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Duality, marginal perturbations, and gauging

Måns Henningson and Chiara R. Nappi
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540
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We study duality transformations for two-dimensional σ models with Abelian chiral isometries and prove that generic such transformations are equivalent to integrated marginal perturbations by bilinears in the chiral currents, thus confirming a recent conjecture by Hassan and Sen formulated in the context of Wess-Zumino-Witten models. Specific duality transformations instead give rise to coset models plus free bosons.

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I. INTRODUCTION

Recently there has been a lot of interest in two-dimensional conformally invariant σ models with Abelian isometries. The space of theories with d Abelian isometries transforms under a group of so-called duality transformations, which is isomorphic to $O(d,d)$. These transformations are a generalization of the transformation introduced by Buscher [1] for the case of a single isometry. Buscher's transformations, in its turn, can be viewed as a generalization of the familiar $R \to 1/R$ symmetry in conformal field theory.

Duality transformations are intriguing and powerful symmetries that may relate conformal string backgrounds with totally different spacetime geometries. Indeed they have been used recently to generate new string solutions from known ones. This paper is an attempt to understand in more detail some properties of this symmetry. In a recent paper [2] Hassan and Sen studied duality transformations of the Wess-Zumino-Witten model and related models. They found that the marginal perturbations by bilinears in the chiral currents of specific such models could be reproduced by suitable duality transformations, and they conjectured that this result should be generalizable to any Wess-Zumino-Witten model. Similar results have also been reported by Kiritsis [3]. In this paper we prove this conjecture in the more general context of σ models with Abelian chiral isometries. An important example is, of course, the Wess-Zumino-Witten model, since such a model based on group G possesses rank G holomorphic and rank G antiholomorphic Abelian chiral isometries, but our considerations will be more general. We will show that generic duality transformations indeed correspond to integrated marginal perturbations.

When this representation in terms of marginal deformation fails, the duality transformation appears to be related to gauged models. Transformations that relate a given model with chiral isometries to its gauged version plus a set of free bosons have already been discussed in the literature. Examples of such duality transformations have been given by Kumar [4], and by Roček and Verlinde [5] in the case of one holomorphic and one antiholomorphic isometry. Here we investigate in detail this latter case, and find that specific duality transformations that yield models related to the corresponding axial and vector coset models are indeed the ones that cannot be represented in terms of marginal perturbations by bilinears in the chiral currents. We conjecture that a similar result should be valid in the general case. We hope that these investigations could be a step on the way to a better understanding of the moduli space of conformal field theories with not only Abelian chiral isometries but more general chiral current algebras.

This paper is organized as follows. In Sec. II, we give a quick review of σ models with Abelian isometries and the corresponding duality group. This discussion is specialized to models with chiral isometries in Sec. III. We show that a generic $O(d,d)$ transformation applied to such a model gives rise to a model of the same type with the same number of chiral isometries. In Sec. IV, we show that infinitesimal duality transformations correspond to marginal perturbations. In Sec. V, we consider in full detail the simplest nontrivial example, i.e., a model with one holomorphic and one antiholomorphic isometry, and determine all models which are related to it by duality. In addition to the previously mentioned models with chiral isometries we also find models which could be obtained by performing the coset construction on the original model plus a set of free bosons. This hints at a deeper relationship between duality and gauging, which we investigate in Sec. VI from a slightly different perspective.

II. ABELIAN ISOMETRIES AND DUALITY TRANSFORMATIONS

In this section we give a brief review of σ models with d Abelian isometries and the associated duality group $O(d,d)$. Readers are referred to the papers by Roček and Verlinde [5] and Giveon and Roček [6] for more details.

We may choose coordinates so that the isometries act by translation of the coordinates $\theta=(\theta^1,\dots,\theta^d)$. The remaining coordinates are denoted x^a. The action may then be written in the form

$$S=\frac{1}{2\pi}\int d^2z[\partial\theta E(x)\bar{\partial}\theta^t+\partial\theta F_{Ra}(x)\bar{\partial}x^a +\partial x^a F_{La}(x)\bar{\partial}\theta^t]+S[x]\ , \tag{1}$$

where $E(x)$, $F_{Ra}(x)$, and $F_{La}(x)$ are matrices of type $d\times d$, $d\times 1$ and $1\times d$, respectively. Matrix transportation is denoted with a superscript t. Henceforth we will often drop the a index on $F_{Ra}(x)$ and $F_{La}(x)$.

The group $O(d,d)$ is defined as the set of all $2d\times 2d$ matrices g that leave a metric J_0 of signature $d+d$ invariant:

$$g^t J_0 g = J_0 , \tag{2}$$

where

$$J_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} . \tag{3}$$

Here I is the d-dimensional identity matrix. If we decompose g in block form as

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \tag{4}$$

where a, b, c, and d are $d\times d$ matrices, this is equivalent to demanding that

$$a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = I . \tag{5}$$

The $O(d,d)$ element g in (4) acts on the σ model defined by (1) by transforming it into a model of the same kind with $E(x)$, $F_R(x)$, $F_L(x)$, and $S[x]$ replaced by

$$\begin{aligned} E'(x) &= [aE(x)+b][cE(x)+d]^{-1} , \\ F'_R(x) &= [a - E'(x)c]F_R(x) , \\ F'_L(x) &= F_L(x)[cE(x)+d]^{-1} , \\ S'[x] &= S[x] - \frac{1}{2\pi}\int d^2z\, \partial x^a F_{La}(x) \\ &\quad \times [cE(x)+d]^{-1} c F_{Rb}(x)\bar{\partial} x^b . \end{aligned} \tag{6}$$

When accompanied by an appropriate shift of the dilaton field, as discussed in [1], these transformations preserve conformal invariance of the model at the one-loop level. Corrections are known to exist to all orders [7] and preserve conformal invariance.

Matrices of the form

$$g = \begin{bmatrix} (\alpha^t)^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \tag{7}$$

constitute a $GL(d)$ subgroup of $O(d,d)$ that acts by linear coordinate transformations among the θ^i coordinates. The matrices

$$g = \begin{bmatrix} I & \beta \\ 0 & I \end{bmatrix}, \quad \beta + \beta^t = 0 \tag{8}$$

form an $\mathbb{R}^{d(d-1)/2}$ subgroup which corresponds to adding total derivative terms to the action. Together these elements generate a subgroup $\Lambda(d)$ of elements of the form

$$\lambda = \begin{bmatrix} (\alpha^t)^{-1} & \beta \\ 0 & \alpha \end{bmatrix}, \quad \alpha^t\beta + \beta^t\alpha = 0 , \tag{9}$$

which act trivially on (1) in the sense that the transformed model is equivalent to the original one up to coordinate transformations and partial integrations.

III. σ MODELS WITH CHIRAL ISOMETRIES

We now specialize the discussion to the case where the d isometries may be decomposed as d_L holomorphic and d_R antiholomorphic chiral isometries with $d = d_L + d_R$. This will allow us to make more specific statements about the dual models. We will see that for a generic $O(d,d)$ transformation the dual model possesses the same number of holomorphic and antiholomorphic chiral isometries.

With an appropriate choice of coordinates, the action may be written in form (1) with

$$\theta = \begin{bmatrix} \theta_R & \theta_L \end{bmatrix} = (\theta_R^1 \cdots \theta_R^{d_r} \;\; \theta_L^1 \cdots \theta_L^{d_L}) \tag{10}$$

and

$$\begin{aligned} E(x) &= \begin{bmatrix} I_R & 2B(x) \\ 0 & I_L \end{bmatrix} , \\ F_R(x) &= \begin{bmatrix} G_R(x) \\ 0 \end{bmatrix} , \\ F_L(x) &= [0 \;\; G_L(x)] . \end{aligned} \tag{11}$$

Here I_L and I_R denote the d_L- and d_R-dimensional identity matrices, and $B(x)$, $G_R(x)$, and $G_L(x)$ are matrices of type $d_R\times d_L$, $d_R\times 1$, and $1\times d_L$, respectively.

Written out explicitly, this action is

$$\begin{aligned} S_{LR} = \frac{1}{2\pi}\int d^2z [&\partial\theta_L\bar{\partial}\theta_L^t + \partial\theta_R\bar{\partial}\theta_R^t + \partial\theta_R 2B(x)\bar{\partial}\theta_L^t \\ &+ \partial\theta_R G_{Ra}(x)\bar{\partial}x^a + \partial x^a G_{La}(x)\bar{\partial}\theta_L] + S[x] . \end{aligned} \tag{12}$$

The equations of motion that follow from a variation of θ_R and θ_L are $\partial\bar{J}_R = 0$ and $\bar{\partial}J_L = 0$, respectively, where the chiral currents are given by

$$\begin{aligned} \bar{J}_R &= (\bar{J}_R^1, \ldots, \bar{J}_R^{d_R}) = \bar{\partial}\theta_R + \bar{\partial}\theta_L B(x)^t + \tfrac{1}{2}G_{Ra}(x)^t\bar{\partial}x^a , \\ J_L &= (J_L^1, \ldots, J_L^{d_L}) = \partial\theta_L + \partial\theta_R B(x) + \tfrac{1}{2}\partial x^a G_{La}(x) . \end{aligned} \tag{13}$$

The conformal dimension of the holomorphic (antiholomorphic) current J_L ($\bar{J}_R$) is (1,0) [(0,1)].

Our object is to analyze the orbit of the action (12) under $O(d,d)$ acting as in (6). However, we are only interested in classically inequivalent models, which could not be obtained from one another by coordinate transformations and partial integrations, so we should rather consider the right coset $\Lambda(d)\backslash O(d,d)$, where the subgroup $\Lambda(d)$ was defined in (9). Furthermore, there is a subgroup $\Omega(d_L, d_R)$ of $O(d,d)$ elements that leave the action (12) invariant. We will construct this subgroup explicitly in a simple case in Sec. V. Our real object of interest is therefore the double coset $\Lambda(d)\backslash O(d,d)/\Omega(d_L,d_R)$, i.e., the set of equivalence classes of $O(d,d)$ under the equivalence relation

$$g \sim \lambda g \omega, \quad \lambda\in\Lambda(d), \omega\in\Omega(d_L,d_R) . \tag{14}$$

To analyze this coset, let us first consider O(d,d) elements such that the submatrix d in (4) is invertible, i.e., $\det d \neq 0$. This is certainly true in a neighborhood of the identity of O(d,d), and for the simplest nontrivial case $d_L = d_R = 1$ we have checked that all O(2,2) elements of the $\det d = 0$ type are equivalent to elements with $\det d \neq 0$ modulo $\Omega(1,1)$ acting from the right . We conjecture that also in the general case one needs only to consider elements of O(d,d) with $\det d \neq 0$, although we have no proof of this.

We may parametrize an O(d,d) element of the form (4) with $\det d \neq 0$ by the $d \times d$ matrices d, e, and f, where e and f are defined by

$$e = d^{-1}c, \quad f = b^t d , \tag{15}$$

so that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (d^t)^{1-}(I - fe) & (d^t)^{-1} f^t \\ de & d \end{bmatrix} . \tag{16}$$

The requirement that g be an element of O(d,d) amounts to e and f being antisymmetric:

$$e + e^t = f + f^t = 0 , \tag{17}$$

while d is unconstrained, apart from the requirement of invertibility.

Multiplication of (16) from the left by an element $\lambda \in \Lambda(d)$ of the form (9) yields a new element $g' = \lambda g$ of form (16) with d, e, and f replaced by

$$d' = \alpha d, \quad e' = e, \quad f' = f + d^t \beta^t \alpha d . \tag{18}$$

We see that e is invariant under such transformations, and furthermore we may transform d and f to any preferred values d_0 and f_0 (satisfying $\det d_0 \neq 0$ and $f_0 + f_0^t = 0$) by choosing

$$\alpha = d_0 d^{-1}, \quad \beta = (d_0^t)^{-1}(f_0^t - f^t) d^{-1} . \tag{19}$$

The equivalence classes of O(d,d) modulo $\Lambda(d)$ may thus be labeled by the matrix e, which is only subject to the constraint of being antisymmetric.

In the case at hand, a convenient choice of representative in (almost) every equivalence class may be described as follows. Introduce the $d \times d$ matrix J as

$$J = \begin{bmatrix} I_R & 0 \\ 0 & -I_L \end{bmatrix} , \tag{20}$$

and define h by

$$h = (J - e)(J + e)^{-1} , \tag{21}$$

which may be inverted to yield

$$e = (I + h)^{-1}(I - h) J . \tag{22}$$

These relations are well defined for generic e and h and at least in a neighborhood of $h = I$, $e = 0$. We will come back to the remaining cases where $J + e$ is not invertible in Sec. V. It is easy to show that the antisymmetry of e is equivalent to h being an element of the group O(d_L, d_R), i.e.,

$$h^t J h = J . \tag{23}$$

We now choose

$$d = \tfrac{1}{2}(I + h), \quad f = \tfrac{1}{4}(I - h^t) J (I + h) . \tag{24}$$

The property (23) implies that f is antisymmetric as required.

To see the advantage of this choice we write h in block form:

$$h = \begin{bmatrix} V_R & T \\ S & V_L \end{bmatrix} , \tag{25}$$

where V_R, V_L, S, and T are matrices of type $d_R \times d_R$, $d_L \times d_L$, $d_L \times d_R$, and $d_R \times d_L$, respectively. The requirement (23) that h be an element of O(d_L, d_R) amounts to

$$V_R^t V_R - S^t S = I_R , \quad V_L^t V_L - T^t T = I_L , \quad V_R^t T - S^t V_L = 0 . \tag{26}$$

Inserting this h in (22), (24), and (16) we get

$$\begin{aligned} a &= \frac{1}{2} \begin{bmatrix} I_R + V_R & -T \\ -S & I_L + V_L \end{bmatrix} , \\ b &= \frac{1}{2} \begin{bmatrix} I_R - V_R & -T \\ S & -I_L + V_L \end{bmatrix} , \\ c &= \frac{1}{2} \begin{bmatrix} I_R - V_R & T \\ -S & -I_L + V_L \end{bmatrix} , \\ d &= \frac{1}{2} \begin{bmatrix} I_R + V_R & T \\ S & I_L + V_L \end{bmatrix} . \end{aligned} \tag{27}$$

We now apply the transformations (6), with a, b, c, and d given by (27), to the model defined by (1) and (11) and get

$$\begin{aligned} E'(x) &= \begin{bmatrix} I_R & 2B'(x) \\ 0 & I_L \end{bmatrix} , \\ F_R'(x) &= \begin{bmatrix} G_R'(x) \\ 0 \end{bmatrix} , \\ F_L'(x) &= [0 \quad G_L'(x)] , \\ S'[x] &= S[x] + \frac{1}{2\pi} \int d^2z \tfrac{1}{2} \partial x^a G_{La}(x) [V_L - SB(x)]^{-1} \\ &\quad \times S G_{Rb}(x) \bar{\partial} x^b , \end{aligned} \tag{28}$$

where

$$\begin{aligned} B'(x) &= [V_R B(x) - T][V_L - SB(x)]^{-1} , \\ G_R'(x) &= [V_R^t - B(x) T^t]^{-1} G_R(x) , \\ G_L'(x) &= G_L(x)[V_L - SB(x)]^{-1} . \end{aligned} \tag{29}$$

We see that the matrices $E'(x)$, $F_R'(x)$, and $F_L'(x)$ are still of form (11), and the transformed model is thus of the same kind as the original one, with d_L holomorphic and d_R antiholomorphic chiral isometries. The only restrictions that we have imposed on the O(d,d) transformation to reach this result is that the submatrix d in (4) and the matrix $J+e$ in (21) are invertible.

IV. INFINITESIMAL TRANSFORMATIONS AND MARGINAL PERTURBATIONS

To get a better understanding of the transformations in the previous section, we will here consider infinitesimal transformations $h=I+\epsilon\tilde{h}+O(\epsilon^2)$ with

$$\tilde{h}=\begin{bmatrix}\tilde{V}_R & \tilde{T}\\ \tilde{S} & \tilde{V}_L\end{bmatrix} . \tag{30}$$

The constraints (26) give

$$\tilde{V}_R+\tilde{V}_R^t=0,\quad \tilde{V}_L+\tilde{V}_L^t=0,\quad \tilde{T}-\tilde{S}^t=0 . \tag{31}$$

Inserting this h in (29) yields the infinitesimal transformation

$$\begin{aligned}B'(x)&=B(x)+\epsilon[\tilde{V}_RB(x)-B(x)\tilde{V}_L\\&\qquad-\tilde{S}^t+B(x)\tilde{S}B(x)]+O(\epsilon^2) ,\\ G_R'(x)&=G_R(x)+\epsilon[-\tilde{V}_R+B(x)\tilde{S}]G_R(x)+O(\epsilon^2) ,\\ G_L'(x)&=G_L(x)+\epsilon G_L(x)[-\tilde{V}_L+\tilde{S}B(x)]+O(\epsilon^2) ,\\ S'[x]&=S[x]+\frac{\epsilon}{2}\frac{1}{2\pi}\int d^2z\, \partial x^aG_{La}(x)\tilde{S}G_{Rb}(x)\bar{\partial}x^b\\&\qquad+O(\epsilon^2) .\end{aligned} \tag{32}$$

These transformations can be interpreted as a marginal perturbation by a bilinear in the holomorphic and antiholomorphic chiral currents (13), as we will now show. For an arbitrary $d_L\times d_R$ matrix $\tilde{S}$, the operator $J_L\tilde{S}J_R^t$ is (classically) of dimension (1,1) and may be added as a perturbation to the Lagrangian (12). One finds that

$$\begin{aligned}S_{LR}'&=S_{LR}+2\epsilon\frac{1}{2\pi}\int d^2zJ_L\tilde{S}J_R^t\\&=\frac{1}{2\pi}\int d^2z\{\partial\theta_R[I_R+2\epsilon B(x)\tilde{S}]\bar{\partial}\theta_R^t+\partial\theta_L[I_L+2\epsilon\tilde{S}B(x)]\bar{\partial}\theta_L^t+\partial\theta_R[2B(x)+2\epsilon B(x)\tilde{S}B(x)]\bar{\partial}\theta_L^t\\&\qquad+\partial\theta_L2\epsilon\tilde{S}\bar{\partial}\theta_R^t+\partial\theta_R[I_R+\epsilon B(x)\tilde{S}]G_{Ra}(x)\bar{\partial}x^a+\partial x^aG_{La}(x)[I_L+\epsilon\tilde{S}B(x)]\bar{\partial}\theta_L^t\\&\qquad+\partial\theta_L\epsilon\tilde{S}G_{Ra}(x)\bar{\partial}x^a+\partial x^aG_{La}(x)\epsilon\tilde{S}\bar{\partial}\theta_R^t+(\epsilon/2)\partial x^aG_{La}(x)\tilde{S}G_{Rb}(x)\bar{\partial}x^b\}+S[x] .\end{aligned} \tag{33}$$

After an infinitesimal coordinate change

$$\begin{aligned}\theta_R&\to\theta_R+\epsilon\theta_R\tilde{V}_R-\epsilon\theta_L\tilde{S} ,\\ \theta_L&\to\theta_L+\epsilon\theta_L\tilde{V}_L-\epsilon\theta_R\tilde{S}^t ,\end{aligned} \tag{34}$$

with $\tilde{V}_R+\tilde{V}_R^t=0$, and $\tilde{V}_L+\tilde{V}_L^t=0$, we get S_{LR}' of the form (12) with $B'(x)$, $G_R'(x)$, $G_L'(x)$, and $S'[x]$ given by (32). The infinitesimal duality transformations (32) are thus equivalent to marginal perturbations (33). Since the marginally perturbed model has the same number of Abelian chiral isometries, we can repeat the process. The result of applying such an "integrated" marginal perturbation to model (12) is a model of form (29), which is obtained by a finite $O(d,d)$ transformation.

Note that this relationship between duality transformations and marginal perturbations provides a simple check that the former preserve conformal invariance. Indeed, Chaudhuri and Schwarz [8] have proved that marginal perturbations by a bilinear in commuting chiral currents preserve conformal invariance.

V. THE CASE OF $d_L=d_R=1$

In Sec. III, we have seen that generic duality transformations of model (12) yield a model of the same type with the couplings given in (29). In this section we will examine the simplest nontrivial example with one holomorphic and one antiholomorphic isometry in somewhat more detail to determine the complete orbit of (12) under duality transformations.

We begin our investigations by determining the group $\Omega(1,1)$ of O(2,2) elements that leave the action S_{LR} invariant under the transformations (6). This turns out to be an Abelian discrete group with four elements:

$$\begin{aligned}\omega_1&=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \omega_2=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{bmatrix},\\ \omega_3&=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix},\quad \omega_4=\begin{bmatrix}0&0&-1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{bmatrix} .\end{aligned} \tag{35}$$

We have already mentioned that all elements g of O(2,2) with the determinant of the submatrix d in (4) vanishing are equivalent modulo $\Omega(1,1)$ acting from the right to elements with $\det d\neq0$. We need therefore only consider the case $\det d\neq0$. From our previous reasoning we know that the equivalence classes of such O(2,2) elements modulo $\Lambda(2)$ acting from the left may be labeled by the 2×2 antisymmetric matrix e given by

$$e=d^{-1}c=\begin{bmatrix}0&-x\\x&0\end{bmatrix} . \tag{36}$$

Here x may take any real value, but the transformation $g\to g\omega_2$, with ω_2 given in (35), induces

$$e \to \begin{bmatrix} 0 & -x^{-1} \\ x^{-1} & 0 \end{bmatrix}, \tag{37}$$

so we may restrict our attention to the interval $-1 \le x \le 1$.

For $-1 < x < 1$ we may use (21) to get

$$h = \begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix}, \tag{38}$$

where x and t are related by $x = (1+\cosh t)^{-1}\sinh t$ and $-\infty < t < \infty$. The corresponding transformations act on the model (12) via (25) and (29) as

$$\begin{aligned} B'(x) &= [\cosh t + B(x)\sinh t]^{-1}[\sinh t + B(x)\cosh t], \\ G'_R(x) &= [\cosh t + B(x)\sinh t]^{-1} G_R(x), \\ G'_L(x) &= [\cosh t + B(x)\sinh t]^{-1} G_L(x), \\ S'[x] &= S[x] - \frac{1}{2\pi}\int d^2z \tfrac{1}{2}[\cosh t + B(x)\sinh t]^{-1} \\ &\quad \times \sinh t\, \partial x^a G_{La}(x) G_{Rb}(x) \bar{\partial} x^b . \end{aligned} \tag{39}$$

As before, this transformation preserves the number of chiral isometries $d_L = d_R = 1$.

The two remaining values $x = \pm 1$ mean that $J+e$ is not invertible so that (21) may not be used. Instead we can choose the representatives

$$\begin{aligned} a &= \frac{1}{2}\begin{bmatrix} -1 & \mp 1 \\ -1 & \pm 1 \end{bmatrix}, \quad b = \frac{1}{2}\begin{bmatrix} -1 & \pm 1 \\ 1 & \pm 1 \end{bmatrix}, \\ c &= \frac{1}{2}\begin{bmatrix} -1 & \pm 1 \\ 1 & \pm 1 \end{bmatrix}, \quad d = \frac{1}{2}\begin{bmatrix} -1 & \mp 1 \\ -1 & \pm 1 \end{bmatrix}, \end{aligned} \tag{40}$$

which, when inserted in (6) together with (11), yield

$$\begin{aligned} E'(x) &= \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1 \mp B(x)}{1 \pm B(x)} \end{bmatrix}, \\ F'_R(x) &= \begin{bmatrix} 0 \\ -\dfrac{G_R(x)}{1 \pm B(x)} \end{bmatrix}, \\ F'_L(x) &= \begin{bmatrix} 0 & \dfrac{G_L(x)}{B(x) \pm 1} \end{bmatrix}, \\ S'[x] &= S[x] - \frac{1}{2\pi}\int d^2z \frac{1}{2}\frac{1}{B(x)\pm 1}\partial x^a G_{La}(x) \\ &\quad \times G_{Rb}(x)\bar{\partial} x^b . \end{aligned} \tag{41}$$

These theories thus consist of a free boson plus a σ model which is, in fact, the coset model S_V and S_A described by Roček and Verlinde [5]:

$$\begin{aligned} S_{V/A} = \frac{1}{2\pi}\int d^2z \Bigg\{ & \frac{1 \mp B(x)}{1 \pm B(x)}\partial\theta\bar{\partial}\theta - \frac{G_{Ra}(x)}{1 \pm B(x)}\partial\theta\bar{\partial}x^a \\ & \pm \frac{G_{La}(x)}{1 \pm B(x)}\partial x^a \bar{\partial}\theta \\ & \mp \frac{1}{2}\frac{G_{La}(x)G_{Rb}(x)}{1 \pm B(x)}\partial x^a \bar{\partial} x^b \Bigg\} + S[x] . \end{aligned} \tag{42}$$

It is probably true that all $O(d,d)$ transformations such that $J+e$ is not invertible give rise to theories with free bosons. In fact, for the case where $d_L = d_R = d/2$, Kumar [4] has given an explicit $O(d,d)$ transformation, which is such that $J+e$ is not invertible, and which transforms model (12) into a model with $d/2$ free bosons.

Note that these models in a sense also have d_L holomorhpic and d_R antiholomorphic Abelian isometries, although in this case the holomorphic isometries are really pairwise identical to the antiholomorphic ones, both acting as translations of the $d/2$ free bosons. We conjecture that, with this definition of the number of chiral isometries, all models obtained by $O(d,d)$ transformations of (12) will have d_L holomorphic and d_R antiholomorphic Abelian isometries.

VI. GAUGING AND DUALITY

In this section we will discuss more explicitly the relation between quotients, quotients by chiral currents, and duality in the $d_L = d_R = 1$ case. First of all, to clarify the constructions in the previous section, we should note that all we have done there is dualizing a combination of holomorphic and antiholomorphic isometries labeled by a mixing angle α. A way to see this is to change variables in (12) from θ_L and θ_R to θ_0 and θ_1 defined as

$$\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \theta_L \\ \theta_R \end{bmatrix} . \tag{43}$$

We then perform a duality transformation with respect to the isometry which acts as a translation of the θ_0 coordinate, according to the prescription of Buscher [1]. Namely, one goes to a first-order form for (12) by introducing $V_\mu = \partial_\mu \theta_0$ via a Lagrange multiplier, and then solves for V_μ through its field equations. The explicit action for the dual model is then

$$\begin{aligned} S_{\rm dual} = \frac{1}{2\pi}\int d^2z \frac{1}{1+B(x)\sin 2\alpha}[& \partial\theta_0\bar{\partial}\theta_0 + \partial\theta_1\bar{\partial}\theta_1 + (1+\cos 2\alpha)B(x)\partial\theta_1\bar{\partial}\theta_0 + (1-\cos 2\alpha)B(x)\partial\theta_0\bar{\partial}\theta_1 \\ & + G_{La}(x)\partial x^a(\cos\alpha\bar{\partial}\theta_0 - \sin\alpha\bar{\partial}\theta_1) + G_{Ra}(x)\bar{\partial}x^a(\cos\alpha\partial\theta_1 - \sin\alpha\partial\theta_0)] + S'[x] \end{aligned} \tag{44}$$

with

$$S'[x] = S[x] - \frac{1}{2}\int d^2z \frac{\sin 2\alpha}{1+B(x)\sin 2\alpha}\,\partial x^a G_{La}(x) G_{Rb}(x)\bar{\partial}x^b \tag{45}$$

To turn (44) into a model of the left-right symmetric type as in (12), we need to make the further change of variables

$$\begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix}=\begin{bmatrix}\sin\alpha & \cos\alpha\\ \cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}\theta'_R\\ \theta'_L\end{bmatrix}, \tag{46}$$

which is nonsingular for $\cos2\alpha\neq0$, i.e., for $-\pi/4<\alpha<\pi/4$. Then (44) yields the chiral model (39) with $\tanh t=\sin2\alpha$. The values $\alpha=0$ and $\pi/2$, corresponding to dualizing a chiral isometry, leave the model invariant, as was noted in [5]. The corresponding O(2,2) transformations, their product, and the identity constitute the subgroup $\Omega(1,1)$ discussed in the previous section. For the values of $\alpha=\pm\pi/4$, instead, the change of variables to θ'_0 and θ'_1 defined by

$$\begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\pm1 & \pm1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\theta'_0\\ \theta'_1\end{bmatrix} \tag{47}$$

leads to free boson θ'_0 plus S_V or S_A coset theories (42), as already discussed in the previous section, and noticed originally in [5].

Before rederiving (44) in a slightly different, although probably more enlightening, way, we will discuss gauging of σ models with left and right chiral isometries. Starting from the action (12) we can gauge any combination of left and right isometries, parametrized by a mixing angle α, by using the minimal coupling prescription

$$\partial\theta_R\to\partial\theta_R+\frac{1}{\sqrt{2}}A\cos\alpha,\quad \partial\theta_L\to\partial\theta_L+\frac{1}{\sqrt{2}}A\sin\alpha\ , \tag{48}$$

and analogously for the antiholomorphic partial derivatives. This gives the gauged action

$$S_{\text{gauged}}=S_{LR}+\frac{1}{2\pi}\int d^2z\left[\frac{1}{2}A\bar{A}[1+B(x)\sin2\alpha]+\frac{1}{\sqrt{2}}A\bar{L}_R+\frac{1}{\sqrt{2}}L_L\bar{A}\right], \tag{49}$$

where

$$\begin{aligned}\bar{L}_R&=\sin\alpha\bar\partial\theta_L+\cos\alpha\bar\partial\theta_R+2B(x)\cos\alpha\bar\partial\theta_L+\cos\alpha G_{Ra}(x)\bar\partial x^a\ ,\\ L_L&=\sin\alpha\partial\theta_L+\cos\alpha\partial\theta_R+2B(x)\sin\alpha\partial\theta_R+\sin\alpha G_{La}(x)\partial x^a\ .\end{aligned} \tag{50}$$

After integrating out the gauge field we get the action

$$\begin{aligned}S_{\text{gauged}}&=S_{LR}-\frac{1}{2\pi}\int d^2z\frac{L_L\bar{L}_R}{1+B(x)\sin2\alpha}\\ &=\frac{1}{2\pi}\int d^2z[1+B(x)\sin2\alpha]^{-1}\\ &\quad\times[\partial\theta\bar\partial\theta-\sin\alpha\partial\theta G_{Ra}(x)\bar\partial x^a+\cos\alpha G_{La}\partial x^a\bar\partial\theta]+S'[x]\ ,\end{aligned} \tag{51}$$

where $S'[x]$ is given in (45) and $\theta=\cos\alpha\theta_L-\sin\alpha\theta_R$ is the gauge-invariant linear combination of θ_L and θ_R. The above gauging procedure is fully gauge invariant for any α, i.e., for any combination of left and right isometries. However, it is not a very interesting one, since it does not automatically lead to conformal theories, even if the starting theory is conformal. From this point of view, a slightly different procedure, that mimics the coset construction and allows to couple the gauge field directly to the chiral currents, seems more interesting. It is possible only for specific "anomaly-free" combinations of holomorphic and antiholomorphic isometries. If the starting model is conformally invariant, the coset model obtained by this procedure is probably conformally invariant as well, as it certainly is in the case of the gauged Wess-Zumino-Witten model.

This new procedure amounts to the addition of the term

$$\frac{1}{2\pi}\int d^2z\frac{1}{\sqrt{2}}(\cos\alpha\theta_R-\sin\alpha\theta_L)(\partial\bar{A}-\bar\partial A) \tag{52}$$

to (49), which completes L_L and $\bar{L}_R$ into the chiral currents J_L and $\bar{J}_R$ defined in (13). The resulting gauged action is

$$S_{\text{coset}}=S_{LR}+\frac{1}{2\pi}\int d^2z\{\tfrac{1}{2}A\bar{A}[1+B(x)\sin2\alpha]+\sqrt{2}\cos\alpha A\bar{J}_R+\sqrt{2}\sin\alpha\bar{A}J_L\}\ . \tag{53}$$

We remark here that it might be possible to understand the connection between (44) and (53) in yet another way. Namely, (53) is reminiscent of the first-order form of (12), which led to (44) upon integrating out an auxiliary gauge field V_μ. It might be that (53) could be thought of as a gauged fixed version of this first-order action, but we have not been able to prove this.

Anyway, upon integrating out the gauge field in (53) we get

$$S_{\text{coset}}=S_{LR}-\frac{1}{2\pi}\int d^2z\frac{2\sin2\alpha}{1+B(x)\sin2\alpha}J_L\bar{J}_R\ . \tag{54}$$

However, this procedure is legitimate only for $\alpha=\pm\pi/4$, since the term (52) is gauge invariant only for these values. In this case we get

$$S_{V/A}=S_{LR}\mp\frac{1}{2\pi}\int d^2z\frac{2}{1\pm B(x)}J_L\bar{J}_R\ . \tag{55}$$

These theories are exactly the axial and vector quotients (42). Notice that gauging by chiral currents (54) and plain gauging (51) yield different results. Indeed, at $\alpha=\pm\pi/4$, (51) and (42) are different. In both cases, gauge invariance of course leads to a reduction of degrees of freedom, and only the gauge-invariant combination $\theta=\theta_R-\theta_L$ survives.

Getting back to the dual action (44), an alternative way to obtain it is by gauging any combination of left and right isometries in (12) by minimal coupling as in (48) and introducing a Lagrange multiplier of the form $(1/2\pi)\phi(\bar\partial A-\partial\bar{A})$. The gauged action is then

$$S_{\rm dual}=S_{LR}+\frac{1}{2\pi}\int d^2z\left\{\frac{1}{2}A\bar{A}[1+B(x)\sin2\alpha]+\frac{1}{\sqrt{2}}A(\bar{L}_R-\bar{\partial}\phi)+\frac{1}{\sqrt{2}}(L_L+\partial\phi)\bar{A}\right\} . \quad (56)$$

After integrating out the gauge fields, one gets the action

$$S_{\rm dual}=S_{LR}-\frac{1}{2\pi}\int d^2z\frac{1}{1+B\sin2\alpha}(\bar{L}_R-\bar{\partial}\phi)(L_L+\partial\phi) . \quad (57)$$

The part of (57) independent of ϕ is exactly the gauged Lagrangian (51), and one recognizes that it is the part of the dual Lagrangian (44) that depends on $\theta_0=\theta$ only. One can furthermore show that the terms in (57) that depend on the Lagrange multiplier ϕ give rise to the remaining pieces in (44), with the identification $\phi=\theta_1$. So dualizing can be interpreted as gauging in the presence of a Lagrangian multiplier.

Obviously, one can add to (56) any term of the form (52), whether it is gauge invariant or not, since it can always be absorbed in a shift of the Lagrange multiplier

$$\phi\to\phi'=\phi+\frac{1}{\sqrt{2}}(\cos\alpha\theta_R-\sin\alpha\theta_L) . \quad (58)$$

The Lagrangian obtained by (56) after the addition of (52) can always be made gauge invariant, by choosing ϕ to transform properly under gauge transformation in such a way to compensate the change of (52). The answer will still be (44).

Moreover, for $\alpha\neq\pm\pi/4$ we can also fix the gauge by setting $\phi=0$, or

$$\phi'=\frac{1}{\sqrt{2}}(\cos\alpha\theta_R-\sin\alpha\theta_L) . \quad (59)$$

Then the gauge fixed action that one gets is exactly (54). In other words, for $\alpha\neq\pm\pi/4$, the action (54) is actually the dual action (44). One can check this by explicit calculation. Indeed

$$S_{LR}-\frac{1}{2\pi}\int d^2z\frac{2\sin2\alpha}{1+B(x)\sin2\alpha}J_L\bar{J}_R=\frac{1}{1+B(x)\sin2\alpha}\{[1-B(x)\sin2\alpha](\partial\theta_L\bar{\partial}\theta_L+\partial\theta_R\bar{\partial}\theta_R)-2\sin2\alpha\partial\theta_L\bar{\partial}\theta_R+2B(x)\partial\theta_R\bar{\partial}\theta_L+G_{La}(x)\partial x^a(\bar{\partial}\theta_L-\sin2\alpha\bar{\partial}\theta_R)+G_{Ra}(x)\bar{\partial}x^a(\partial\theta_R-\sin2\alpha\partial\theta_L)\}+S'[x] . \quad (60)$$

By introducing the variables θ_0 and θ_1 through the relations

$$\theta_R=\frac{1}{\cos2\alpha}(\sin\alpha\theta_0+\cos\alpha\theta_1) , \quad \theta_L=\frac{1}{\cos2\alpha}(\cos\alpha\theta_0+\sin\alpha\theta_1) , \quad (61)$$

which are nonsingular for $\alpha\neq\pm\pi/4$, we get exactly (44).

The representation (54) of the dual action is interesting first of all because it makes the relation with marginal perturbations explicit. The integrability of marginal perturbations is also evident if one notes that from (44) the chiral currents of the dualized model are given by

$$\bar{J}_R(\alpha)=\frac{\cos2\alpha}{1+B(x)\sin2\alpha}\bar{J}_R , \quad J_L(\alpha)=\frac{\cos2\alpha}{1+B(x)\sin2\alpha}J_L . \quad (62)$$

Then, combining (54) and (62), one immediately gets that

$$S_{\rm dual}(\alpha+\delta\alpha)=S_{\rm dual}(\alpha)-\frac{1}{2\pi}\int d^2z\frac{4\delta\alpha}{\cos2\alpha}J_L(\alpha)\bar{J}_R(\alpha) , \quad (63)$$

which is a marginal perturbation around $\alpha\neq0$. More interestingly, perhaps, the action (54) seems to be a natural definition of duality as well as of gauging by a chiral current, in the case $d_L=d_R=1$. For $\alpha\neq\pm\pi/4$, (54) obtained from (53) by integrating over the gauge fields, is the dual Lagrangian. At $\alpha=\pm\pi/4$ (53) develops a gauge invariance which permits gauging away one of the field, and it turns into the gauged model $S_{V/A}$. This observation puts duality and gauging on the same footing; gauging is duality at a point where a gauge invariance develops. This interpretation is probably generalizable to the general case of $O(d,d)$ and to any gauging of Abelian isometries.

We would like to point out that an alternative explanation of the connection between duality and gauging has been given in [2] for the specific case of an SU(2)/U(1) coset model.

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