

Charge quantization of axion-dilaton black holes

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We present axion-dilaton black-hole and multi-black-hole solutions of the low-energy string effective action. Under $SL(2, R)$ electric-magnetic duality rotations only the “hair” (charges and asymptotic values of the fields) of our solutions is transformed. The functional form of the solutions is duality invariant. Axion-dilaton black holes with zero entropy and zero area of the horizon form a family of stable particlelike objects, which we call *holons*. We study the quantization of the charges of these objects and its compatibility with duality symmetry. In general the spectrum of black-hole solutions with quantized charges is not invariant under $SL(2, R)$ but only under $SL(2, Z)$ or one of its subgroups Γ_l . Because of their transformation properties, the asymptotic value of the axion-dilaton field of a black hole may be associated with the modular parameter τ of some complex torus and the integer numbers (n, m) that label its quantized electric and magnetic charges may be associated with winding numbers.

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In this paper we are going to present general four-dimensional static axion-dilaton black-hole solutions. The independent parameters characterizing such solutions are the black-hole mass, electric and magnetic charges, and the values of dilaton and axion fields at infinity. The dilaton and axion charges of these solutions are functions of these independent parameters. Electric and magnetic charges of these black holes will be quantized.

Our solutions include zero-area, zero-entropy axion-dilaton black holes, carrying quantized electric and magnetic charges. In this sense such objects resemble elementary particles: We will call them *holons*.

The $SL(2, R)$ -duality symmetry of the classical equations of motion of the low-energy string effective action [1, 2] has been used in [3], [4], and [5] to find new solutions with a nontrivial axion field. However, their final form was not quite satisfactory either because the axion and dilaton fields at infinity were not independent parameters or because the new solutions were expressed in terms of the old charges and the parameters of the transformation that have no physical meaning. Here we present the most general static solutions that can be obtained in this way, but expressed in terms of the new charges only. Under a new duality transformation only the “hair” (charges and asymptotic values of axion and dilaton fields, i.e., the boundary conditions) will change in a well-established way, while the functional form of the fields will remain invariant.

We also obtained axion-dilaton multi-black-hole solutions. All fields in the solutions are naturally built out of two complex harmonic functions, which are real when the axion field identically vanishes [6, 7].

We will treat the $SL(2, R)$ transformations in a way that will stress the analogies and differences with the

well-known $U(1)$ duality group of the Einstein-Maxwell theory. In the Einstein-Maxwell case, with the electric and magnetic charges Q, P , one can build a two-component duality vector (Q, P) , as explained in [8]. With axion and dilaton fields, the canonical electric and magnetic charges do not form $SL(2, R)$ -duality vectors. We will present combinations (\tilde{q}, p) of the canonical charges Q, P and axion-dilaton asymptotic value λ_0 , which do transform as duality vectors.

The issue of quantization of electric and magnetic charges and the question of which subgroup of $SL(2, R)$ will be compatible with the spectrum of black-hole charges can be addressed in the context of these families of black-hole solutions. In the context of general string-theory solutions this problem has been studied by Sen [9] by embedding the $U(1)$ into the $SU(2)$ gauge group. We will find the spectra of axion-dilaton black-hole charges with and without non-Abelian embedding, and will classify the symmetry groups of the spectra allowed by Dirac quantization condition.

For $SL(2, Z)$ -invariant spectra, the $SL(2, R)$ -duality vectors (\tilde{q}, p) , when properly normalized, take integer values (n, m) and label naturally the allowed states. We will see that it is useful to describe the action of $SL(2, Z)$ on these spectra in terms of $-1/\lambda_0 = \tau$, the modular parameter of a complex torus and the winding numbers (n, m) of nontrivial homology cycles on this torus.

Our conventions and our action are those of Refs. [7] and [5], but here we will use the complex scalar $\lambda = iz = a + ie^{-2\phi}$, where a is the axion field, and ϕ is the dilaton field, and we will extend our analysis to a set of $U(1)$ vector fields $A_\mu^{(n)}$, $n = 1, 2, \dots, N$.

We find it convenient to define the $SL(2, R)$ duals¹ to the fields $F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$,

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¹The spacetime duals are $*F^{(n)\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, with $\epsilon^{0123} = \epsilon_{0123} = +i$.

$$\tilde{F}^{(n)} = e^{-2\phi} \star F^{(n)} - iaF^{(n)}, \quad (1)$$

in terms of which the action reads

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -R + \frac{1}{2} \frac{\partial_\mu \lambda \partial^\mu \bar{\lambda}}{(\text{Im } \lambda)^2} - \sum_{n=1}^N F_{\mu\nu}^{(n)} \star \tilde{F}^{(n)\mu\nu} \right\}. \quad (2)$$

In terms of the component fields, we have

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -R + 2(\partial\phi)^2 + \frac{1}{2} e^{4\phi} (\partial a)^2 - e^{-2\phi} \sum_{n=1}^N (F^{(n)})^2 + ia \sum_{n=1}^N F^{(n)} \star F^{(n)} \right\}. \quad (3)$$

The advantage of using $\tilde{F}^{(n)}$ is that the equations of motion imply the local existence of N real vector potentials $\tilde{A}^{(n)}$ such that

$$\tilde{F}^{(n)} = i d\tilde{A}^{(n)}. \quad (4)$$

The analogous equation $F^{(n)} = dA^{(n)}$ is not a consequence of the equations of motion but a consequence of the Bianchi identity. If the timelike components $A_t^{(n)}$ play the role of electrostatic potentials, then the $\tilde{A}_t^{(n)}$ will play the role of magnetostatic potentials. As we will see in the next section, the $SL(2, R)$ -duality transformations consist in the mixing of $A^{(n)}$ with $\tilde{A}^{(n)}$ and of the equations of motion with Bianchi identities, as in the Einstein-Maxwell case.

Let us stress at this point that the fields $\tilde{A}^{(n)}$ are *not* dynamical fields of this theory. They exist only on shell. But they are particularly useful to describe the field of magnetic monopoles from a strictly classical point of view. The $A^{(n)}$'s are the fields we are ultimately interested in, and when we study the quantization problem, the description provided by the $\tilde{A}^{(n)}$'s will not be sufficient.

Here we present two different kinds of static solutions to the equations of motion of the action (2), (3): spherical black-hole solutions and multi-black-hole solutions, both with nontrivial axion, dilaton, and $U(1)$ fields. All the previously known solutions of these kinds [Schwarzschild, (multi-)Reissner-Nordström, the purely electric and magnetic dilaton black holes of Refs. [6], [10], and [11], the electric-magnetic black holes of Refs. [7], and [5], and the axion-dilaton black holes of Refs. [3] and [5]] are particular cases of them.

The static spherically symmetric black-hole solutions are

$$ds^2 = e^{2U} dt^2 - e^{-2U} dr^2 - R^2 d\Omega^2,$$

$$\lambda(r) = \frac{\lambda_0 r + \bar{\lambda}_0 \Upsilon}{r + \Upsilon},$$

$$A_t^{(n)}(r) = e^{\phi_0} R^{-2} [\Gamma^{(n)}(r + \Upsilon) + \text{c.c.}],$$

$$\tilde{A}_t^{(n)}(r) = -e^{\phi_0} R^{-2} [\Gamma^{(n)}(\lambda_0 r + \bar{\lambda}_0 \Upsilon) + \text{c.c.}], \quad (5)$$

where

$$e^{2U}(r) = R^{-2}(r - r_+)(r - r_-), \quad r_\pm = M \pm r_0,$$

$$R^2(r) = r^2 - |\Upsilon|^2,$$

$$r_0^2 = M^2 + |\Upsilon|^2 - 4 \sum_{n=1}^N |\Gamma^{(n)}|^2. \quad (6)$$

The complex electromagnetic charge Γ , axion-dilaton charge Υ , and asymptotic values of the fields are defined in the Appendix. The singularity is hidden under a horizon if $r_0^2 > 0$, and it is hidden or coincides with it (but still is invisible for external observers) if $r_0 = 0$. The conditions $r_0^2 \geq 0$ and $M \geq |\Upsilon|$ can be related to supersymmetry bounds [7, 5, 12]. All solutions given above have the entropy

$$S = \pi(r_+^2 - |\Upsilon|^2). \quad (7)$$

When all supersymmetric bounds are saturated, i.e., $r_+ = M = |\Upsilon|$, the objects described by this solution have zero area of the horizon and vanishing entropy. They may be considered as the ground states of the theory, and we will call them *holons*. A detailed discussion of thermodynamics of all solutions and their physical interpretation will be given in [12].

Our second kind of solutions describes axion-dilaton extreme multi-black-hole solutions. The fields are

$$ds^2 = e^{2U} dt^2 - e^{-2U} d\mathbf{x}^2,$$

$$e^{-2U}(\mathbf{x}) = 2 \text{Im} [\mathcal{H}_1(\mathbf{x}) \bar{\mathcal{H}}_2(\mathbf{x})],$$

$$\lambda(\mathbf{x}) = \frac{\mathcal{H}_1(\mathbf{x})}{\mathcal{H}_2(\mathbf{x})},$$

$$A_t^{(n)}(\mathbf{x}) = e^{2U} [k^{(n)} \mathcal{H}_2(\mathbf{x}) + \text{c.c.}],$$

$$\tilde{A}_t^{(n)}(\mathbf{x}) = -e^{2U} [k^{(n)} \mathcal{H}_1(\mathbf{x}) + \text{c.c.}], \quad (8)$$

where $\mathcal{H}_1(\mathbf{x}), \mathcal{H}_2(\mathbf{x})$ are two complex harmonic functions:

$$\mathcal{H}_1(\mathbf{x}) = \frac{e^{\phi_0}}{\sqrt{2}} \left\{ \lambda_0 + \sum_{i=1}^I \frac{\lambda_0 M_i + \bar{\lambda}_0 \Upsilon_i}{|\mathbf{x} - \mathbf{x}_i|} \right\},$$

$$\mathcal{H}_2(\mathbf{x}) = \frac{e^{\phi_0}}{\sqrt{2}} \left\{ 1 + \sum_{i=1}^I \frac{M_i + \Upsilon_i}{|\mathbf{x} - \mathbf{x}_i|} \right\}. \quad (9)$$

The horizon of the i th (extreme) black hole is at \mathbf{x}_i (in these isotropic coordinates the horizons look like single points) and has mass M_i , electromagnetic charge Γ_i , etc., as can be seen by using the definitions in the Appendix

in the limit $|\mathbf{x} - \mathbf{x}_i| \rightarrow \infty$. Charges without a label are total charges. The constants $k^{(n)}$ are

$$k^{(n)} = -\sqrt{2} \left(\frac{\Gamma^{(n)} M + \overline{\Gamma^{(n)} \Upsilon}}{M^2 - |\Upsilon|^2} \right). \quad (10)$$

The consistency of the solution requires, for every i ,

$$k_i^{(n)} = k^{(n)}, \quad \text{Arg}(\Upsilon_i) = \text{Arg}(\Upsilon). \quad (11)$$

Finally, for each i and also for the total charges, the supersymmetric Bogomolny bound is saturated:

$$M^2 + |\Upsilon|^2 - 4 \sum_{n=1}^N |\Gamma^{(n)}|^2 = 0. \quad (12)$$

For a single $U(1)$ vector field the extreme solution simplifies to ($k \equiv k^1$, $\Gamma \equiv \Gamma^1$)

$$M^2 = |\Upsilon|^2, \quad k = -\frac{\sqrt{2} \Gamma}{M}. \quad (13)$$

As a consequence of all the identities obeyed by the charges, it is possible to derive the following expression of equilibrium of forces between two extreme black holes [12]:

$$M_1 M_2 + \Sigma_1 \Sigma_2 + \Delta_1 \Delta_2 = Q_1 Q_2 + P_1 P_2. \quad (14)$$

$SL(2, R)$ duality acts on the fields of our theory as

$$\begin{aligned} A^{(n)'}(x) &= \delta A(x) - \gamma \tilde{A}(x), \\ \tilde{A}^{(n)'}(x) &= -\beta A(x) + \alpha \tilde{A}(x), \end{aligned} \quad (15)$$

where α , β , γ , and δ are the elements of an $SL(2, R)$ matrix:

$$R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (16)$$

Note that, since the $\tilde{A}^{(n)}$'s are not independent fields, the consistency of Eqs. (15) implies the transformation law of λ :

$$\lambda'(x) = \frac{\alpha \lambda(x) + \beta}{\gamma \lambda(x) + \delta}. \quad (17)$$

With no dilaton nor axion ($\lambda = i$) our theory coincides with Einstein-Maxwell theory. In this case $\tilde{F} = *F$ and the consistency of Eqs. (15) would imply that R is an $SO(2)$ matrix, the duality group being just $U(1)$.

As in the Einstein-Maxwell case, the duality transformations (15) rotate continuously the equations of motion into Bianchi identities. In both cases the equations of motion are invariant under duality, but the actions are not.

Many objects in this theory have well-defined transformation properties under duality; i.e., they transform according to some representation of $SL(2, R)$, usually the vector representation or its dual. We can speak about duality vectors (pairs that transform with R) and duality forms (pairs that transform with R^{-1}), in the language of Ref. [8].

Using the definitions of the charges in the Appendix and Eqs. (15), which can be written as

$$(A^{(n)'}(x), \tilde{A}^{(n)'}(x)) = (A^{(n)}(x), \tilde{A}^{(n)}(x)) R^{-1}, \quad (18)$$

we get the following transformation laws for the “hair” of any field configuration and, in particular, of our solutions:

$$\begin{pmatrix} \tilde{q}' \\ p' \end{pmatrix} = R \begin{pmatrix} \tilde{q} \\ p \end{pmatrix}, \quad (q', p') = (q, p) R^{-1}, \quad (19)$$

$$\lambda'_0 = \frac{\alpha \lambda_0 + \beta}{\gamma \lambda_0 + \delta}, \quad \Upsilon' = e^{-2i \text{Arg}(\gamma \lambda_0 + \delta)} \Upsilon, \quad (20)$$

$$\Gamma' = e^{+i \text{Arg}(\gamma \lambda_0 + \delta)} \Gamma. \quad (21)$$

The pairs $(q^{(n)}, \tilde{p}^{(n)})$ and $(A^{(n)}(x), \tilde{A}^{(n)}(x))$ are duality forms and the pairs $(\tilde{q}^{(n)}, p^{(n)})$ are duality vectors. It should be possible to express any observable property of a system of two dyons in terms of $SL(2, R)$ -invariant bilinears of their charges. The first invariant is the scalar product of two duality vectors, which is equivalent to the action of a duality form on a duality vector:

$$(q_1, \tilde{p}_1) \begin{pmatrix} \tilde{q}_2 \\ p_2 \end{pmatrix} = q_1 \tilde{q}_2 + \tilde{p}_1 p_2 = Q_1 Q_2 + P_1 P_2. \quad (22)$$

This is symmetric and appears in the expression of the “Coulomb” force between two static dyons [see Eq. (14)]. The second invariant is the exterior product of two duality vectors or forms, which corresponds to the symplectic form

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (23)$$

It is obviously invariant under $Sp(2, R) = SL(2, R)$:

$$(\tilde{q}_1, p_1) S \begin{pmatrix} \tilde{q}_2 \\ p_2 \end{pmatrix} = (q_1, \tilde{p}_1) S \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = Q_1 P_2 - Q_2 P_1. \quad (24)$$

This is antisymmetric and will appear in the quantization condition.

Now, using Eqs. (15) in the solutions presented in the previous section, one can check that the effect of an $SL(2, R)$ transformation on them is equivalent to an $SL(2, R)$ transformation of the “hair” according to Eq. (21). The transformation of the fields at any point of the spacetime is equivalent to the transformation of the boundary conditions only.

It is well known that, in general, the introduction of magnetic charges is not compatible with quantum mechanics. In the case of two dyons interacting through their electric and magnetic monopole fields, the quantization of the system will be consistent only if the charges obey the Dirac-Schwinger-Zwanziger condition [13]

$$Q_1 P_2 - Q_2 P_1 = \frac{n}{2}, \quad (25)$$

where n is any integer number. Naively, Eq. (25) can

be obtained by quantization of the angular momentum of the electromagnetic field. This condition ensures the single valuedness of the system's wave function, i.e., that the string singularity in the vector potential cannot be detected in an Aharonov-Bohm-like experiment. On the other hand, Eq. (25) is invariant under the U(1)-duality rotations of Maxwell's theory whose preservation was the main idea behind Dirac's work.

We have described our solutions in terms of the "dual potentials" $\tilde{A}_\mu^{(n)}$, but the fundamental objects in quantum mechanics are the potentials $A_\mu^{(n)}$. Their spacelike components $A_i^{(n)}$ can be expressed in terms of the magnetostatic potentials $\tilde{A}_0^{(n)}$. In the presence of magnetic charges they will exhibit string singularities or will have to be defined in different gauges in different regions of the space [14]. To achieve consistency, conditions such as Eq. (25) will have to be satisfied.

If we want to study a quantum system consisting of two of our dyons moving in each other's very distant field,² only the asymptotic behavior of the potentials matters and the black-hole nature of our dyons becomes irrelevant. But this is just the same of dyons in Minkowski space, and the same string singularities must be present. Then, the consistency condition (calculated with our charge conventions given in the Appendix) takes the standard form of Eq. (25).

In terms of our duality vectors and duality forms the quantization condition is

$$\tilde{q}_1 p_2 - \tilde{q}_2 p_1 = q_1 \tilde{p}_2 - q_2 \tilde{p}_1 = \frac{n}{2}; \quad (26)$$

that is, the exterior product [8] of the charge duality vectors or forms of the two dyons is a half-integer. This statement is invariant under the full $SL(2, R)$.

However, these conditions only restrict the product of the charges and do not tell us anything about what the individual charges might be. To go further we need additional information (basically the spectrum of electric charges allowed by gauge invariance). In Ref. [4] information about the spectrum of electrically charged states was taken from string theory. Here we will study two cases: with and without non-Abelian embedding. In the first case we essentially follow the approach of Ref. [4], embedding our U(1) vector field in SU(2): $A_\mu = B_{3\mu}$. Asymptotically, our theory is equivalent to a spontaneously broken SU(2) theory with a CP-violating term of the form $\sim \theta F^* F$. The relation between g , the coupling constant,³ and θ of this theory and the asymptotic values of the dilaton field ϕ_0 and axion field a_0 are given by

$$e^{-2\phi_0} = \frac{4\pi}{g^2}, \quad a_0 = \frac{\theta}{2\pi}. \quad (27)$$

²For the sake of simplicity we consider a single U(1) field in this section.

³Only in the framework of a non-Abelian theory is this identification possible, since in the Abelian theory the term F^2 does not contain couplings between charged particles.

In this theory there is a natural unit of charge c , which with our charge conventions is

$$c = \frac{g}{\sqrt{4\pi}} = e^{+\phi_0}. \quad (28)$$

When $a_0 = 0$ the electric charge of a dyon is an integer multiple of this elementary charge $Q = nc$. However, Witten showed in Ref. [15] that in presence of the CP-violating θ and magnetic charge P , the value of Q is corrected by a term proportional to θ . In our case the corrected charge is given by

$$Q = nc - \frac{\theta}{2\pi} c^2 P = e^{+\phi_0} (n - a_0 e^{+\phi_0} P), \quad (29)$$

which means that $\tilde{q} = n$. Using Eq. (26) for a purely electric state and a state with magnetic charge, we see that the allowed values for P are $P = m/2c$, that is, $p = m/2$. However, since this theory admits particles with elementary charge $c/2$, the half-integer values of p have to be excluded in case such particles exist (all the magnetic charges have to be multiples of the elementary charge of the 't Hooft-Polyakov monopole). Hence, if we denote each dyon state by the duality vector (\tilde{q}, p) , then the spectrum will be given by

$$Q + iP = e^{+\phi_0} (n - \overline{\lambda_0} m), \quad (\tilde{q}, p) = (n, m), \quad n, m \in Z. \quad (30)$$

This is a one-dimensional version of the spectrum of Ref. [4]. It is clear that the form of the spectrum in Eq. (30) is not respected by arbitrary $SL(2, R)$ transformations, but only when $\alpha, \beta, \gamma, \delta$ are all integers. Then, the spectrum (30) is invariant under $SL(2, Z)$ (the transformations act on the states as permutations). In particular, when $a_0 = 0$, the spectrum is invariant under the Z_2 subgroup $Q \rightarrow P, P \rightarrow -Q$. This is possible because $c = e^{\phi_0}$ does change according to $c \rightarrow c^{-1}$ under the former transformation. This invariance is not present in the Maxwell-Dirac case $Q = ne, P = me^{-1}$ because e does not change under the U(1)-duality group.

How do our results depend on the embedding of U(1) in a non-Abelian⁴ group? The most important difference would be the absence of a natural unit of charge. Once a unit of charge c is defined, because of U(1) gauge symmetry, the electric charge is quantized in integer multiples of it. In general, the spectrum will break completely the duality invariance as in the Einstein-Maxwell case, since the duality group does not act on c . A more interesting starting point would be to assume that the elementary charge is a factor ξ times $e^{+\phi_0}$, $c = \xi e^{+\phi_0}$, and so the electric charge is again given by Eq. (29). Using the most general magnetic charges allowed by Eq. (25), we are led to the spectrum

⁴This case is particularly important for supersymmetric black holes [7, 12]. It was also stressed by Maharana and Schwarz [2] that it is not known how to maintain the $SL(2, R)$ symmetry when non-Abelian vector fields and/or charged fields are introduced.

$$(\tilde{q}, p) = \left(n\xi, \frac{m}{2\xi} \right), \quad n, m \in \mathbb{Z}. \quad (31)$$

This spectrum includes states $\tilde{q}p = \frac{1}{2}(\text{mod } \mathbb{Z})$ that behave as fermions [16], while (30) does not. Again, for general ξ , the spectrum (31) breaks $\text{SL}(2, R)$ completely because, in principle, $\text{SL}(2, R)$ does not act on ξ . Naively, the introduction of ξ seems to be completely equivalent to a rescaling of $e^{+\phi_0}$ such that $e^{+\phi'_0} = \xi e^{+\phi_0}$. However, there is no reason why $\xi e^{+\phi_0}$ should transform like e^{ϕ_0} . We can consider both possibilities [ξ invariant and ξ transforming under $\text{SL}(2, R)$ in the right way] by using explicitly an $\text{SL}(2, R)$ -invariant ξ . The case in which it is legitimate to absorb ξ in a rescaling of $e^{+\phi_0}$, that is, the case in which the elementary charge c transforms as $e^{+\phi_0}$, will be just the case $\xi = 1$ in what follows.

If $2\xi^2 = l \in \mathbb{Z}$, it is not hard to check that (31) is still invariant under Γ_l , the subgroup of matrices of $\text{SL}(2, \mathbb{Z})$ with $\beta = 0 \pmod{l}$. The case $l = 1$ is special, since $\Gamma_1 = \text{SL}(2, \mathbb{Z})$. This means that for $\xi = 1/\sqrt{2}$ the spectrum (31) has the maximum amount of symmetry.

When $l \neq 1$, the spectrum has less symmetry than $\Gamma_1 = \text{SL}(2, \mathbb{Z})$. Nevertheless, if we restrict m to be a multiple of l , $m = rl$, projecting out of the spectrum the other states, we recover full $\text{SL}(2, \mathbb{Z})$ invariance and the spectrum will have the form

$$(\tilde{q}, p) = \sqrt{\frac{l}{2}} (n, r), \quad n, r, l \in \mathbb{Z}. \quad (32)$$

The spectrum will contain fermionic states if l is odd. The first spectrum (30) which results from the embedding in $\text{SU}(2)$ is the case $l = 2$ of (32). On the other hand, (32) seems to be the most general form of a spectrum invariant under the full $\text{SL}(2, \mathbb{Z})$. The lesson is that no part of the original $\text{SL}(2, R)$ -duality symmetry will be respected by the spectrum after quantization if the elementary electric and magnetic charges (which are essentially the inverse of each other) do not have the right transformation properties under duality. However, if they do have special transformation properties under duality, the largest subgroup of the classical symmetry, which is compatible with quantization, is $\text{SL}(2, \mathbb{Z})$.

The solutions presented in this paper provide explicit realizations of the states in all these spectra.

A useful way of thinking of the transformation rules of the hair of our black holes after quantization is the following. It is always possible to interpret λ_0 as the modular parameter of some complex torus, transforming under the modular group as described in Eq. (21). But it is also possible to consider $-1/\lambda_0$ as the modular parameter τ of another (conformally equivalent) torus. This means that we have chosen a canonical basis of homology cycles A, B on the torus and we have normalized the unique Abelian differential $\omega(z)dz$ by

$$\oint_A \omega(z)dz = 1, \quad (33)$$

and when we integrate over the B cycle we get

$$\oint_B \omega(z)dz = \tau = -1/\lambda_0. \quad (34)$$

Under the $\text{SL}(2, \mathbb{Z})$ transformation (21), τ transforms as

$$\tau' = \frac{\delta\tau - \beta}{\gamma\tau + \alpha}, \quad (35)$$

which corresponds to the following transformation of the homology basis:

$$\begin{pmatrix} B' \\ A' \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}. \quad (36)$$

Now, if we have any homology cycle C on the surface of the torus, it can be expressed in terms of the basis A, B as $C = mA + nB$, where the equal sign means ‘‘homologically equivalent.’’ The integers n and m count the number of times the cycle C wraps around the cycles B and A , respectively, and are called *winding numbers*. In terms of the transformed basis A', B' , we will have $C = m'A' + n'B'$, and the new winding numbers will be related to the old ones by

$$\begin{pmatrix} n' \\ m' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}. \quad (37)$$

For any $\text{SL}(2, \mathbb{Z})$ -invariant spectrum of the form (32) this is exactly the way our integer quantum numbers (n, m) transform. Then, in this framework, we can state our result in this way: To any static spherically symmetric dilaton-axion black hole with the hair $M, Q(n, m), P(m), \lambda_0$, we can associate a complex torus of modular parameter $\tau = -1/\lambda_0$ and a homology cycle C of winding numbers (n, m) on it.

We hope to present a detailed discussion of different aspects of quantized axion-dilaton black holes in a subsequent publication [12].

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APPENDIX: DEFINITIONS OF THE CHARGES

We define the complex charges in terms of the asymptotic behavior ($r \rightarrow \infty$) of the different complex fields

$$\begin{aligned} g_{tt} &\sim 1 - \frac{2M}{r}, \quad \lambda \sim \lambda_0 - ie^{-2\phi_0} \frac{2\Upsilon}{r}, \\ F_{tr} &\sim \frac{e^{+\phi_0} Q}{r^2} = \frac{q}{r^2}, \quad *F_{tr} \sim \frac{ie^{+\phi_0} P}{r^2} = \frac{ip}{r^2}. \end{aligned} \quad (A1)$$

The relation between these charges and the canonically normalized $\text{SU}(2)$ charges of Ref. [15] is

$$Q = \frac{Q^{\text{can}}}{\sqrt{4\pi}}, \quad P = \frac{P^{\text{can}}}{\sqrt{4\pi}}. \quad (A2)$$

The real axion (Δ), dilaton (Σ), electric (Q), and magnetic (P) charges and the asymptotic values of the axion (a_0) and dilaton (ϕ_0) are

$$\Upsilon = \Sigma - i\Delta, \quad \Gamma = \frac{1}{2}(Q + iP), \quad \lambda_0 = a_0 + ie^{-2\phi_0}. \quad (A3)$$

$$q = e^{+\phi_0}Q, \quad \tilde{q} = (e^{-\phi_0}Q + a_0e^{+\phi_0}P), \quad (A5)$$

$$\tilde{p} = (e^{-\phi_0}P - a_0e^{+\phi_0}Q), \quad p = e^{+\phi_0}P.$$

The definition of the charges associated with \tilde{F} are

$$\tilde{F}_{\text{tr}} \sim \frac{i\tilde{p}}{r^2}, \quad {}^*\tilde{F}_{\text{tr}} \sim \frac{\tilde{q}}{r^2}, \quad (A4)$$

and so we have, for the components of the duality vectors and forms,

In every single black hole in our solutions the charge of the complex scalar is related to the N gauge charges by

$$\Upsilon = -\frac{2}{M} \sum_{n=1}^{n=N} (\bar{\Gamma}_n)^2. \quad (A6)$$

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