Phase transitions out of equilibrium: Domain formation and growth

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We study the dynamics of phase transitions out of equilibrium in weakly coupled scalar field theories. We consider the case in which there is a rapid supercooling from an initial symmetric phase in thermal equilibrium at temperature $T_i > T_c$ to a final state at low temperature $T_f \approx 0$. In particular we study the formation and growth of correlated domains out of equilibrium. It is shown that the dynamics of the process of domain formation and growth (spinodal decomposition) cannot be studied in perturbation theory, and a nonperturbative self-consistent Hartree approximation is used to study the long time evolution. We find in weakly coupled theories that the size of domains grows at long times as $\xi_D(t) \approx \sqrt{t\xi(0)}$. The size of the domains and the amplitude of the fluctuations grow up to a maximum time t_s which in weakly coupled theories is estimated to be

$$t_s \approx -\xi(0) \ln \left[\left(\frac{3\lambda}{4\pi^3}\right)^{\frac{1}{2}} \left(\frac{(\frac{T_i}{2T_c})^3}{[\frac{T_i^2}{T_c^2} - 1]} \right) \right]$$

with $\xi(0)$ the zero-temperature correlation length. For very weakly coupled theories, their final size is several times the zero-temperature correlation length. For strongly coupled theories the final size of the domains is comparable to the zero-temperature correlation length and the transition proceeds faster.

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I. INTRODUCTION AND MOTIVATION

Phase transitions play a fundamental role in our understanding of the interplay between cosmology and particle physics in extreme environments. It is widely accepted that many different phase transitions took place in the early Universe at different energy (temperature) scales and with remarkable consequences at low temperatures and energies, in particular, broken symmetries, and possibly the observed baryon asymmetry in the Universe.

Phase transitions are an essential ingredient in inflationary models of the early Universe [1–6]. The importance of the description of phase transitions in extreme environments was recognized a long time ago and efforts were devoted to their description in relativistic quantum field theory at finite temperature [7–9]. For a very thorough account of phase transitions in the early Universe see the reviews by Brandenberger [10], Kolb and Turner [11], and Linde [12].

The methods used to study the *equilibrium* properties of phase transitions are by now well understood and widely used, in particular field theory at finite temperature and effective potentials [14].

These methods, however, are restricted to a *static* description of the consequences of the phase transition, but can hardly be used to understand the *dynamics* of the processes involved during the phase transition. In particular, for example, the effective potential that is widely used to determine the nature of a phase transition and static quantities such as critical temperatures, etc., is *irrelevant* for the description of the dynamics. The effective potential corresponds to the *equilibrium* free-energy density as a function of the order parameter. This is a static quantity, calculated in equilibrium, and in particular to one-loop order it is complex within the region of homogeneous field configurations in which $V''(\phi) < 0$, where $V(\phi)$ is the classical potential. This was already recognized in the early treatments by Dolan and Jackiw [8].

In statistical mechanics, this region is referred to as the "spinodal" and corresponds to a sequence of states which are thermodynamically unstable.

At zero temperature, the imaginary part of the effective potential has been identified with the decay rate of this particular unstable state [26].

The use of the static effective potential to describe the dynamics of phase transitions has been criticized by many authors, among them Mazenko, Unruh, and Wald [15]. These authors argued that phase transitions in typical theories will occur via the formation and growth of correlated domains inside which the field will relax to the value of the minimum of the equilibrium free energy fairly quickly. It is now believed that this picture may be correct for *strongly coupled theories* but is not accurate for weak couplings.

The mechanism that is responsible for a typical second-order phase transition from an initially symmetric high-temperature state is fairly well known. When the temperature becomes lower than the critical temperature, long-wavelength fluctuations become unstable and begin to grow and the field becomes correlated inside "domains," the order parameter (the expectation value of the volume average of the field) remaining zero all throughout the transition.

Recently, de Vega and one of the authors have studied the influence of these instabilities in the evolution of the order parameter out of equilibrium [16] in the case when a nonzero (but small) initial value of the order parameter was assumed.

In this work we continue the study of the *non-equilibrium* aspects of second-order phase transitions in typical scalar field theories, with a view towards a deeper understanding of the *dynamics* of phase transitions in inflationary scenarios of the early Universe. In particular trying to describe the process of domain formation and growth in the case in which the initial state is symmetric and in equilibrium at a temperature higher than the critical temperature and cooled down below the critical temperature.

Although there have been several attempts to study the time evolution of the scalar field either in flat space-time or de Sitter space [17–21], to our knowledge there has not as yet been a consistent treatment of the *dynamics* of domain formation and growth.

Recently, Kolb and Wang [22] have reported on an equilibrium study of the static properties of domains produced in late-time phase transitions, but it becomes a pressing issue to understand the process of domain formation and growth, especially the time scales involved and the size of the correlations.

Although our initial motivation, and ultimate goal, is to study the dynamics of the phase transition in an expanding universe, in this work we report our studies on the dynamics of the phase transitions in Minkowski space. We do not attempt in this article to study the nonequilibrium properties in the necessarily more complicated setting of inflationary cosmologies, and restrict ourselves to introducing the methods of nonequilibrium quantum statistical mechanics and apply them to the study of formation and growth of domains in flat space-time.

We would like to point out at this stage, that the situation under consideration is very different from the classical description of the process of spinodal decomposition in statistical mechanics. The classical approach to spinodal decomposition is based on a "coarse grained" time-dependent Landau-Ginzburg equation, which is first order in time and purely dissipative. Thermal fluctuations are usually introduced as a Langevin noise, typically uncorrelated [23, 24], that obeys the fluctuation dissipation relation.

In our case we are studying a *quantum field theory*, the Heisenberg field equations are second order, nondissipative (in Minkowski space), and both quantum and thermal fluctuations are present in the initial state (density matrix), furthermore, as is typical in these scalar theories, the order parameter is not conserved.

This article is organized as follows. In Sec. II we present our arguments that suggest that phase transitions in expanding cosmologies must be studied away from equilibrium for weakly coupled theories. We emphasize that the important long-wavelength modes that become unstable below the critical temperature and whose dynamics is relevant for the process of phase separation and domain growth will easily be out of equilibrium during the transition for weakly coupled theories.

In Sec. III, we introduce our model and the methods of non-equilibrium statistical mechanics as applied to the description of the dynamics of the phase transition.

In Sec. IV, we analyze the real time correlation functions in zeroth order and obtain the first quantitative expressions, a scaling law for the size of the domains and the growth of the amplitude of the fluctuations. In Sec. V, we carry out a perturbative calculation and show quantitatively that the dynamics cannot be studied within a perturbative framework.

In Sec. VI we introduce a nonperturbative self-consistent Hartree approximation to study the evolution of correlations and growth of domains. We provide an analytic and numerical analysis of the process of domain growth and establish a scaling law for the size of the domains at long times, and an estimate for the maximum size of the domains for very weakly coupled theories.

We summarize our findings and pose further questions in Sec. VII.

II. THE CASE FOR A NONEQUILIBRIUM DESCRIPTION

Before entering into the technical details, let us sketch the arguments that suggest that when the temperature is very near the critical temperature, the relevant dynamics *must* be studied out of equilibrium.

As in any dynamical process, in order to try to describe the time evolution of the system, one must first try to determine the typical time scales involved in the different dynamical processes. This understanding becomes more pressing when one tries to understand the dynamics of phase transitions.

In particular, is it possible to describe the phase transition in an environment in which the temperature is changing at some particular rate, in local thermodynamic equilibrium? To address this issue one must compare the typical collisional relaxational rates of the particles to the rate of change of the temperature.

The typical collisional relaxation rate for a process at energy E in the heat bath is given by $\Gamma(E) = \tau^{-1}(E) \approx n(E)\sigma(E)v(E)$ with $n(E)$ the number density of particles with this energy E, $\sigma(E)$ the scattering cross section at this energy, and $v(E)$ the velocity of the incident beam of particles. The lowest-order (Born approximation) scattering cross section in a typical scalar theory with a quartic interaction is $\sigma(E) \approx \lambda^2/E^2$. At very high temperatures, $T \gg m_\Phi$, with m_Φ the mass of the field, $n(T) \approx T^3$, the internal energy density is $U/V \approx T^4$, and the average energy per particle is $\langle E \rangle \approx T$, and $v(T) \approx 1$. Thus the typical collisional relaxation rate is $\Gamma(T) \approx \lambda^2 T$. In an expanding universe, the conditions for local equilibrium will prevail provided that $\Gamma(T) \gg [\dot{a}(t)/a(t)] = H(t)$, with $a(t)$ the Friedmann-

Robertson-Walker (FRW) scale factor. If this is the case, the collisions occur very quickly compared to the expansion rate, and particles will equilibrate. This argument applies to the collisional relaxation of high frequency (short-wavelength) modes for which $k \gg m_\Phi$. This may be understood as follows. Each "external leg" in the scattering process considered carries typical momentum and energy $k, E \approx T > T_c$. But in these typical theories $T_c \approx m_\Phi/\sqrt{\lambda}$. Thus for weakly coupled theories $T_c \gg m_\Phi$.

To obtain an order of magnitude estimate, we concentrate near the phase transition at $T \approx T_c \approx 10^{14}$ GeV, $H \approx T^2/M_{\rm Pl} \approx 10^{-5}T$, this implies that for $\lambda > 0.01$ the conditions for local equilibrium may prevail. However, in weakly coupled inflaton models of inflation, phenomenologically the coupling is bound by the spectrum of density fluctuations to be $\lambda \approx 10^{-12} - 10^{-14}$ [12, 13]. Thus, in these weakly coupled theories the conditions for local equilibrium of high-energy modes may not be achieved. One may, however, assume that although the scalar field is weakly self-coupled, it has strong coupling to the heat bath (presumably other fields in the theory) and thus remains in local thermodynamic equilibrium.

This argument, however, applies to the collisional relaxation of *short-wavelength modes.* We observe, however, that these type of arguments are *not valid* for the dynamics of the *long-wavelength modes* at temperatures *below* T_c for the following reason.

At very high temperatures, and in local equilibrium, the system is in the disordered phase with $\langle\Phi\rangle = 0$ and *short-ranged correlations*, as measured by the equal time correlation function (properly subtracted)

$$\langle\Phi(\mathbf{r},t)\Phi(\mathbf{0},t)\rangle \approx T^2 \exp[-|\mathbf{r}|/\xi(T)], \tag{2.1}$$

$$\xi(T) \approx \frac{1}{\sqrt{\lambda}T} \approx \xi(0)\left(\frac{T_c}{T}\right). \tag{2.2}$$

As the temperature drops near the critical temperature, and below, the phase transition occurs. The onset of the phase transition is characterized by the instabilities of long-wavelength fluctuations, and the ensuing growth of correlations. The field begins to correlate over larger distances, and correlated domains will form and grow. If the initial value of the order parameter is zero, it will remain zero throughout the transition. This is the process of spinodal decomposition, or phase separation. This growth of correlations cannot be described as a process in local thermodynamic equilibrium.

These instabilities are manifest in the *equilibrium* free energy in the form of imaginary parts, and the equilibrium free energy is not a relevant quantity to study the *dynamics.*

These long-wavelength modes whose instabilities trigger the phase transitions have very slow dynamics. This is the phenomenon of critical slowing down that is observed experimentally in binary mixtures and numerically in typical simulations of phase transitions. The long-wavelength fluctuations correspond to coherent collective behavior in which degrees of freedom become correlated over large distances. These collective long-wavelength modes have extremely slow relaxation near the phase transition, and they do not have many available low-energy decay channels. Certainly through the phase transition, high frequency, short-wavelength modes may still remain in local equilibrium by the arguments presented above (if the coupling is sufficiently strong), they have many channels for decay, and thus will maintain local equilibrium through the phase transition.

To make this argument more quantitative, consider the situation in which the final temperature is below the critical value and early times after the transition. For small amplitude fluctuations of the field, long-wavelength modes "see" an inverted harmonic oscillator and the amplitude fluctuations begin to grow as (see below)

$$\langle\Phi_k(t)\Phi_{-\mathbf{k}}(t)\rangle \approx \exp[2W(k)t], \tag{2.3}$$

$$W(k) = \sqrt{\mu^2(T) - \mathbf{k}^2}, \tag{2.4}$$

$$\mu^2(T) = \mu^2(0)\left[1 - \left(\frac{T_f}{T_c}\right)^2\right], \tag{2.5}$$

for $\mathbf{k}^2 < \mu^2(T)$.

In particular this situation, modeled with the "inverted harmonic oscillators," is precisely the situation thoroughly and clearly studied by Guth and Pi [25] and Weinberg and Wu [26].

The time scales that must be compared for the dynamics of these instabilities are now the growth rate $\Gamma(k) \approx \sqrt{\mu^2(T) - \mathbf{k}^2}$ and the expansion rate $H \approx T^2/M_{\rm Pl} \approx (10^{-5})T$, if the expansion rate is comparable to the growth rate, then the long-wavelength modes that are unstable and begin to correlate may be in local thermodynamic equilibrium through the cooling down process. Using $T_c \approx \mu(0)/\sqrt{\lambda}$, we must compare $[1-(\frac{T}{T_c})^2]^{\frac{1}{2}}$ to $10^{-5}/\sqrt{\lambda}$. Clearly for weakly coupled theories, or very near the critical temperature, the growth rate of the unstable modes will be much slower than the rate of cooling down, and the phase transition will be supercooled, similarly to a "quenching process" from a high-temperature, disordered phase to a supercooled low-temperature situation. For example, for $\lambda \approx 10^{-12}$ the growth rate of long-wavelength fluctuations is much smaller than the expansion rate even for a final temperature $T_f \approx 0$, and the long-wavelength modes will be strongly supercooled.

As mentioned previously, we do not attempt in this work to study the situation in an expanding gravitational background, and limit ourselves to studying the dynamics of the phase transition in Minkowski space by modeling the important features that are relevant for the phase transition in a weakly coupled theory. Our goals here are to introduce the methods to study this type of phase transition out of equilibrium, and to study the physics of domain formation and growth within a simplified situation. Eventually we propose to extend these methods to the case of an inflationary background.

We do not envisage here to account for the cooling down process (which requires a clear understanding of gravitational effects, and time scales) and restrict ourselves to *assuming* a supercooled phase transition in a weakly coupled theory and propose a particular model to understand this situation in Minkowski space.

III. STATISTICAL MECHANICS OUT OF EQUILIBRIUM

A period of rapid temperature change may be modeled by considering a time-dependent Hamiltonian with a *time-dependent mass term* of the form

$$H(t) = \int_\Omega d^3x \left\{ \frac{1}{2}\Pi^2(x) + \frac{1}{2}[\nabla\Phi(x)]^2 + \frac{1}{2}m^2(t)\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x) \right\}, \quad (3.1)$$

$$m^2(t) = m_i^2\Theta(-t) - m_f^2\Theta(t), \quad (3.2)$$

$$m_i^2 = \mu^2\left(\frac{T_i^2}{T_c^2} - 1\right), \quad (3.3)$$

$$m_f^2 = \mu^2\left(1 - \frac{T_f^2}{T_c^2}\right), \quad (3.4)$$

with $\mu^2 > 0, T_i > T_c, T_f \ll T_c$. The introduction of the T_i, T_f in the above mass term is just a *parametrization* of the model, which incorporates the ingredients of a high-temperature state at $t < 0$ and a low-temperature situation for $t > 0$. Again, the mechanism that drives the phase transition may either be a period of rapid inflation or a sudden coupling to a heat bath at a much lower temperature. The above parametrization incorporates by hand this "rapid supercooling" situation.

Clearly this is a simplification, but in view of the above comments, we believe that this approximation is justified insofar as we are trying to understand the dynamics of the instabilities of the long-wavelength modes and the growth of correlations in weakly coupled theories. This assumption of a rapid "quench" may be relaxed at the expense of complications. As will become clear below, this approximation will allow us to obtain analytic results and to perform explicit calculations. Furthermore, we assume that for all times $t < 0$ the system is in thermal equilibrium at the initial temperature T_i, thus described by the density matrix

$$\hat{\rho}_i = e^{-\beta_i H_i}, \quad (3.5)$$

$$H_i = H(t < 0). \quad (3.6)$$

In the Schrödinger picture, the density matrix evolves in time as

$$\hat{\rho(t)} = U(t)\hat{\rho}_i U^{-1}(t) \quad (3.7)$$

with $U(t)$ the time evolution operator.

An alternative and equally valid interpretation is that we consider an initial condition in which the system is in thermodynamic equilibrium at temperature $T_i > T_c$ in the symmetric phase for $t < 0$, and evolved in time with a Hamiltonian with a "negative mass squared" that allows for broken symmetry states for $t > 0$.

This interpretation in fact describes the situation studied by Guth and Pi [25] and Weinberg and Wu [26]. These authors prepare an initial Gaussian state or density matrix, and study the time evolution of this initially prepared state with a Hamiltonian for a collection of "inverted harmonic oscillators." Preparing an initial state, and evolving it with a Hamiltonian of which the initial state is *not an eigenstate* (in the language of density matrices, the density matrix does not commute with the Hamiltonian), is the quantum mechanic equivalent of a "quenching process" or a "sudden approximation." It is in this sense that we are thus generalizing the situation studied by the above authors.

The expectation value of any operator is thus

$$\langle\mathcal{O}\rangle(t) = \mathrm{Tr}e^{-\beta_i H_i}U^{-1}(t)\mathcal{O}U(t)/\mathrm{Tr}e^{-\beta_i H_i}. \quad (3.8)$$

This expression may be written in a more illuminating form by choosing an arbitrary time $T < 0$ for which $U(T) = \exp[-iTH_i]$ then we may write $\exp[-\beta_i H_i] = \exp[-iH_i(T - i\beta_i - T)] = U(T - i\beta_i, T)$. Inserting in the trace $U^{-1}(T)U(T) = 1$, commuting $U^{-1}(T)$ with $\hat{\rho}_i$ and using the composition property of the evolution operator, we may write (3.8) as

$$\langle\mathcal{O}\rangle(t) = \mathrm{Tr}U(T - i\beta_i, t)\mathcal{O}U(t, T)/\mathrm{Tr}U(T - i\beta_i, T). \quad (3.9)$$

The numerator of the above expression has a simple meaning: start at time $T < 0$, evolve to time t, insert the operator $\mathcal{O}$, and evolve backwards in time from t to $T < 0$, and along the negative imaginary axis from T to $T - i\beta_i$. The denominator just evolves along the negative imaginary axis from T to $T - i\beta_i$. The contour in the numerator may be extended to an arbitrary large positive time T' by inserting $U(t, T')U(T', t) = 1$ to the left of $\mathcal{O}$ in (3.9), thus becoming

$$\langle\mathcal{O}\rangle(t) = \mathrm{Tr}U(T - i\beta_i, T)U(T, T')U(T', t)\mathcal{O}U(t, T)/\mathrm{Tr}U(T - i\beta_i, T). \quad (3.10)$$

The numerator now represents the process of evolving from $T < 0$ to t, inserting the operator $\mathcal{O}$, evolving further to T', and backwards from T' to T and down the negative imaginary axis to $T - i\beta_i$. This process is depicted in the contour of Fig. 1. Eventually we take $T \to -\infty$, $T' \to \infty$. It is straightforward to generalize to real time correlation functions of Heisenberg picture operators.

This formalism allows us also to study the general case in which both the mass and the coupling depend on time. The insertion of the operator $\mathcal{O}$, may be achieved as usual by introducing currents and taking variational derivatives with respect to them.

Because the time evolution operators have the interaction terms in them, and we would like to generate a perturbative expansion and Feynman diagrams, it is convenient to introduce source terms for *all* the time evolution operators in the above trace. Thus we are led to consider the generating functional

$$Z[J^+,J^-,J^\beta] = \mathrm{Tr} U(T-i\beta_i,T;J^\beta)U(T,T';J^-)U(T',T;J^+). \tag{3.11}$$

The denominator in (3.9) is simply $Z[0,0,0]$ and may be obtained in a series expansion in the interaction by considering $Z[0,0,J^\beta]$. By inserting a complete set of field eigenstates between the time evolution operators, finally, the generating functional $Z[J^+,J^-,J^\beta]$ may be written as

$$Z[J^+,J^-,J^\beta] = \int D\Phi D\Phi_1 D\Phi_2 \int \mathcal{D}\Phi^+\mathcal{D}\Phi^-\mathcal{D}\Phi^\beta e^{i\int_T^{T'}\{\mathcal{L}[\Phi^+,J^+]-\mathcal{L}[\Phi^-,J^-]\}} e^{i\int_T^{T-i\beta_i}\mathcal{L}[\Phi^\beta,J^\beta]} \tag{3.12}$$

with the boundary conditions $\Phi^+(T)=\Phi^\beta(T-i\beta_i)=\Phi, \Phi^+(T')=\Phi^-(T')=\Phi_2, \Phi^-(T)=\Phi^\beta(T)=\Phi_1$. This may be recognized as a path integral along the contour in complex time shown in Fig. 1. As usual the path integrals over the quadratic forms may be evaluated and one obtains the final result for the partition function:

$$\begin{aligned} Z[J^+,J^-,J^\beta] = {} & \exp\left\{i\int_T^{T'} dt\,[\mathcal{L}_{\text{int}}(-i\delta/\delta J^+)-\mathcal{L}_{\text{int}}(i\delta/\delta J^-)]\right\} \\ & \times\exp\left\{i\int_T^{T-i\beta_i} dt\,\mathcal{L}_{\text{int}}(-i\delta/\delta J^\beta)\right\}\exp\left\{\frac{i}{2}\int_c dt_1\int_c dt_2 J_c(t_1)J_c(t_2)G_c(t_1,t_2)\right\} \end{aligned} \tag{3.13}$$

where J_c stands for the currents on the contour as shown in Fig. 1, G_c are the Green's functions on the contour [27], and again the spatial arguments were suppressed.

In the limit $T\to-\infty$, the contributions from the terms in which one of the currents is J^+ or J^- and the other is J^β vanish when computing correlation functions in which the external legs are at finite *real time*, as a consequence of the Riemann-Lebesgue lemma. For this *real time* correlation function, there is no contribution from the J^β terms that cancel between numerator and denominator. Finite temperature enters through the boundary conditions on the Green's functions (see below). For the calculation of finite *real time* correlation functions, the generating functional simplifies to [28, 29]

$$Z[J^+,J^-] = \exp\left\{i\int_T^{T'} dt\,[\mathcal{L}_{\text{int}}(-i\delta/\delta J^+)-\mathcal{L}_{\text{int}}(i\delta/\delta J^-)]\right\}\exp\left\{\frac{i}{2}\int_T^{T'} dt_1\int_T^{T'} dt_2 J_a(t_1)J_b(t_2)G_{ab}(t_1,t_2)\right\}, \tag{3.14}$$

with $a,b=+,-$.

The Green's functions that enter in the integrals along the contours in Eqs. (3.13) and (3.14) are given by (see above references)

$$G^{++}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = G^>(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\Theta(t_1-t_2)+G^<(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\Theta(t_2-t_1), \tag{3.15}$$
$$G^{--}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = G^>(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\Theta(t_2-t_1)+G^<(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\Theta(t_1-t_2), \tag{3.16}$$
$$G^{+-}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = -G^<(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2), \tag{3.17}$$
$$G^{-+}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = -G^>(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = -G^<(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2), \tag{3.18}$$
$$G^>(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = \langle\Phi(\mathbf{r}_1,t_1)\Phi(\mathbf{r}_2,t_2)\rangle, \tag{3.19}$$
$$G^<(\mathbf{r}_1,T;\mathbf{r}_2,t_2) = G^>(\mathbf{r}_1,T-i\beta_i;\mathbf{r}_2,t_2). \tag{3.20}$$

Condition (3.20) is recognized as the periodicity condition in imaginary time and is a result of considering an equilibrium situation for $t<0$. The functions $G^>, G^<$ are homogeneous solutions of the quadratic form, with appropriate boundary conditions and will be constructed explicitly below.

This formulation in terms of time evolution along a contour in complex time has been used many times in nonequilibrium statistical mechanics. To our knowledge the first to introduce this formulation were Schwinger [30] and Keldysh [31] (for an early account see Mills [32]). There are many clear articles in the literature using this technique to study real time correlation functions [28, 29, 33–37].

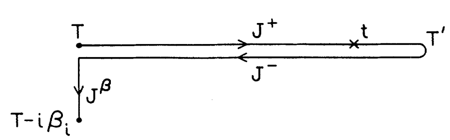

FIG. 1. Contour in a complex time plane to evaluate the generating functional for nonequilibrium Green's functions.

Our goal is to study the formation and growth of domains and the time evolution of the correlation functions. In particular, the relevant quantity of interest is the *equal time* correlation function

$$S(\mathbf{r};t) = \langle\Phi(\mathbf{r},t)\Phi(\mathbf{0},t)\rangle, \tag{3.21}$$
$$S(\mathbf{r};t) = \int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}S(\mathbf{k};t), \tag{3.22}$$
$$S(\mathbf{k};t) = \langle\Phi_{\mathbf{k}}(t)\Phi_{-\mathbf{k}}(t)\rangle = [-iG^{++}_{\mathbf{k}}(t;t)], \tag{3.23}$$

where we have performed the Fourier transform in the spatial coordinates (there still is spatial translational and rotational invariance). Notice that at equal times, all the Green's functions are equal, and we may compute any of them.

Clearly in an equilibrium situation this equal time correlation function will be time independent, and will only measure the *static correlations.* In the present case, however, there is a nontrivial time evolution arising from the departure from equilibrium of the initial state. This correlation function will measure the correlations in space, and their time dependence.

The function $G^{>}_{\mathbf{k}}(t,t')$ is constructed from the homogeneous solutions to the operator of quadratic fluctuations

$$\left[\frac{d^2}{dt^2}+\mathbf{k}^2+m^2(t)\right]\mathcal{U}^{\pm}_k=0 \tag{3.24}$$

with $m^2(t)$ given by (3.2).

The boundary conditions on the homogeneous solutions are

$$\mathcal{U}^{\pm}_k(t<0)=e^{\mp i\omega_<(k)t}, \tag{3.25}$$

$$\omega_<(k)=\left[\mathbf{k}^2+m_i^2\right]^{\frac{1}{2}}, \tag{3.26}$$

corresponding to positive frequency (particles) and negative frequency (antiparticles) $[\mathcal{U}^{+}_k(t);\mathcal{U}^{-}_k(t)$, respectively].

The solutions are as follows: (i) stable modes ($\mathbf{k}^2 > m_f^2$)

$$\mathcal{U}^{+}{}_k(t)=e^{-i\omega_<(k)t}\Theta(-t) + \left(a_k e^{-i\omega_>(k)t}+b_k e^{i\omega_>(k)t}\right)\Theta(t), \tag{3.27}$$

$$\mathcal{U}^{-}{}_k(t)=\left[\mathcal{U}^{+}{}_k(t)\right]^*, \tag{3.28}$$

$$\omega_>(k)=\sqrt{\mathbf{k}^2-m_f^2}, \tag{3.29}$$

$$a_k=\frac{1}{2}\left(1+\frac{\omega_<(k)}{\omega_>(k)}\right), \tag{3.30}$$

$$b_k=\frac{1}{2}\left(1-\frac{\omega_<(k)}{\omega_>(k)}\right), \tag{3.31}$$

(ii) unstable modes ($\mathbf{k}^2 < m_f^2$)

$$\mathcal{U}^{+}{}_k(t)=e^{-i\omega_<(k)t}\Theta(-t) + \left(A_k e^{W(k)t}+B_k e^{-W(k)t}\right)\Theta(t), \tag{3.32}$$

$$\mathcal{U}^{-}{}_k(t)=\left[\mathcal{U}^{+}{}_k(t)\right]^*, \tag{3.33}$$

$$W(k)=\sqrt{m_f^2-\mathbf{k}^2} \tag{3.34}$$

$$A_k=\frac{1}{2}\left(1-i\frac{\omega_<(k)}{W(k)}\right), B_k=(A_k)^*. \tag{3.35}$$

With these mode functions, and the periodicity condition (3.20), we find

$$G^{>}_k(t,t')=\frac{i}{2\omega_<(k)}\frac{1}{1-e^{-\beta_i\omega_<(k)}}\left[\mathcal{U}^{+}_k(t)\mathcal{U}^{-}_k(t')+e^{-\beta_i\omega_<(k)}\mathcal{U}^{-}_k(t)\mathcal{U}^{+}_k(t')\right], \tag{3.36}$$

$$G^{<}_k(t,t')=G^{>}(t',t). \tag{3.37}$$

The zeroth-order equal time Green's function becomes

$$[G^{>}_{\mathbf{k}}(t;t)]=\frac{i}{2\omega_<(k)}\coth[\beta_i\omega_<(k)/2] \tag{3.38}$$

for $t<0$, and

$$[G^{>}_{\mathbf{k}}(t;t)]=\frac{i}{2\omega_<(k)}[\ (1+2A_kB_k\{\cosh[2W(k)t]-1\})\Theta(m_f^2-\mathbf{k}^2) +(1+2a_kb_k\{\cos[2\omega_>(k)t]-1\})\Theta(\mathbf{k}^2-m_f^2)]\coth[\beta_i\omega_<(k)/2], \tag{3.39}$$

for $t>0$.

The first term, the contribution of the unstable modes, reflects the growth of correlations because of the instabilities and will be the dominant term at long times.

IV. ZERO-ORDER CORRELATIONS

Before proceeding to study the correlations in higher orders in the coupling constant, it will prove to be very illuminating to understand the behavior of the equal time nonequilibrium correlation functions at the tree level. Because we are interested in the growth of correlations, we will study only the contributions of the unstable modes.

The integral of the equal time correlation function over all wave vectors shows the familiar short distance divergences. From the above expression, however, it is clear that these may be removed by subtracting (and also multiplicatively renormalizing) this correlation function at $t=0$. The contribution of the stable modes to the subtracted and multiplicatively renormalized correlation function is always bounded in time and thus uninteresting for the purpose of understanding the growth of the fluctuations.

We are thus led to study *only* the contributions of the unstable modes to the subtracted and renormalized correlation function; this contribution is finite and unambiguous.

For this purpose it is convenient to introduce the dimensionless quantities

$$\kappa = \frac{k}{m_f}, \quad L^2 = \frac{m_i^2}{m_f^2} = \frac{\left[T_i^2 - T_c^2\right]}{\left[T_c^2 - T_f^2\right]}, \quad \tau = m_f t, \quad \mathbf{x} = m_f \mathbf{r}. \tag{4.1}$$

Furthermore for the unstable modes $\mathbf{k}^2 < m_f^2$, and for initial temperatures larger than the critical temperature $T_c^2 = 24\mu^2/\lambda$, we can approximate $\coth[\beta_i \omega_<(k)/2] \approx 2T_i/\omega_<(k)$. Then, at the tree level, the contribution of the unstable modes to the subtracted structure factor (3.23) $S^{(0)}(k,t) - S^{(0)}(k,0) = (1/m_f)\mathcal{S}^{(0)}(\kappa,\tau)$ becomes

$$\mathcal{S}^{(0)}(\kappa,\tau) = \left(\frac{24}{\lambda[1 - \frac{T_f^2}{T_c^2}]}\right)^{\frac{1}{2}} \left(\frac{T_i}{T_c}\right) \frac{1}{2\omega_\kappa^2} \left(1 + \frac{\omega_\kappa^2}{W_\kappa^2}\right) [\cosh(2W_\kappa \tau) - 1], \tag{4.2}$$

$$\omega_\kappa^2 = \kappa^2 + L^2, \tag{4.3}$$

$$W_\kappa = 1 - \kappa^2. \tag{4.4}$$

To obtain a better idea of the growth of correlations, it is convenient to introduce the scaled correlation function

$$\mathcal{D}(x,\tau) = \frac{\lambda}{6m_f^2} \int_0^{m_f} \frac{k^2 dk}{2\pi^2} \frac{\sin(kr)}{(kr)} [S(k,t) - S(k,0)]. \tag{4.5}$$

The reason for this is that the minimum of the tree-level potential occurs at $\lambda\Phi^2/6m_f^2 = 1$, and the inflection (spinodal) point, at $\lambda\Phi^2/2m_f^2 = 1$, so that $\mathcal{D}(0,\tau)$ measures the excursion of the fluctuations to the spinodal point and beyond as the correlations grow in time.

At large τ (large times), the product $\kappa^2 \mathcal{S}(\kappa,\tau)$ in (4.5) has a very sharp peak at $\kappa_s = 1/\sqrt{\tau}$. In the region $x < \sqrt{\tau}$ the integration may be done by the saddle point approximation and we obtain for $T_f/T_c \approx 0$ the large time behavior

$$\mathcal{D}(x,\tau) \approx \mathcal{D}(0,\tau) \exp\left(-\frac{x^2}{8\tau}\right) \frac{\sin(x/\sqrt{\tau})}{(x/\sqrt{\tau})}, \tag{4.6}$$

$$\mathcal{D}(0,\tau) \approx \left(\frac{\lambda}{12\pi^3}\right)^{\frac{1}{2}} \left(\frac{(\frac{T_i}{2T_c})^3}{[\frac{T_i^2}{T_c^2} - 1]}\right) \frac{\exp[2\tau]}{\tau^{\frac{3}{2}}}. \tag{4.7}$$

Restoring dimensions, and recalling that the zero-temperature correlation length is $\xi(0) = 1/\sqrt{2}\mu$, we find that for $T_f \approx 0$ the amplitude of the fluctuation inside a "domain" $\langle\Phi^2(t)\rangle$, and the "size" of a domain $\xi_D(t)$ grows as

$$\langle\Phi^2(t)\rangle \approx \frac{\exp[\sqrt{2}t/\xi(0)]}{[\sqrt{2}t/\xi(0)]^{\frac{3}{2}}}, \tag{4.8}$$

$$\xi_D(t) \approx (8\sqrt{2})^{\frac{1}{2}} \xi(0) \sqrt{\frac{t}{\xi(0)}}. \tag{4.9}$$

An important time scale corresponds to the time τ_s at which the fluctuations of the field sample beyond the spinodal point. Roughly speaking, when this happens the instabilities should shut off as the mean square root fluctuation of the field $\sqrt{\langle\Phi^2(t)\rangle}$ is now probing the stable region. This will be seen explicitly below when we study the evolution nonperturbatively in the Hartree approximation and the fluctuations are incorporated self-consistently in the evolution equations. In zero order we estimate this time from the condition $3\mathcal{D}(0,t) = 1$, we use $\lambda = 10^{-12}$, $T_i/T_c = 2$, as representative parameters (this value of the initial temperature does not have any particular physical meaning and was chosen only as representative). We find

$$\tau_s \approx 10.15 \tag{4.10}$$

or, in units of the zero-temperature correlation length

$$t \approx 14.2\xi(0) \tag{4.11}$$

for other values of the parameters τ_s is found from the above condition on (4.7).

These are some of the main results of this work.

V. PERTURBATION THEORY AND ITS DEMISE

The results presented in the previous section rely on a zero-order (tree-level) analysis of the nonequilibrium correlation function. Clearly one needs to incorporate the effects of the interaction. The nonequilibrium formalism introduced above lends itself to a diagrammatic expansion of the nonequilibrium correlation functions. We now present a one-loop calculation of the equal time correlation function $\langle\Phi_{\mathbf{k}}(t)\Phi_{-\mathbf{k}}(t)\rangle$.

There are two vertices, corresponding to forward (+) and backward (−) time propagation, with opposite couplings and four different propagators as given in Eqs. (3.15)–(3.18) [see Fig. 2(a)]. The Feynman rules are the standard ones. The Feynman diagrams that contribute up to one loop to the structure factor (3.23),

$$\langle \Phi^+_{\mathbf{k}}(t)\Phi^-_{-\mathbf{k}}(t)\rangle = \langle \Phi_{\mathbf{k}}(t)\Phi_{-\mathbf{k}}(t)\rangle = iG_k^{+-}(t,t),$$

are depicted in Fig. 2(b). Then up to one loop, and in terms of the zero-order Green's functions, we find

$$\langle \Phi_{\mathbf{k}}(t)\Phi_{-\mathbf{k}}(t)\rangle = [-iG_k^<(t,t)] + \frac{\lambda}{2}\int_{-\infty}^{t} dt_1 \int \frac{d^3q}{(2\pi)^3}\Big(G^>_{0,q}(t_1,t_1)\{[G^>_{0,k}(t,t_1)]^2 - [G^<_{0,k}(t,t_1)]^2\} \times \coth[\beta_i\omega_<(q)/2]\coth[\beta_i\omega_<(k)/2]\Big), \tag{5.1}$$

where in the one-loop integral we wrote the finite temperature Green's functions in terms of the zero-temperature ones $G^>_{0,q}$. Clearly because of the complicated time dependence, the Dyson's series for the propagator may not be summed exactly and we only analyze here the one-loop contribution given above. Before proceeding further with the analysis, let us understand the renormalizations that are necessary.

It becomes more illuminating to write the one-loop contribution explicitly in terms of the mode functions:

$$\langle \Phi^+_{\mathbf{k}}(t)\Phi^-_{-\mathbf{k}}(t)\rangle = \frac{\mathcal{U}_k^+(t)\mathcal{U}_k^-(t)}{2\omega_<(k)}\coth[\beta_i\omega_<(k)/2] + \frac{\lambda}{2}\int_{-\infty}^{t} dt_1 \int \frac{d^3q}{(2\pi)^3}\left(\frac{i}{2\omega_<(q)}\right)\left(\frac{i}{2\omega_<(k)}\right)^2 \mathcal{U}_q^+(t_1)\mathcal{U}_q^-(t_1) \times\left\{\left[\mathcal{U}_k^+(t)\mathcal{U}_k^-(t_1)\right]^2 - \left[\mathcal{U}_k^+(t_1)\mathcal{U}_k^-(t)\right]^2\right\} \times \coth[\beta_i\omega_<(q)/2]\coth[\beta_i\omega_<(k)/2]. \tag{5.2}$$

Clearly, in the one-loop contribution, the terms with wave vectors k correspond to the "external legs," whereas the terms with q which are integrated over correspond to the loop line in Fig. 2(b) (because at equal time the $++$ and $--$ terms are equal).

First let us study the above contribution for $t<0$, in which case we should recover the usual result. In this case both t, $t_1<0$, and performing the time integration with an adiabatic cutoff we find

$$\langle \Phi_{\mathbf{k}}(t)\Phi_{-\mathbf{k}}(t)\rangle = \frac{1}{2\omega_<(k)}\coth[\beta_i\omega_<(k)/2] - \frac{\lambda}{2}\int \frac{d^3q}{(2\pi)^3}\left(\frac{1}{2\omega_<(q)}\right)\left(\frac{1}{2\omega_<(k)}\right)^3 \coth[\beta_i\omega_<(q)/2]\coth[\beta_i\omega_<(k)/2]. \tag{5.3}$$

In fact this is the familiar equal time Green's function up to one loop of the time-independent theory. The one-loop term has a temperature-independent ultraviolet divergent contribution arising from the q-integral. Introducing an upper momentum cutoff Λ and an arbitrary renormalization scale $\mathcal{K}$ we obtain

$$I_{\rm div}(t_1<0) = \int \frac{d^3q}{(2\pi)^3}\left(\frac{1}{2\omega_<(q)}\right) = \frac{1}{8\pi^2}\left[\Lambda^2 - m_i^2\ln\left(\frac{\Lambda}{\mathcal{K}}\right)\right]. \tag{5.4}$$

In any case, it becomes clear that the potential divergences of the one-loop contribution can be traced to the integral (again only the zero-temperature contribution is divergent)

$$I_{\rm div}(t_1) = \int \frac{d^3q}{(2\pi)^3}\left(\frac{1}{2\omega_<(q)}\right)\mathcal{U}_q^+(t_1)\mathcal{U}_q^-(t_1). \tag{5.5}$$

For $t>0$, the time integral in the one-loop correction can be split into the integral $\int_{-\infty}^0 dt_1$ and $\int_0^t dt_1$. In the first integral (from $-\infty$ to 0), $\mathcal{U}_q^+(t_1)\mathcal{U}_q^-(t_1)=1$ and the divergence structure is the same as that analyzed for $t<0$. In the second integral (from 0 to t) the divergent contribution arises solely from the *stable modes* as the unstable modes are cut off at $q=m_f$. By analyzing the product of the mode functions, it becomes clear that the time-dependent part $(2a_qb_q\cos[2\omega_>(q)t_1])$ will yield to a finite contribution because the strong oscillations (for $t_1>0$) ensure convergence at large momenta. Thus the divergent term arises only from the time-independent term, $(a_q^2+b_q^2)$, in the product of mode functions. Finally we find the divergent term to be temperature independent and for $t_1>0$ given by

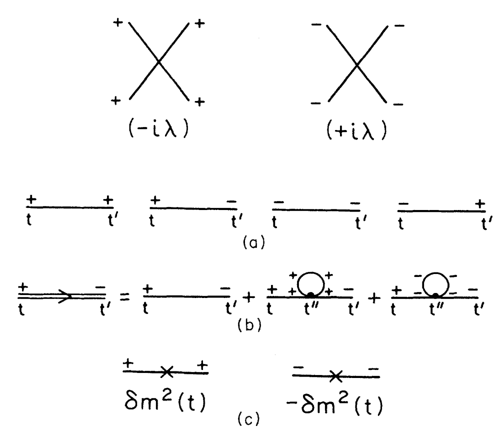

FIG. 2. (a) Two vertices and four propagators generate the Feynman diagrammatic expansion for nonequilibrium Green's functions. (b) Diagrams that contribute up to one loop to $\langle\Phi_{\mathbf{k}}(t)\Phi_{-\mathbf{k}}(t')\rangle$. (c) Two mass counterterms.

$$I_{\rm div} = \frac{1}{4\pi^2}\int_{m_f}^{\infty} dq \left(\frac{q^2}{\sqrt{q^2+m_i^2}}\right)\left\{1+\frac{1}{2}\left(\frac{m_i^2+m_f^2}{q^2-m_f^2}\right)\right\}. \tag{5.6}$$

Again in terms of an ultraviolet cutoff (Λ) and renormalization scale ($\mathcal{K}$), we find

$$I_{\rm div}(t_1>0) = \frac{1}{8\pi^2}\left[\Lambda^2 - (-m_f^2)\ln\left(\frac{\Lambda}{\mathcal{K}}\right)\right]. \tag{5.7}$$

These divergences may be canceled by introducing a local but *time dependent* counterterm in the original Lagrangian density:

$$\mathcal{L}_{\rm ct} = \delta m^2(t)\Phi^2(\mathbf{r},t), \tag{5.8}$$

$$\delta m^2(t) = -\frac{\lambda}{8\pi^2}\int dq \frac{q^2}{\sqrt{q^2+m_i^2}}\left\{\Theta(-t)+\Theta(t)\Theta(q^2-m_f^2)(a_q^2+b_q^2)\right\}. \tag{5.9}$$

On the forward and backward time contour for the nonequilibrium theory, this counterterm translates in the two counterterm insertions shown in Fig. 2(c). The introduction of these counterterms renders finite the one-loop contribution to all the nonequilibrium one-particle Green's functions as may now be easily checked.

Having disposed of the renormalization problem, we must however address the issue of the instabilities. The instabilities and growth of correlations at zero-order had been analyzed before. We now realize that in the loop integral there is a contribution to the loop from the integration over the unstable modes which will enhance the exponential growth in the correlation functions.

The maximum instability in the one-loop term is when the mode functions for both momenta q,k are unstable ($q^2,k^2 < m_f^2$). For the initial temperature $T_i > T_c$, for these values of the momenta we use the high-temperature approximation $\coth[\beta_i\omega_</2] \approx 2T_i/\omega_<$. It is convenient to introduce the dimensionless quantities defined in Eq. (4.1) and $Q = q/m_f$. Using Eqs. (3.3) and (3.4) and $T_c^2 = 24\mu^2/\lambda$, and the same conventions as for the zero-order structure factor given by Eqs. (4.3), (4.4), and (4.1), we obtain, for the *most unstable contribution* to the one-loop correction to the structure factor $S^{(1)}(k,t) = (1/m_f)\mathcal{S}^{(1)}(\kappa,\tau)$

$$\mathcal{S}^{(1)}(\kappa,\tau) = \left(-\frac{6}{\pi^2}\right)\frac{T_i^2}{T_c^2[1-(T_f^2/T_c^2)]}\int_0^{\tau} d\tau_1\int_0^1 dQ\frac{Q^2}{\omega_Q^2\omega_\kappa^2 W_\kappa}\sinh[W_\kappa(\tau-\tau_1)]\left[1+\frac{1}{2}\left(1+\frac{\omega_Q^2}{W_Q^2}\right)[\cosh(2W_Q\tau_1)-1]\right]$$
$$\times\left[\left(1+\frac{\omega_\kappa^2}{W_\kappa^2}\right)\cosh[W_\kappa(\tau+\tau_1)]+\left(1-\frac{\omega_\kappa^2}{W_\kappa^2}\right)\cosh[W_\kappa(\tau-\tau_1)]\right]. \tag{5.10}$$

The integral over τ_1 may be carried out yielding a rather cumbersome result, but it becomes clear that this result will grow roughly as the square of the zero-order result at large τ. Introducing the scaled correlation function as in Eq. (4.5) both for the zero-order and the one-loop order $\mathcal{D}^{(0)}(x,\tau)$ and $\mathcal{D}^{(1)}(x,\tau)$, respectively, in Fig. 3(a) we show the behavior for $3\mathcal{D}^{(0)}(0,\tau)$ (solid lines) and $3\mathcal{D}^{(1)}(0,\tau)$ (dashed lines) for the values of the parameters $\lambda = 10^{-12}$, $(T_i/T_c) = 2$. It is clear that eventually the one-loop term becomes *much larger* than the tree-level term even in the case of very weak coupling. This is a consequence of the instabilities and the growth of correlations that are a hallmark of the phase transition. Clearly the *dynamics* of the phase transition *cannot be studied in perturbation theory.* In fact this result in a very quantitative manner confirms the ideas that the onset of the transition and the time evolution of the system after the phase transition cannot be studied perturbatively.

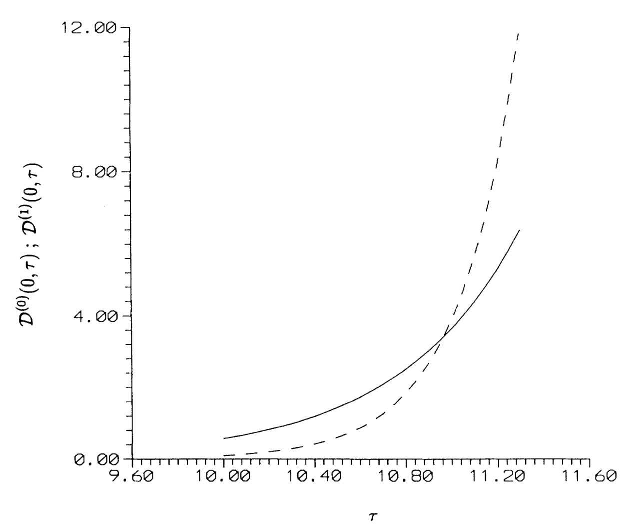

FIG. 3. Zero- and one-loop contributions to the structure factor. The solid line represents $3\mathcal{D}^{(0)}(0,\tau)$, the dashed line represents $3\mathcal{D}^{(1)}(0,\tau)$.

VI. BEYOND PERTURBATION THEORY: HARTREE APPROXIMATION

It became clear from the analysis of the previous section that perturbation theory is inadequate to describe the nonequilibrium dynamics of the phase transition, precisely because of the instabilities and the growth of correlations. This growth is manifest in the Green's functions that enter in any perturbative expansion thus invalidating any perturbative approach. Higher-order corrections will have terms that grow exponentially and faster than the previous term in the expansion. And even for very weakly coupled theories, the higher-order corrections eventually become of the same order as the lower-order terms.

As the correlations and fluctuations grow, field configurations start sampling the stable region beyond the spinodal point. This will result in a slowdown in the growth of correlations, and eventually the unstable growth will shut off. When this happens, the state may be described by correlated domains with equal probability for both phases inside the domains. The expectation value of the field in this configuration will be zero, but inside each domain, the field will acquire a value very close to the value in equilibrium at the minimum of the *effective potential.* The size of the domain in this picture will depend on the time during which correlations had grown enough so that fluctuations start sampling beyond the spinodal point.

Since this physical picture may not be studied within perturbation theory, we now introduce a *nonperturbative* method based on a self-consistent Hartree approximation [38–40].

The self-consistent Hartree approximation is implemented as follows: in the initial Lagrangian write

$$\frac{\lambda}{4!}\Phi^4(\mathbf{r},t) = \frac{\lambda}{4}\langle\Phi^2(\mathbf{r},t)\rangle\Phi^2(\mathbf{r},t) + \left(\frac{\lambda}{4!}\Phi^4(\mathbf{r},t) - \frac{\lambda}{4}\langle\Phi^2(\mathbf{r},t)\rangle\Phi^2(\mathbf{r},t)\right) \tag{6.1}$$

the first term is absorbed in a shift of the mass term

$$m^2(t) \to m^2(t) + \frac{\lambda}{2}\langle\Phi^2(t)\rangle$$

(where we used spatial translational invariance). The second term in (6.1) is taken as an interaction with the term $\langle\Phi^2(t)\rangle\Phi^2(\mathbf{r},t)$ as a "mass counterterm." The Hartree approximation consists of requiring that the one-loop correction to the two-point Green's functions must be canceled by the "mass counterterm." This leads to the self-consistent set of equations

$$\langle\Phi^2(t)\rangle = \int\frac{d^3k}{(2\pi)^3}\left[-iG_k^<(t,t)\right] = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_<(k)}\mathcal{U}_k^+(t)\mathcal{U}_k^-(t)\coth[\beta_i\omega_<(k)/2], \tag{6.2}$$

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) + \frac{\lambda}{2}\langle\Phi^2(t)\rangle\right]\mathcal{U}_k^\pm = 0. \tag{6.3}$$

Before proceeding any further, we must address the fact that the composite operator $\langle\Phi^2(\mathbf{r},t)\rangle$ needs one subtraction and multiplicative renormalization. As usual the subtraction is absorbed in a renormalization of the bare mass and the multiplicative renormalization into a renormalization of the coupling constant. We must also point out that the Hartree approximation is uncontrolled in this scalar theory; it becomes equivalent to the large-N limit in theories in which the field is in the vector representation of $O(N)$ (see, for example, [8]).

At this stage our justification for using this approximation is based on the fact that it provides a nonperturbative framework to sum an infinite series of Feynman diagrams of the cactus type [8, 40].

In principle one may improve on this approximation by using the Hartree propagators in a loop expansion. The cactus-type diagrams will still be canceled by the counterterms (Hartree condition), but other diagrams with loops (for example diagrams with multiparticle thresholds) may be computed by using the Hartree propagators on the lines. This approach will have the advantage that the Hartree propagators will only be unstable for a finite time $t \leq t_s$. It is not presently clear to these authors, however, what, if any, would be the expansion parameter in this case.

It is clear that for $t < 0$ there is a self-consistent solution to the Hartree equations with Eq. (6.2) and

$$\langle\Phi^2(t)\rangle = \langle\Phi^2(0^-)\rangle,$$
$$\mathcal{U}_k^\pm = \exp[\mp i\omega_<(k)], \tag{6.4}$$

$$\omega_<^2(k) = \mathbf{k}^2 + m_i^2 + \frac{\lambda}{2} + \langle\Phi^2(0^-)\rangle = \mathbf{k}^2 + m_{i,R}^2, \tag{6.5}$$

where the composite operator has been absorbed in a renormalization of the initial mass, which is now parametrized as $m_{i,R}^2 = \mu_R^2[(T_i^2/T_c^2) - 1]$. For $t > 0$ we subtract the composite operator at $t = 0^+$ absorbing the subtraction into a renormalization of m_f^2 which we now parametrize as $m_{f,R}^2 = \mu_R^2[1 - (T_f^2/T_c^2)]$. We should point out that this choice of parametrization only represents a choice of the bare parameters, which can always be chosen to satisfy this condition. The logarithmic multiplicative divergence of the composite operator will be absorbed in a coupling constant renormalization consistent with the Hartree approximation [41]; however, for the purpose of understanding the dynamics of growth of instabilities associated with the long-wavelength fluctuations, we will not need to specify this procedure. After this subtraction procedure, the Hartree equations read

$$[\langle\Phi^2(t)\rangle - \langle\Phi^2(0)\rangle] = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_<(k)}[\mathcal{U}_k^+(t)\mathcal{U}_k^-(t) - 1]\coth[\beta_i\omega_<(k)/2], \tag{6.6}$$

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m_R^2(t) + \frac{\lambda_R}{2}\left[\langle\Phi^2(t)\rangle - \langle\Phi^2(0)\rangle\right]\right]\mathcal{U}_k^\pm(t) = 0, \tag{6.7}$$

$$m_R^2(t) = \mu_R^2\left[\frac{T_i^2}{T_c^2} - 1\right]\Theta(-t) - \mu_R^2\left[1 - \frac{T_f^2}{T_c^2}\right]\Theta(t), \tag{6.8}$$

with $T_i > T_c$ and $T_f \ll T_c$. With the self-consistent solution and boundary condition for $t < 0$,

$$[\langle\Phi^2(t<0)\rangle - \langle\Phi^2(0)\rangle] = 0, \tag{6.9}$$

$$\mathcal{U}_k^\pm(t<0) = \exp[\mp i\omega_<(k)t], \tag{6.10}$$

$$\omega_<(k) = \sqrt{\mathbf{k}^2 + m_{iR}^2}. \tag{6.11}$$

This set of Hartree equations is extremely complicated to be solved exactly. However it has the correct physics in it. Consider the equations for $t > 0$, at very early times, when (the renormalized) $\langle\Phi^2(t)\rangle - \langle\Phi^2(0)\rangle \approx 0$ the mode functions are the same as in the zero-order approximation, and the unstable modes grow exponentially. By computing expression (6.6) self-consistently with these zero-order unstable modes, we see that the fluctuation operator begins to grow exponentially.

As $[\langle\Phi^2(t)\rangle - \langle\Phi^2(0)\rangle]$ grows larger, its contribution to the Hartree equation tends to balance the negative mass term, thus weakening the unstabilities, so that only longer wavelengths can become unstable. Even for very weak coupling constants, the exponentially growing modes make the Hartree term in the equation of motion for the mode functions become large and compensate for the negative mass term. Thus when

$$\frac{\lambda_R}{2m_{f,R}^2}\left[\langle\Phi^2(t)\rangle - \langle\Phi^2(0)\rangle\right] \approx 1$$

the instabilities shut off, this equality determines the "spinodal time" t_s. The modes will still continue to grow further after this point because the time derivatives are fairly (exponentially) large, but eventually the growth will slow down when fluctuations sample deep inside the stable region.

After the subtraction, and multiplicative renormalization (absorbed in a coupling constant renormalization), the composite operator is finite. The stable mode functions will make a *perturbative* contribution to the fluctuation which will be always bounded in time. The most important contribution will be that of the *unstable modes.* These will grow exponentially at early times and their effect will dominate the dynamics of growth and formation of correlated domains. The full set of Hartree equations is extremely difficult to solve, even numerically, so we will restrict ourselves to account *only* for the unstable modes. From the above discussion it should be clear that these are the only relevant modes for the dynamics of formation and growth of domains, whereas the stable modes will always contribute perturbatively for weak coupling after renormalization.

Introducing the dimensionless ratios (4.1) in terms of $m_{f,R}$, $m_{i,R}$ (all momenta are now expressed in units of $m_{f,R}$), dividing (6.7) by $m_{f,R}^2$, using the high-temperature approximation $\coth[\beta_i\omega_<(k)/2] \approx 2T_i/\omega_<(k)$ for the unstable modes, and expressing the critical temperature as $T_c^2 = 24\mu_R^2/\lambda_R$, the set of Hartree equations (6.6) and (6.7) become the following integro-differential equation for the mode functions for $t > 0$:

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + g\int_0^1 dp\left\{\frac{p^2}{p^2 + L_R^2}[\mathcal{U}_p^+(t)\mathcal{U}_p^-(t) - 1]\right\}\right]\mathcal{U}_q^\pm(t) = 0 \tag{6.12}$$

with

$$\mathcal{U}_q^\pm(t<0) = \exp[\mp i\omega_<(q)t], \tag{6.13}$$

$$\omega_<(q) = \sqrt{q^2 + L_R^2}, \tag{6.14}$$

$$L_R^2 = \frac{m_{i,R}^2}{m_{f,R}^2} = \frac{[T_i^2 - T_c^2]}{[T_c^2 - T_f^2]}, \tag{6.15}$$

$$g = \frac{\sqrt{24\lambda_R}}{4\pi^2}\frac{T_i}{[T_c^2 - T_f^2]^{\frac{1}{2}}}. \tag{6.16}$$

The effective coupling (6.16) reflects the enhancement of quantum fluctuations by high-temperature effects; for $T_f/T_c \approx 0$, and for couplings as weak as $\lambda_R \approx 10^{-12}$, $g \approx 10^{-7}(T_i/T_c)$.

Eqs. (6.12) may now be integrated numerically for the mode functions; once we find these, we can then compute the contribution of the unstable modes to the subtracted correlation function equivalent to (4.5):

$$\mathcal{D}^{(HF)}(x,\tau)=\frac{\lambda_R}{6m^2_{f,R}}\left[\langle\Phi(\mathbf{r},t)\Phi(\mathbf{0},t)\rangle-\langle\Phi(\mathbf{r},0)\Phi(\mathbf{0},0)\rangle\right], \tag{6.17}$$

$$3\mathcal{D}^{(HF)}(x,\tau)=g\int_0^1 dp\left(\frac{p^2}{p^2+L_R^2}\right)\frac{\sin(px)}{(px)}\left[\mathcal{U}_p^+(t)\mathcal{U}_p^-(t)-1\right]. \tag{6.18}$$

In Fig. 4 we show

$$\frac{\lambda_R}{2m^2_{f,R}}[\langle\Phi^2(\tau)\rangle-\langle\Phi^2(0)\rangle]=3[\mathcal{D}^{\rm HF}(0,\tau)-\mathcal{D}^{\rm HF}(0,0)]$$

(solid line) and also for comparison, its zero-order counterpart $3[\mathcal{D}^{(0)}(0,\tau)-\mathcal{D}^{(0)}(0,0)]$ (dashed line) for $\lambda_R=10^{-12}$, $T_i/T_c=2$. (Again, this value of the initial temperature does not have any particular physical significance and was chosen as a representative.) We clearly see what we expected; whereas the zero-order correlation grows indefinitely, the Hartree correlation function is bounded in time and oscillatory. At $\tau\approx10.52$, $3[\mathcal{D}^{(\rm HF)}(0,\tau)-\mathcal{D}^{(\rm HF)}(0,\tau)]=1$, fluctuations are sampling field configurations near the classical spinodal, fluctuations continue to grow, however, because the derivatives are still fairly large. However, after this time, the modes begin to probe the stable region in which there is no exponential growth. At this point $\frac{\lambda_R}{2m^2_{f,R}}[\langle\Phi^2(\tau)\rangle-\langle\Phi^2(0)\rangle]$ becomes small again because of the small coupling $g\approx10^{-7}$, and the correction term becomes small. When it becomes smaller than one, the instabilities set in again, modes begin to grow, and the process repeats. This gives rise to an oscillatory behavior around $\frac{\lambda_R}{2m^2_{f,R}}[\langle\Phi^2(\tau)\rangle-\langle\Phi^2(0)\rangle]=1$ as shown in Fig. 4.

In Figs. 5(a)–(d), we show the structure factors as a function of x for $\tau=6$, 8, 10, 12, both for zero-order (tree level) $\mathcal{D}^{(0)}$ (dashed lines) and Hartree $\mathcal{D}^{(\rm HF)}$ (solid lines). These correlation functions clearly show that correlations grow in amplitude and that the size of the region in which the fields are correlated increases with time. Clearly this region may be interpreted as a "domain," inside which the fields have strong correlations, and outside which the fields are uncorrelated.

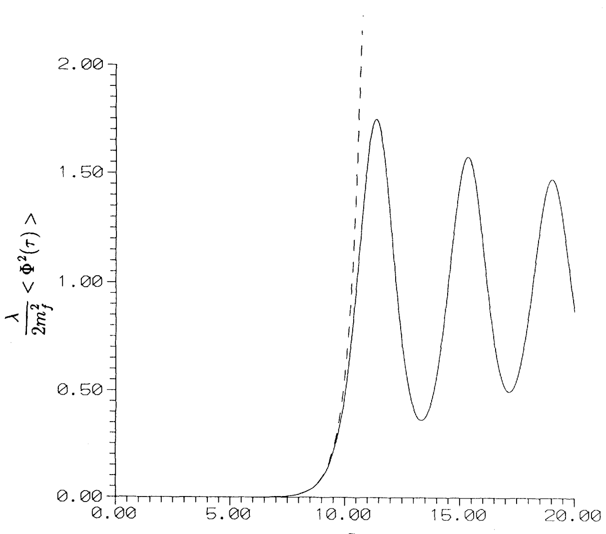

FIG. 4. Hartree (solid line) and zero-order (dashed line) results for $\frac{\lambda_R}{2m_f^2}[\langle\Phi^2(\tau)\rangle-\langle\Phi^2(0)\rangle]=\mathcal{D}(0,\tau)$, for $\lambda=10^{-12}$, $\frac{T_i}{T_c}=2$.

We see that up to the spinodal time $\tau_s\approx10.52$ at which $\frac{\lambda_R}{2m^2_{f,R}}[\langle\Phi^2(\tau_s)\rangle-\langle\Phi^2(0)\rangle]=1$, the zero-order correlation $3\mathcal{D}^{(0)}(0,\tau)$ is very close to the Hartree result. In fact, at τ_s, the difference is less than 15%. In particular for these values of the coupling and initial temperature, the zero-order correlation function leads to $\tau_s\approx10.15$, and we may use the zero-order correlations to provide an analytic estimate for τ_s, as well as the form of the correlation functions and the size of the domains.

The fact that the zero-order correlation remains very close to the Hartree-corrected correlations up to times comparable to the spinodal is clearly a consequence of the very small coupling.

To illustrate this fact, we show in Figs. (6) and (7) the same correlation functions but for $\lambda=0.01$, $T_i/T_c=2$. Clearly the stronger coupling makes the growth of domains much faster and the departure from tree-level correlations more dramatic. Thus, it becomes clear that for strong couplings, domains will form very rapidly and only grow to sizes of the order of the zero-temperature correlation length. The phase transition will occur very rapidly, and clearly our initial assumption of a rapid supercooling will be unjustified. This situation for strong couplings, of domains forming very rapidly to sizes of the order of the zero-temperature correlation length, is the picture presented by Mazenko and collaborators [15]. However, for very weak couplings (consistent with the bounds from density fluctuations), our results indicate that the phase transition will proceed very slowly, domains will grow for a long time and become fairly large, with a typical size several times the zero-temperature correlation length. In a sense, this is a self-consistent check of our initial assumptions on a rapid supercooling in the case of weak couplings.

Thus, as we argued above, for very weak coupling, we may use the tree-level result to give an approximate bound to the correlation functions up to times close to the spinodal time using the result given by Eq. (4.7), for $T_f\approx0$. Thus, we conclude that for large times, and very weakly coupled theories ($\lambda_R\leq10^{-12}$) and for initial temperatures of the order of the critical temperature, the size of the domains $\xi_D(t)$ will grow typically in time as

$$\xi_D(t)\approx(8\sqrt{2})^{\frac{1}{2}}\xi(0)\sqrt{\frac{t}{\xi(0)}} \tag{6.19}$$

with $\xi(0)$ the zero-temperature correlation length. The maximum size of a domain is approximately determined by the time at which fluctuations begin probing the stable region, this is the spinodal time t_s and the maximum size of the domains is approximately $\xi_D(t_s)$.

An estimate for the spinodal time is obtained from Eq. (4.7) by the condition $3\mathcal{D}(\tau_s) = 1$, then for weakly coupled theories and $T_f \approx 0$, we obtain

$$\tau_s = \frac{t_s}{\sqrt{2}\xi(0)} \approx -\ln\left[\left(\frac{3\lambda}{4\pi^3}\right)^{\frac{1}{2}}\left(\frac{(\frac{T_i}{2T_c})^3}{[\frac{T_i^2}{T_c^2}-1]}\right)\right]. \quad (6.20)$$

It is remarkable that the domain size scales as $\xi_D(t) \approx t^{\frac{1}{2}}$ just like in classical theories of spinodal decomposition, when the order parameter *is not conserved*, as is the case in the scalar relativistic field theory under consideration, but certainly for completely different reasons. At the tree level, we can identify this scaling behavior as arising from the relativistic dispersion relation, and second-order time derivatives in the equations of motion, a situation very different from the classical description of the Allen-Cahn-Lifshitz [23, 24, 42] theory of spinodal decomposition based on a time-dependent Landau-Ginzburg model.

A. Beyond Hartree

The Hartree approximation, keeping only the unstable modes in the self-consistent equation, clearly cannot be accurate for times beyond the spinodal time. When the oscillations in the Hartree solution begin, the field fluc-

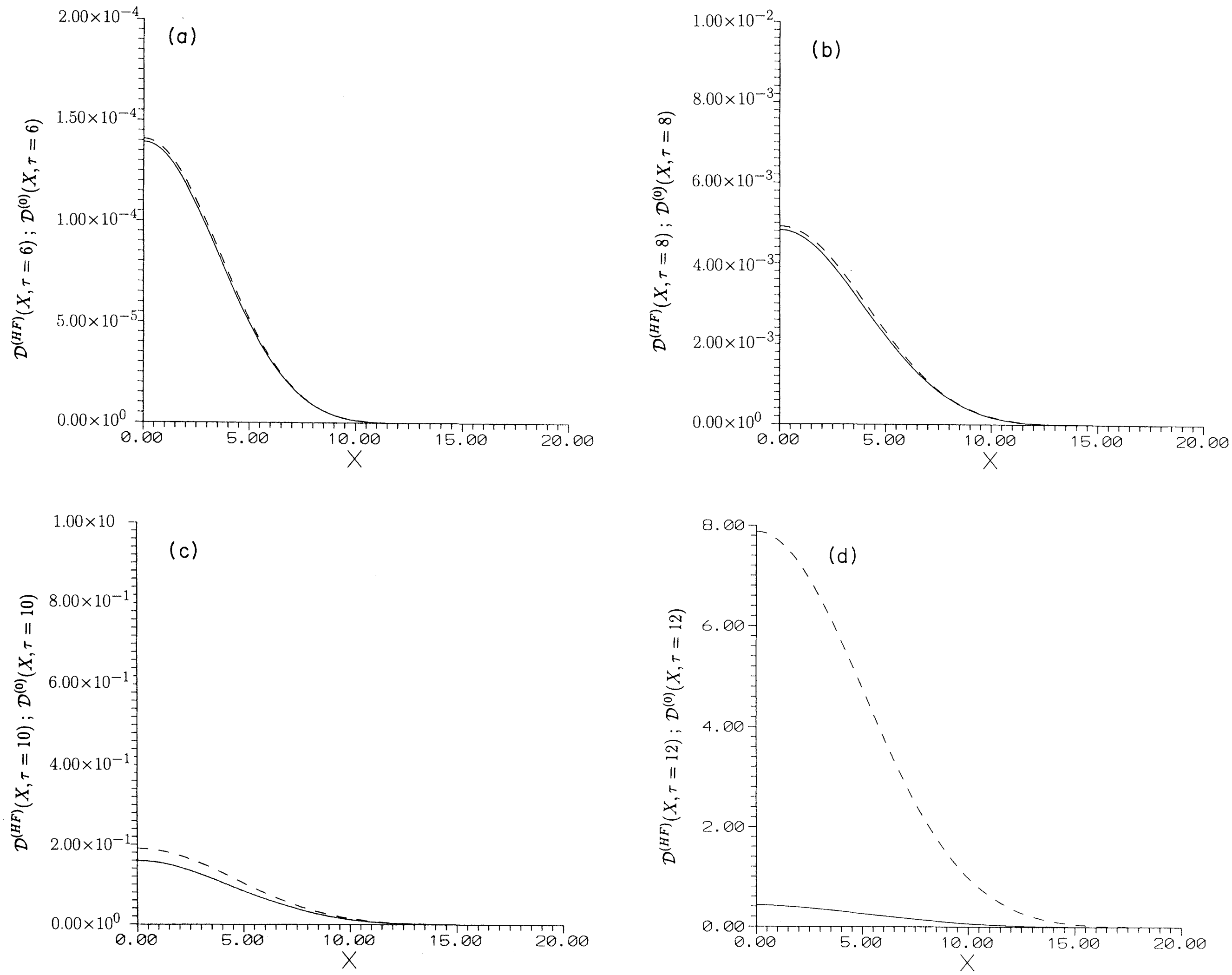

FIG. 5. (a) Scaled correlation functions for $\tau = 6$, as function of x, $\mathcal{D}^{(\mathrm{HF})}(x,\tau)$ (solid line), and $\mathcal{D}^{(0)}(x,\tau)$ (dashed line). $\lambda = 10^{-12}$, $\frac{T_i}{T_c} = 2$. (b) Scaled correlation functions for $\tau = 8$, as function of x, $\mathcal{D}^{(\mathrm{HF})}(x,\tau)$ (solid line), and $\mathcal{D}^{(0)}(x,\tau)$ (dashed line). $\lambda = 10^{-12}$, $\frac{T_i}{T_c} = 2$. (c) Scaled correlation functions for $\tau = 10$, as function of x, $\mathcal{D}^{(\mathrm{HF})}(x,\tau)$ (solid line), and $\mathcal{D}^{(0)}(x,\tau)$ (dashed line). $\lambda = 10^{-12}$, $\frac{T_i}{T_c} = 2$. (d) Scaled correlation functions for $\tau = 12$, as function of x, $\mathcal{D}^{(\mathrm{HF})}(x,\tau)$ (solid line), and $\mathcal{D}^{(0)}(x,\tau)$ (dashed line). $\lambda = 10^{-12}$, $\frac{T_i}{T_c} = 2$.

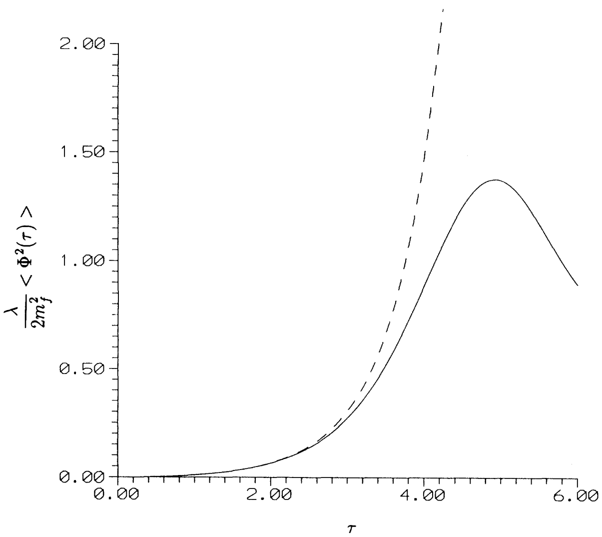

FIG. 6. Hartree (solid line) and zero-order (dashed line) results for $\frac{\lambda_R}{2m_f^2}[\langle\Phi^2(\tau)\rangle - \langle\Phi^2(\tau)\rangle] = 3\mathcal{D}(0,\tau)$, for $\lambda = 0.01$, $\frac{T_i}{T_c} = 2$.

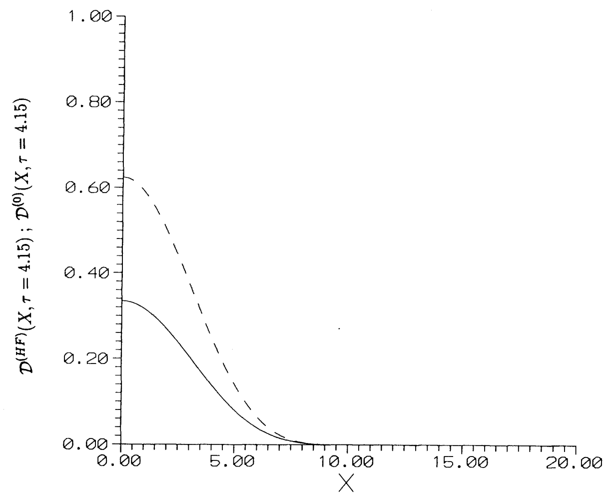

FIG. 7. Scaled correlation functions for $\tau = 4.15$, as function of x, $\mathcal{D}^{(\mathrm{HF})}(x,\tau)$ (solid line), and $\mathcal{D}^{(0)}(x,\tau)$ (dashed line). $\lambda = 0.01$, $\frac{T_i}{T_c} = 2$.

tuations are probing the stable region. This should correspond to the onset of the "reheating" epoch, in which dissipative effects become important for processes of particle and entropy production. Clearly the Hartree approximation ignores all dissipative processes, as may be understood from the fact that this approximation sums the cactus-type diagrams for which there are no multiparticle thresholds. Furthermore, in this region, the contribution of the stable modes to the Hartree equation becomes important for the subsequent evolution beyond the spinodal point and clearly will contribute to the "reheating" process. A possible approach to incorporate the contribution of the stable modes may be that explored by Avan and de Vega [43] in terms of the effective action for the composite operator.

Thus, although the Hartree approximation may give a fairly accurate picture of the process of domain formation and growth, one must go beyond this approximation at times later than the spinodal time, to incorporate dissipative effects and to study the "reheating" period. Clearly, one must also attempt to study the possibility of "percolation of domains." Furthermore, the Hartree approximation is essentially a Gaussian approximation, as the wave functional (or in this case the functional density matrix) is Gaussian with kernels that are obtained self-consistently. The wave functional must include non-Gaussian correlations that will account for the corrections to the Hartree approximation and will be important to obtain the long time behavior for $t > t_s$.

VII. CONCLUSIONS AND LOOKING AHEAD

The motivations of this work were twofold. First we pointed out that the dynamics of typical phase transitions in weakly coupled theories must be studied away from thermodynamic equilibrium, and introduced the methods and techniques of nonequilibrium quantum statistical mechanics to study this situation.

Second we studied both analytically and numerically the case of a strongly supercooled phase transition in which the system initially in thermal equilibrium at an initial temperature larger than the critical is cooled down to temperatures well below the transition temperature. The motivation here was to model a period of rapid inflation in a weakly coupled theory and to study the formation and growth of correlated domains. We indicated that the *dynamics* of the phase transition cannot be studied within perturbation theory because of the instabilities that drive the process of domain formation and growth, that is spinodal decomposition.

We used a nonperturbative self-consistent Hartree approximation to study the time evolution of domain growth, reflected in the equal time two-point correlation function $\langle\Phi(\mathbf{r},t)\Phi(\mathbf{0},t)\rangle$.

We conclude that, for *weakly* coupled theories at long times (and distances),

$$\langle\Phi(\mathbf{r},t)\Phi(\mathbf{0},t)\rangle \approx \frac{\exp[\sqrt{2}t/\xi(0)]}{[\sqrt{2}t/\xi(0)]^{\frac{3}{2}}}\exp[-r^2/\xi_D^2(t)]\frac{\sin[\sqrt{8}r/\xi_D(t)]}{[r/\xi_D(t)]}, \tag{7.1}$$

$$\xi_D(t) \approx (8\sqrt{2})^{\frac{1}{2}}\xi(0)\sqrt{\frac{t}{\xi(0)}}, \tag{7.2}$$

with $\xi(0)$ the zero-temperature correlation length. The domains, however, will grow up to a maximum time at which the fluctuations begin sampling the stable region. This maximum "spinodal time" is approximately given for weakly coupled theories by

$$t_s \approx -\sqrt{2}\xi(0)\ln\left[\left(\frac{3\lambda}{4\pi^3}\right)^{\frac{1}{2}}\left(\frac{(\frac{T_i}{2T_c})^3}{[\frac{T_i^2}{T_c^2}-1]}\right)\right]. \tag{7.3}$$

When the self-couplings are strong, the phase transition proceeds rapidly, and domains will not have time to grow substantially, and their sizes will be of the order of the zero-temperature correlation length.

In principle the "sudden approximation" (quenching) may be relaxed at the expense of complications; however, the formalism presented in this paper is completely general, once the initial state is specified and the boundary conditions for the mode functions are understood, the time evolution of correlation functions may be studied numerically.

Clearly, the next step is to study the dynamics of the phase transition in FRW cosmologies. In this case, there are several physical effects that will play a very important role in the dynamics. In particular, the redshift of physical wave vectors will tend to enhance the instabilities, because more wave vectors are entering the unstable region as time evolves. On the other hand, the presence of a horizon, and the "friction" term in the Heisenberg equations of motion, will prevent domains from growing bigger than the horizon size. Thus there seems to be a competition between the different time scales that must be studied carefully to obtain any meaningful conclusion about formation and growth of correlated domains in FRW cosmologies.

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$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_<(k)} [\mathcal{U}_k^+(t)\mathcal{U}_k^-(t) - 1] \coth[\beta_i \omega_<(k)/2]|_{\rm div} = \frac{\lambda}{8\pi^2} [\langle \Phi^2(t) \rangle - \langle \Phi^2(0) \rangle] \ln\left(\frac{\Lambda}{\kappa}\right) ;$$

the logarithmic divergence will be absorbed in the renormalization of the coupling constant. We will be cavalier about the renormalization procedure as we are only interested in the unstable fluctuations which will always yield to a finite and unambiguous contribution, and will refer to the renormalized (subtracted and multiplicatively renormalized) composite operator.
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