Estimating perturbative coefficients in quantum field theory using Padé approximants

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We show how one can accelerate the convergence of a perturbation series by using Padé approximants. We use the first few coefficients of each perturbation series to predict the next term. We first check our method for known results and then predict the value of, as yet, unknown terms. Our results for $a_\mu - a_e$, a_e, a_μ, R_τ, and the QCD β function are remarkably good.

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It has long been a hope in perturbative quantum field theory, first expressed by Feynman, to be able to estimate, in a given order, the result for the coefficient, without the brute force evaluation of all of the Feynman diagrams contributing in this order. As one goes to higher and higher order, the number of diagrams, and the complexity of each, increases very rapidly. Feynman suggested that even a way of determining the sign of the contribution would be useful.

An attempt to do this in the case of e^+e^- annihilation to hadrons for the quantity

$$R = \frac{\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{1}$$

was made recently by West [1]. Although this method worked well for $N_f = 5$, where N_f is the number of fermions (quarks), it failed [2] for other values of N_f. He is now attempting to calculate corrections to his result [3].

In this paper we first outline our method and then apply it to some known series to illustrate how it works. Then we will apply it to several perturbation series in quantum electrodynamics (QED) and quantum chromodynamics (QCD).

We begin by defining the Padé approximant (type I)

$$[n,m] = \frac{a_0 + a_1 x + \cdots + a_n x^n}{1 + b_1 x + \cdots + b_m x^m} \tag{2}$$

to the series

$$S = S_0 + S_1 x + \cdots + S_{n+m} x^{n+m} , \tag{3}$$

where we set

$$[n,m] = S + O(x^{n+m+1}) . \tag{4}$$

We have written a computer program which solves Eq. (4) and then predicts the coefficient of the next term,

TABLE I. This table illustrates the acceleration of convergence one gets from the Padé approximants by comparing the estimated next term (ESNT) and the known exact next term (EXNT).

Series	$[n,m]$	ESNT	EXNT
$\sum \frac{(x)^k}{k!} = e^x$	[2,1]	0.556×10^{-1}	0.417×10^{-1}
$\sum \frac{(-)^{k+1}(x)^k}{k} = \ln(1+x)$	[2,1]	-0.22	-0.25
	[10,8]	0.052 631 578 92	0.052 631 578 95
$\sum \frac{1}{k^2}$	[3,2]	0.0202	0.0204
	[10,8]	0.002 499 999 996	0.0025
$\sum \frac{(-)^k}{k^4}$	[2,1]	-0.0012	-0.0016
	[6,4]	0.4821×10^{-4}	0.4823×10^{-4}
	[10,8]	$0.624\,999\,987 \times 10^{-5}$	0.625×10^{-5}
$4\sum \frac{(-)^{k+1}}{2k-1}$	[3,2]	0.306	0.308
	[6,4]	$-0.173\,911$	$-0.172\,913$
	[10,8]	$-0.102\,564\,102\,53$	$-0.102\,564\,102\,56$
$\sum \frac{1}{4k^2-1}$	[15,13]	$0.277\,854\,959\,70 \times 10^{-3}$	$0.277\,854\,959\,71 \times 10^{-3}$
$\sum \frac{1}{k^2}$	[15,13]	0.001 111 111 111 1111	0.001 111 111 111 1111
$\sum \frac{1}{k^4}$	[10,9]	$0.514\,189\,044 \times 10^{-5}$	$0.514\,189\,047 \times 10^{-5}$
$\sum \frac{1}{2^k k^4}$	[10,9]	$0.245\,184\,443 \times 10^{-11}$	$0.245\,184\,444 \times 10^{-11}$
$\sum \frac{1}{k(k+1)(k+2)}$	[10,9]	$0.941\,087\,8970 \times 10^{-4}$	$0.941\,087\,8976 \times 10^{-4}$

S_{n+m+1}. Some illustrative results are presented in Table I. It can be seen that the Padé approximant predicts the next coefficient very accurately, with the accuracy of the prediction increasing as n and m increase.

We now turn to the application of Padé approximants to perturbation series in quantum field theory. Our results which we shall use are

$$\text{I}\quad S_3=S_2^2/S_1\quad [1,1]\ ,$$

$$S_4=S_3^2/S_2\quad [2,1]\ ,$$

$$\text{II}\quad S_3=2S_1S_2/S_0-S_1^3/S_0^2\quad [0,2]\ ,$$

$$\text{III}\quad S_4=\frac{2S_1S_2S_3-S_0S_3^2-S_2^3}{S_1^2-S_0S_2}\quad [1,2]\ . \tag{5}$$

We shall now apply these results to perturbation series in QED and QCD. We shall use the last 2 terms of the series for I, 3 terms for II, and 4 terms for III.

If we define

$$R=S_1S_3/S_2^2\ , \tag{6}$$

then it can be seen that the prediction

$$S_4=S_3^2/S_2 \tag{7}$$

agrees with the prediction

$$S_4'=2S_2S_3/S_1-S_2^3/S_1^2\ ,\quad S_0=0\ , \tag{8}$$

if

$$R+R^{-1}=2\ . \tag{9}$$

Note that if

$$R=1+\epsilon\ , \tag{10}$$

$$R+R^{-1}=2+\epsilon^2\ , \tag{11}$$

and

$$\frac{S_4-S_4'}{S_4}=\frac{\epsilon^2}{(1+\epsilon)^2}\ , \tag{12}$$

the two predictions will agree very well. It can also easily be seen that the condition $R+R^{-1}=2$ also ensures that Eq. III agrees with Eqs. I and II [see Eq. (5)]. Equations (7) and (8) ensure that a positive-definite series remains positive definite, a negative-definite series remains negative definite, and an oscillating series remains oscillating. For other unusual series, although the method still works, it requires more terms.

TABLE II. This table compares the Padé estimate for $a_\mu-a_e$ with the known results. NT means the next (unknown) term. The entry 1600–3400 NT comes from Eq. (14) and the equation numbers refer to Eq. (5). The entry 8500–16 500 comes from Eq. (15). NNT means next-next term or second unknown term.

$a_\mu-a_e$ Equation	Estimate	Known result
I	705	570(140)
	2558	1600–3400 NT
	11 480	8500–16 500 NNT
II	2415	1600–3400 NT
	11 480	8500–16 500 NNT
III	2362	1600–3400 NT
	11 480	8500–16 500 NNT

As we shall see, our method works best for higher-order terms. But this is just where good estimates are badly needed, since one has hundreds of Feynman diagrams and the calculations are very complicated.

We begin with the difference between the muon and the electron anomalous magnetic moments (QED contribution) [4,6]:

$$a_\mu-a_e=1.094x^2+22.87x^3+127x^4+570(140)x^5\ , \tag{13}$$

where $x=(\alpha/\pi)$ and 570(140) means 570±140, and the x^5 coefficient is a conservative estimate. The results are given in Table II. It can be seen that there is beautiful agreement with the known results. Moreover the next term is predicted to be about 2500 and this agrees very well with the estimate

$$a_\mu^{(12)}=10k^3a_\mu^{(6)}(\gamma\gamma)=2500(900)\ , \tag{14}$$

where we take $2\le k\le 2.5$. (See Kinoshita, Nizic, and Okamoto [5] for a discussion of this method.) For the next-next term, we estimate, using $k=2.5$,

$$a_\mu^{(14)}=15k^4a_\mu^{(6)}(\gamma\gamma)=12\,500(4000)\ . \tag{15}$$

Next we consider the anomalous magnetic moment of the electron [6]:

$$a_e=\frac{x}{2}-0.3285x^2+1.176x^3-1.43x^4\ . \tag{16}$$

The results are given in Table III.

Again there is good agreement with the known results,

TABLE III. Padé estimates for a_e are compared with the known results.

a_e Equation	Estimate	Known result
I	−4.21	−1.43
	1.74	NT
	−2.12	NNT
II	−1.40	−1.43
III	3.22	NT
	−2.13	NNT

TABLE IV. Padé estimates for a_μ(QED) are compared with the known results.

a_μ Equation	Estimate	Known result
I	656	573(140)
	2614	1600–3400 NT
	11 925	8500–16 500 NNT
II	71.8	125.6
	2559	1600–3400 NT
	11 925	8500–16 500 NNT
III	2548	1600–3400 NT
	11 929	8500–16 500 NNT

TABLE V. Padé estimates for R_τ are compared with the known results.

R_τ Equation	Estimate	Known result
I	27.0	26.38
	133.8	NT
	678.6	NNT
II	9.4	26.38
	133.7	NT
	678.7	NNT
III	133.8	NT
	678.8	NNT

especially for the eighth-order coefficient from II, where the prediction is -1.40 and the known result is -1.43. Moreover the next term may be about $+2.5$. It is interesting to note that, if this is correct, the perturbation series for a_e continues to be an oscillating series. The next term, predicted to be -2.12, continues this pattern.

We now consider the perturbation expansion for

$$a_\mu = \frac{x}{2} + 0.7655x^2 + 24.05x^3 + 125.6x^4 + 573(140)x^5 , \tag{17}$$

where the x^5 coefficient is a conservative estimate. The results are shown in Table IV. It can be seen that the agreement with the known values is quite good and the prediction for the next term and the next-next term agree very well with the estimates using Kinoshita's method. [See Eqs. (14) and (15).] Now we consider the result from the QCD for

$$R_\tau = \frac{\Gamma(\tau \to \text{hadrons})}{\Gamma(\tau \to e\nu\bar{\nu})} . \tag{18}$$

The known coefficients are given by [7]

$$R_\tau = 1 + y + 5.2y^2 + 26.38y^3 , \tag{19}$$

where $y = \alpha_S/\pi$. Our results are presented in Table V. The first estimate from I agrees incredibly well with the known result. Moreover the three predictions for the next term are remarkably consistent, with the value being 133.8.

As our final example, we turn to the β function of QCD. It is given by [8]

$$\beta_{\text{QCD}}/g^2 = (-11 + \tfrac{2}{3}N_f)z + (-102 + \tfrac{38}{3}N_f)z^2 + (-\tfrac{2857}{2} + \tfrac{5033}{18}N_f - \tfrac{325}{54}N_f^2)z^3 , \tag{20}$$

where $z = g^2/(4\pi)^2$ and N_f is the number of fermions (quarks). We give our results in Table VI for $SU(3)_c$ and $N_f = 5$, 3, and 1. The results for $N_f = 5$ are extremely good. The results for $N_f = 3$, and 1, although not as good, are still quite reasonable. The method also works for $N_f = 2$, and 4, but not for $N_f = 6$, since, in that case, the z^3 term changes sign and the series becomes an unusual series $(-, -, +)$. However, the magnitude of the prediction for $N_f = 6$, which is -96.6, does agree very well with the z^3 coefficient, which is 109.9.

TABLE VI. Padé estimates for β_{QCD} are compared with the known results for $N_f = 5$, 3, and 1. NT refers to the next (unknown) term and NNT denotes the next-next unknown term or the second unknown term.

β_{QCD}/g^2	Estimate	Known result
$N_f = 5$		
I	-195.0	-180.9
	-846.3	NT
	-3959	NNT
II	-841.2	NT
	-3959	NNT
III	-3960	NNT
$N_f = 3$		
I	-455.1	-643.9
	-6480	NT
	$-65\,213$	NNT
II	-5921	NT
	$-65\,213$	NNT
III	$-65\,216$	NNT
$N_f = 1$		
I	-772.3	-1155
	$-14\,931$	NT
	$-193\,034$	NNT
II	$-13\,292$	NT
	$-193\,033$	NNT
III	$-193\,036$	NNT

In conclusion, we have shown how one can estimate coefficients of perturbation series in perturbative quantum field theory. Our results agree well with known results for $a_\mu - a_e$, a_e, a_μ, R_τ, and β_{QCD}. In addition, in each case, we obtain predictions for the next (unknown) terms.

After this work was completed, we were made aware of two earlier papers on this subject. Luban and Chew [9] consider a_e, and Fleisher, Pindor, Raczka, and Raczka [10] discuss the R ratio in QCD.

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