j Z_{ij} - 2i \sum_{a,j} \epsilon_a p_j \dot{Z}_{aj} + i \sum_{a,b} \epsilon_a \epsilon_b \ddot{Z}_{ab} \right\} \right\rangle, \quad (62)$$

where

$$\dot{Z}_{aj} = \partial_a Z_{aj}, \quad \ddot{Z}_{ab} = \partial_a \partial_b Z_{ab}, \quad (63)$$

with $Z_{ii} = 0$ and ∂_a defined in (37). A similar but slightly more complicated formula [(76) below] exists when derivative couplings are present, as long as the technical restriction mentioned earlier is obeyed.

We will proceed now to demonstrate (62). The primitive spin factor $S_0(J)$ for any standard diagram is now given by $S_0(J) = S'_0(J)S''_0$, where

$$\begin{aligned} S'_0(J) &= \prod_a C_a = \prod_a [e\epsilon_a(J_{a'} + J_{a''})] \\ &= (-ie)^n \left\langle \exp \left[i \sum_a \epsilon_a (J_{a'} + J_{a''}) \right] \right\rangle \end{aligned} \quad (64)$$

is the primitive spin factor for the external cubic electromagnetic vertices, and S''_0 is the momentum-independent vertex factors of the rest. Using (43) this becomes

$$S'_0(J) = (-ie)^n \left\langle \exp \left(-2i \sum_{a,j} \epsilon_a p_j \dot{Z}_{aj} \right) \right\rangle. \quad (65)$$

To compute the modified spin factor $S(J)$, the contraction rule (9) is first used to compute the contraction of a pair of electromagnetic vertices. After using (47) this becomes

$$C_a C_b \rightarrow -2ie^2 \epsilon_a \epsilon_b H_{ab} = -2ie^2 \epsilon_a \epsilon_b \ddot{Z}_{ab}. \quad (66)$$

Then (8) is used to sum up all the contractions, giving

$$S(J) = (-ie)^n S''_0 \left\langle \exp \left\{ i \sum_{a,j} [-2\epsilon_a p_j \dot{Z}_{aj}] + i \sum_{a,b} [\epsilon_a \epsilon_b \ddot{Z}_{ab}] \right\} \right\rangle. \quad (67)$$

Substituting this into (2) we obtain the stringlike formula (62).

The string formula (62) can be simplified by noting that

$$S(J) \exp(iP) = (-ie)^n \left\langle \exp \left[- \sum_a \epsilon_a D_a \right] \right\rangle \times S''_0 \exp(iP), \quad (68)$$

$$P = \sum_{i,j} p_i Z_{ij} p_j, \quad (69)$$

where D_a is defined in (44).

From the discussions at the end of Sec. II, one expects the amplitude to be particularly simple when expressed in the configuration space. Indeed, using (69) and the definition of the configuration-space amplitude (49), one gets

$$M'(x) = (-ie)^n \left\langle \exp \left(-i \sum_a \epsilon_a \partial_a x_a \right) \right\rangle \prod_{r=1}^N \Delta_+(y_r). \quad (70)$$

The stringlike formula (62) exhibits a simple gauge transformation property, which is probably the reason why it is inherently important. This is the *gauge characteristic* we have been talking about. This gauge property is particularly transparent when (62) is written in the form of (68), for then a gauge transformation

$$\epsilon_a \rightarrow \epsilon_a + p_a \lambda_a \quad (71)$$

simply brings on the following change of the integrand of (2):

$$(-ie)^n \left\langle \exp \left(- \sum_a \epsilon_a D_a \right) \right\rangle \Delta^{-2} S''_0 \exp(-iD) \quad (72)$$

$$\rightarrow (-ie)^n \left\langle \exp \left(- \sum_a (\epsilon_a + \lambda_a p_a) D_a \right) \right\rangle \Delta^{-2} S''_0 \exp(-iD). \quad (73)$$

Note that the first equation in (38) has been used, and we have also used the fact that lines a' and a'' have the same mass. Since the gauge parameters λ_a are multiplied by the derivatives $\partial_a = \partial/\partial\alpha_{a'} - \partial/\partial\alpha_{a''}$, they appear only in surface terms corresponding to diagrams with $\alpha_{a'}$ or $\alpha_{a''}$ short circuited. It is these surface terms that connect the permuted standard diagrams and the seagull diagrams to enable gauge-dependent terms to be

canceled out at the end. In this way the stringlike formula makes the Ward-Takahashi identity in Feynman-parameter space almost explicit as the one seen in momentum space, and it is also this same gauge characteristic which enables the IBP technique to be applied.

We turn now to the general case where internal photon lines and "strong interaction" with derivative couplings are allowed to be present in the standard diagram, pro-

vided the technical restriction mentioned earlier is met. The primitive spin factor is then given by

$$S_0(J) = S'_0(J)S''_0(J), \quad (74)$$

where $S'_0(J)$ is the contribution from the external electromagnetic vertices which is given by (64), and $S''_0(J)$ is the contribution from the rest of the vertex factors which is now momentum dependent. The modified spin factor $S(J)$ is obtained from (74) by binary contractions with the rule of (9). When the technical restriction is met, contractions with the currents at an external vertex a may be accomplished by the operator D_a as in (48),

hence we obtain a relation similar to (68), which now reads

$$S(J) \exp(iP) = (-ie)^n \left\langle \exp \left[- \sum_a \epsilon_a D_a \right] \right\rangle S''(J) \exp(iP), \quad (75)$$

where $S''(J)$ is the modified spin factor corresponding to the primitive spin factor $S'_0(J)$ of (74). Substituting this into (2), we finally obtain the stringlike formula in the general case to be

$$M(p) = \frac{i^{N-n} e^n}{(-16\pi^2)^l} \int_0^\infty \left(\prod_{r=1}^N d\alpha_r \right) \left\langle \exp \left[- \sum_a \epsilon_a D_a \right] \right\rangle \Delta(\alpha)^{-2} S''(J) \exp \left[-i \sum_{r=1}^N \alpha_r m_r^2 + i \sum_{i,j} p_i p_j Z_{ij} \right]. \quad (76)$$

Again, the first equality of (38) as well as the fact electromagnetic interaction is diagonal in mass have been used. The gauge transformation property of this is similar to (73). Note that $\exp[-\sum_a \epsilon_a D_a]$ as defined in (44) is a translation operator shifting momentum p_a by an amount $-\epsilon_a \partial_a$. If we carry out this momentum shift in (76), the exponential will return to the form (62), but the momenta p implicitly contained in $S''(J)$ must be so shifted as well, and this shift contains the gauge-dependent quantity ϵ_a . We prefer not to write it in this shifted form for it makes the gauge transformation property more obscure.

IV. SPINOR QED

In spinor electrodynamics, an n -photon one-loop amplitude has a stringlike formula given in [23,25]. To obtain its multiloop generalization, a Gordon decomposition has to be made to separate the current into a convective part and a spin part.

Consider a fermion propagating in the presence of a background electromagnetic potential $A^\mu(x)$ and a background neutral scalar field $\phi(x)$. Depending on what is required, we can later on replace the background $A^\mu(x)$ by a polarization vector or one end of an internal photon propagator, and the background $\phi(x)$ by an external or internal "neutral scalar meson" coupled to the fermion by strong interaction. For definiteness we shall assume a Yukawa coupling for the strong interaction, but this point is not crucial for the following discussion. The fermion propagator is $[m - i\gamma(\partial - ieA) + \phi]^{-1} \delta^4(x - y)$; perturbation series are obtained by expanding this in power series of A and ϕ . Gordon decomposition is accomplished by noting that

$$[m + \phi - i\gamma(\partial - ieA)]^{-1} = [m + \phi + i\gamma(\partial - ieA)]I, \quad (77)$$

where

$$\begin{aligned} I &= \{[m + \phi - i\gamma(\partial - ieA)][m + \phi + i\gamma(\partial - ieA)]\}^{-1} \\ &= \{(m + \phi)^2 - i\gamma(\partial\phi) + (\partial - ieA)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\}^{-1} \\ &= (m^2 + \partial^2)^{-1} \sum_{n=0}^{\infty} [(C + Q + S + M_1 \\ &\quad + M_2 + M_3)(m^2 + \partial^2)^{-1}]^n \end{aligned} \quad (78)$$

and

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu], \\ C &= ie(\partial A + A\partial), \\ Q &= e^2 A^2, \\ S &= \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}, \\ M_1 &= -2m\phi, \\ M_2 &= -\phi^2, \\ M_3 &= i\gamma(\partial\phi). \end{aligned} \quad (79)$$

C and Q are just the cubic and the seagull electromagnetic vertices for a scalar particle. They give rise to the convective part of the current. On top of these, there is the spin vertex S which gives rise to the magnetic moment of the fermion. The vertices M_i are strong interaction vertices between the fermion and the neutral scalar meson. Note that M_3 is both momentum and spin dependent.

In a scattering diagram, a fermion propagator can end in two external fermions, in which case the relevant factor is

$$\bar{u}(m - i\gamma\partial)[m - i\gamma(\partial - ieA) + \phi]^{-1}(m - i\gamma\partial)u = \frac{1}{2m} \bar{u} \sum_{n=0}^{\infty} [(C + Q + S + M_1 + M_2 + M_3)(m^2 + \partial^2)^{-1}]^n (m^2 + \partial^2)u, \tag{80}$$

or with the wave functions u, \bar{u} replaced by v, \bar{v} . Otherwise, it can close on itself in a loop, in which case the proper factor is

$$\ln\{\det[m - i\gamma(\partial - ieA) + \phi]\} = \frac{1}{2} \ln\{\det[I]\} = \frac{1}{2} \text{Tr} \left\{ \ln \left[1 + \sum_{n=1}^{\infty} [(C + Q + S + M_1 + M_2 + M_3)(m^2 + \partial^2)^{-1}]^n \right] \right\}. \tag{81}$$

A constant irrelevant normalization factor has been dropped in the last expression. In other words, other than signs associated with statistics, the electromagnetic interaction of spinor QED differs from that of scalar QED only in having the extra vertex S .

The rest of the discussion is identical to the scalar case, leading up to the stringlike equation (76) for standard diagrams, where $S''(J)$ is now the modified spin factor for the vertices S, M_i , and the internal C 's. The technical restriction mentioned in Sec. III is automatically fulfilled for the derivatively coupled vertex M_3 because it depends only on the momentum of the neutral meson but not of the fermions.

To establish contact with the formula in [22–25] let us specialize to a one-fermion-loop n -photon amplitude in the absence of all strong interactions. Then

$$S''(J) = \prod_c S_c = \text{tr} \left\{ \prod_c [-ie\sigma_{\mu\nu}\epsilon_c^\mu p_c^\nu] \right\}. \tag{82}$$

The trace of a product of $2m$ γ matrices is given by 4 times the sum of all signed contractions, with each signed contraction given by a product of m factors of $(\pm g_{\alpha\beta})$'s, corresponding to the contraction of $(\cdots \gamma_\alpha \cdots \gamma_\beta \cdots)$. Using this rule it is easy to compute the trace

$$\frac{1}{4} \left(\frac{2}{i}\right)^m \text{tr} [(\epsilon_1 \sigma p_1) \cdots (\epsilon_m \sigma p_m)] \tag{83}$$

as follows. Take any permutation $t \in P_m$ of m objects and express it into cycles: $t = (t_1 t_2 \cdots t_k)(\cdots) \cdots$. Then (83) is given by

$$\eta_t [(\epsilon_{t_1} p_{t_2})(\epsilon_{t_2} p_{t_3})(\cdots)(\epsilon_{t_k} p_{t_1})] [\cdots] \tag{84}$$

summed over all permutations $t \in P_m$, and summed over all possible interchanges between every pair

$$\epsilon_{t_i} \leftrightarrow p_{t_i}. \tag{85}$$

η_t is the signature of the permutation, being ± 1 for even and/or odd permutations, and a minus sign is to be associated with each interchange (85).

It is these terms in (85) that give rise to the functions G_F^{ij} and their associated rules in the formula of [22–25].

V. AN EXAMPLE

To illustrate some of the circuit quantities and relations let us consider the two-loop Compton amplitude of Fig. 4. The dashed lines are photons and the solid lines are charged scalar mesons. The external vertices are $a = 1, 3$, and the “internal” vertices are $j = 2, 4$. Using (35) one obtains the impedance matrix in the zero-diagonal scheme to be

$$\begin{aligned} \Delta &= (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \\ Z_{ii} &= 0, \\ Z_{ij} &= Z_{ji}, \\ -2\Delta Z_{12} &= \alpha_2[(\alpha_3 + \alpha_4)(\alpha_1 + \alpha_5) + \alpha_1\alpha_5], \\ -2\Delta Z_{13} &= (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4) \\ &\quad + \alpha_5(\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3), \\ -2\Delta Z_{14} &= \alpha_1[(\alpha_3 + \alpha_4)(\alpha_2 + \alpha_5) + \alpha_2\alpha_5], \\ -2\Delta Z_{23} &= \alpha_3[(\alpha_1 + \alpha_2)(\alpha_4 + \alpha_5) + \alpha_4\alpha_5], \\ -2\Delta Z_{24} &= \alpha_5(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4), \\ -2\Delta Z_{34} &= \alpha_4[(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_5) + \alpha_3\alpha_5]. \end{aligned} \tag{86}$$

From these and definition (37) one can compute the various derivatives $\dot{Z}_{aj} = \partial_a Z_{aj}$ and $\ddot{Z}_{ab} = \partial_a \partial_b Z_{ab}$ to be

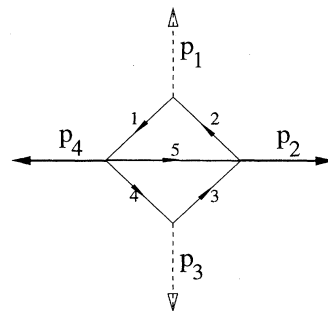


FIG. 4. A two-loop Compton scattering amplitude in scalar QED. The p_i are the external outgoing momenta, and the five internal lines are numbered as shown.

$$\begin{aligned}
-2\Delta\dot{Z}_{12} &= (\alpha_2 - \alpha_1)(\alpha_3 + \alpha_4) \\
&\quad + \alpha_5(-\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4), \\
-2\Delta\dot{Z}_{13} &= (\alpha_2 - \alpha_1)(\alpha_3 + \alpha_4) \\
&\quad + \alpha_5(-\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4), \\
-2\Delta\dot{Z}_{14} &= (\alpha_2 - \alpha_1)(\alpha_3 + \alpha_4) \\
&\quad + \alpha_5(-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \\
-2\Delta\dot{Z}_{31} &= (\alpha_4 - \alpha_3)(\alpha_1 + \alpha_2) + \alpha_5(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) \\
&\quad (87) \\
-2\Delta\dot{Z}_{32} &= (\alpha_4 - \alpha_3)(\alpha_1 + \alpha_2) + \alpha_5(\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4), \\
-2\Delta\dot{Z}_{34} &= (\alpha_4 - \alpha_3)(\alpha_1 + \alpha_2) \\
&\quad + \alpha_5(-\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4), \\
-2\Delta\ddot{Z}_{13} &= 2\alpha_5.
\end{aligned}$$

Note that the one-loop n -photon relationship $\dot{Z}_{13} = -\dot{Z}_{31}$ is no longer valid.

These are all the quantities needed in the stringlike formula (62). The stringlike formula is obtained by using the external-vertex formulas (38) and (39), as well as the level-dependent relations (43), (47), and (48). Let us look into the explicit form of these relations for the external vertex $a = 1$ in the present example. For $a = 1$ we see from Fig. 4 that $a' = 1$ and $a'' = 2$, so $\partial_1 = \partial/\partial\alpha_1 - \partial/\partial\alpha_2$. The quantities relevant to the level-dependent relations are

$$\begin{aligned}
\Delta(J_1 + J_2) &= 2\dot{V}_1 \\
&= (\alpha_1 - \alpha_2)(\alpha_3 + \alpha_4 + \alpha_5)p_1 \\
&\quad + \alpha_5[(\alpha_3 + \alpha_4)(p_4 - p_2) \\
&\quad + (\alpha_3 - \alpha_4)p_3], \\
\Delta(D_1^\mu J_1^\nu) &= \Delta(D_1^\mu J_2^\nu) = \frac{1}{2}(\alpha_3 + \alpha_4 + \alpha_5)g^{\mu\nu} \\
&= -\frac{1}{2}\Delta H_{11}g^{\mu\nu} \\
&= -\frac{1}{2}\Delta H_{12}g^{\mu\nu},
\end{aligned} \tag{88}$$

$$\begin{aligned}
\Delta(D_1^\mu J_3^\nu) &= \Delta(D_1^\mu J_4^\nu) = \alpha_5 g^{\mu\nu} \\
&= -\Delta H_{13}g^{\mu\nu} \\
&= -\Delta H_{14}g^{\mu\nu} = -\Delta\ddot{Z}_{13}g^{\mu\nu}, \\
\Delta(D_1^\mu J_5^\nu) &= (\alpha_3 + \alpha_4)g^{\mu\nu} = -\Delta H_{15}g^{\mu\nu}.
\end{aligned}$$

Consequently (43) and (47) are valid, as they should be, and (48) is true when $a' \neq s \neq a''$, but is not true when $s = a'$ or a'' . In the latter cases an extra factor of $\frac{1}{2}$ appears. From (86) and (88) it is also easy to check explicitly the external-vertex relations (38) to be correct. We have not written down H_{2s} , but they can be calculated explicitly to see that (39) is indeed valid.

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