New singularity in anisotropic, time-dependent, maximally Gauss-Bonnet extended gravity

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Among the solutions for anisotropic, time-dependent, maximally Gauss-Bonnet extended gravity, we find a class of curvature singularities for which the metric components remain finite. These new singularities therefore differ in type from the standard Kasner-like divergences expected for this class of theories. We study perturbative solutions near the singularity and show that there exist solutions with timelike paths that reach the singularity in finite proper time. Solving the equation of geodesic deviation in the same approximation, we show that the comoving coordinate system does not break down at the singularity. A brief classification of the corresponding singularity types in Robertson-Walker cosmologies is also provided.

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I. INTRODUCTION

Many unification theories, including superstring, supergravity, and Kaluza-Klein theories, suggest spacetimes of higher dimensions than four. The maximally Gauss-Bonnet extended Einstein, or Lovelock, theory is a viable generalization of general relativity to such higher-dimensional spacetimes. Specifically, it gives the most general theory of gravity with field equations which contain no more than second derivatives of the metric [1].

The Gauss-Bonnet extended gravity Lagrangian has the form

$$L=\sum_{k=0}^{k_{\max}}\tilde{c}_k L_k\ , \tag{1}$$

in d dimensions, where $k_{\max}$ is the integer part of $(d-1)/2$ and

$$L_k=\mathbf{R}^{ab}\mathbf{R}^{cd}\cdots\mathbf{R}^{ef}\mathbf{e}^{g}\cdots\mathbf{e}^{h}\varepsilon_{abcd\ldots efg\ldots h}\ . \tag{2}$$

There are k factors of the curvature two-form $\mathbf{R}^{ab}$, and $(d-2k)$ factors of the vielbein one-form $\mathbf{e}^a$. The constants $\tilde{c}_k$ are arbitrary and $\varepsilon_{abcd\ldots efg\ldots h}$ is the d-dimensional Levi-Civita tensor. Differential forms are represented by boldface symbols and are always assumed to be multiplied using the wedge product. For d even, the integral of $L_{d/2}$ is the Euler character. When $0\le k<d/2$, L_k is called a dimensionally extended Euler characteristic density.

Besides providing the most general second-order theory of gravity and the most general gravitational action constructible from $\mathbf{R}^{ab}$, $\mathbf{e}^a$, and invariant tensors, several properties recommend the use of L as the gravitational Lagrangian. The first-order density L_1 is the usual Einstein Lagrangian, while the lowest-order density L_0 provides a cosmological constant. L_2 arises as the order α' correction in the low-energy expansion of string models [2–4]. Any theory using L therefore reduces to Einstein or low-energy string gravity when the curvature is small. Furthermore, given the quadratic and higher-order terms in the curvature tensor, the extra dimensions can spontaneously compactify [6]. Finally, L_k is free of ghosts [1,5] when the theory is quantized.

Several solutions to this class of theories have been studied previously. Static, spherically symmetric solutions have been found with [7–11] or without [12–14] the presence of other fields coupled to gravity. Cosmological solutions have been studied extensively by various authors [6,11,15–28]. See Deruelle and Fariña-Busto [30] for a recent review of cosmological studies in extended gravity.

The generalization of the Kasner geometry was studied up to $k=2$ by Lee and Lee [25], and Deruelle [31] perturbatively analyzed the Kasner-like singularities to all orders. In [32], we achieved the complete exact integration of the Kasner-symmetry problem, and distinguished two types of singularities which occur. The type-I subclass of singularities may be characterized by the singularity of the metric (as well as the curvature). These divergent solutions had been found perturbatively by Deruelle [31], and are of the sort found in the usual Kasner solution to general relativity. Type-II solutions, by contrast, retain finite metrics at the curvature singularity. While the solution of Ref. [32] clearly displays the possibility of such solutions, their existence and character was not fully established. In the present work, we explicitly study these new type-II singularities demonstrating conclusively their existence and verifying that they can be reached in a finite proper time.

Section II begins with a review of the Bianchi type-I extended gravity solution and the definition of the two types of singularity, continues with a proof by example of the existence of type-II singularities and concludes with a discussion of general properties of such solutions. Then, in Sec. III, the field equations are linearized and perturbative solutions both emerging from and evolving toward the type-II singularities are obtained. Based on these linearized solutions we show, in Sec. IV, that the metric components stay finite and that the universe undergoes an inflationary state near a singularity. We also show, by solving the equation of geodesic deviation perturbatively,

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that, generically, the comoving coordinate system does not break down at a singularity. Section V gives a brief discussion of type-II singularities occurring in other extended gravity solutions, including a classification of the singularities of the $k_{\max}=2$ spatially flat Robertson-Walker solutions. Section VI includes a summary of the results.

II. FIELD EQUATIONS AND SINGULARITY BEHAVIOR

Kitaura and Wheeler [32] give the details of how the solution is obtained. The following is a brief summary of the results.

We use the Palatini variation. The variation of L with respect to the connection vanishes identically when we assume the vanishing of the torsion two-form, $\mathbf{T}^a=\boldsymbol{D}\mathbf{e}^a=0$, and use the Bianchi identity $\boldsymbol{D}\mathbf{R}^{ab}=0$. The vielbein variation gives the field equations

$$\sum_{k=0}^{k_{\max}}(d-2k)\tilde{c}_k\mathbf{R}^{ab}\ldots\mathbf{R}^{cd}\mathbf{e}^g\ldots\mathbf{e}^h\varepsilon_{ab\ldots cdfg\ldots h}=0 \ . \quad (3)$$

We evaluate these equations of motion for a time-dependent, homogeneous, anisotropic, diagonal metric

$$g_{ab}=\begin{bmatrix}-1 & & & \\ & A_1^2 & & \\ & & \ddots & \\ & & & A_{d-1}^2\end{bmatrix}, \quad (4)$$

where A_i $(i=1,2,\ldots,d-1)$ are functions of time, t.

Next, we define new variables α_i by

$$\alpha_i\equiv\frac{d}{dt}\ln(A_i) \ . \quad (5)$$

The equations are then simplest when written in terms of the fundamental symmetric symbols C^k and the associated symbols $C^k_{m;\ldots n}$. C^k is the sum of all possible products of k different α's,

$$C^k=\sum_\sigma\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k} \ , \quad (6)$$

with the sum over $\sigma=\{(i_1,i_2,\ldots,i_k)|0<i_1<i_2\ldots<i_k\leq k_{\max}\}$, while subscripts, as in $C^k_{m;\ldots n}$, indicate the absence of the indices $m\ldots n$ from the set σ.

Now the field equations reduce to

$$\sum_{k=0}^{k_{\max}}c_kC^{2k}=0 \ , \quad (7a)$$

$$\mathbf{N}\frac{d\alpha}{dt}=\beta \ , \quad (7b)$$

where $c_k\equiv(2k)!(d-2k)!\tilde{c}_k$, $\mathbf{N}$ is the symmetric matrix

$$N_{ij}\equiv\sum_{k=1}^{k_{\max}}\frac{c_k}{2k-1}C^{2k-2}_{i;j}=N_{ji} \ , \quad i\neq j \ , \quad (8)$$

satisfying $N_{ii}=0$ for each i. α is the vector with components α_i, and the components of β are given by

$$\beta_j=-c_0-\sum_{k=1}^{k_{\max}}\frac{c_k}{2k-1}\left[\sum_{i\neq j}\alpha_i^2C^{2k-2}_{i;j}+(2k-1)C^{2k}_j\right] \ . \quad (9)$$

When $\mathbf{N}^{-1}$ exists, Eq. (7b) can be solved by integrating $d\alpha/dt=\mathbf{N}^{-1}\beta$, or, in components,

$$\frac{d\alpha_j}{dt}=\frac{1}{\det\mathbf{N}}\sum_iD_{ji}\beta_i \ . \quad (10)$$

Here $\mathbf{D}$ is the transpose of the cofactor matrix of $\mathbf{N}$. Note that since $\mathbf{N}$ is symmetric, $\mathbf{D}$ is also symmetric.

Equation (10) clearly shows the distinction between type-I and type-II singularities. If the solution of this equation for α_i gives a diverging function for some value of i, there results a type-I singularity and both the corresponding metric component, A_i, and various curvature invariants will diverge. But, even if all of the components, α_i, remain finite it is possible to have $\det\mathbf{N}=0$ and $\Sigma D_{ij}\beta_j$ nonzero, so that $d\alpha/dt$ diverges. This leads to the divergence of the curvature invariant

$$R^{abcd}R_{abcd}=4\sum_i\left[\frac{d\alpha_i}{dt}+\alpha_i^2\right]^2+4\sum_{ij}\alpha_i^2\alpha_j^2 \ , \quad (11)$$

yielding a type of singularity distinct from the usual Kasner-like singularities and not present in general relativity. These are the type-II cases.

To prove conclusively that this type of singularity actually occurs, we consider a specific example. In a five-dimensional spacetime, if $c_0=c_1=1$, and $c_2=\frac{3}{2}$, then $\alpha_1=-1$, $\alpha_2=\sqrt{7}/2$, $\alpha_3=1$, $\alpha_4=0$ satisfy both Eq. (7a) and $\det\mathbf{N}=0$. The expression $\sum_jD_{ij}\beta_j$ does not vanish. Therefore, the time derivatives of α's diverge at this point. A similar point can be found for $c_0=0$. A numerical study of the five-dimensional case shows that these same conclusions hold for a large range of parameter values. We further find that the gradient of the surface equation, $\det\mathbf{N}=0$, with respect to α_i does not vanish, so type-II singularities do not always lie on a kink of this surface.

Some observations about the new class of singularity are in order. We note that when $k_{\max}=1$, that is, in the usual Einstein case, $\mathbf{N}_{ij}=c_1$ with $\mathbf{N}_{ii}=0$, so $\det\mathbf{N}$ never vanishes and this type of singularity never arises.

To generalize our study of the type-II singularities we let Ω denote the $(d-2)$-dimensional surfaces in α-space defined by Eq. (7a), and let the $(d-2)$-dimensional surfaces where $\det\mathbf{N}=0$ be called S_d. Then we have a solution at all points on the intersection, $\Omega\cap CS_d$, where C denotes the complement.

Now, since both S_d and Ω are $(d-2)$-dimensional hypersurfaces in the $(d-1)$-dimensional α-space, the intersection $I=\Omega\cap S_d$ is, in general, a $(d-3)$-dimensional surface, unless the surfaces intersect tangentially. Since I is where the determinant vanishes, I divides Ω into two separate regions, characterized by the sign of $\det\mathbf{N}$.

To establish the degree of importance of this type of breakdown, it is necessary to show that generic type-II solutions move from the allowed solution space inside Ω

to I. Passing through each point on Ω there exists exactly one solution curve. Given such a solution curve, $\alpha_i(t)$, the $\det\mathbf{N}=0$ condition becomes a polynomial equation in t. Suppose this equation has n real roots: $t_1<t_2<\ldots t_n$. Then the spacetime can exist only between two consecutive roots. Since, as shown later in this chapter, the metric components do not diverge in general at a type-II singularity, we have a picture of a $(d-1)$-dimensional hypersurface evolving for, say, $t>t_2$, until $t=t_3$, at which time the hypersurface ceases to exist. The universe starting at $t=t_n$ can exist forever unless it encounters other types of singularities such as a Kasner singularity. Another interesting possibility is that a Kasner singularity can be placed at $t=0$ and the constants c_k can be chosen so that one of the roots is positive. Then we have a universe that emerges from a Kasner singularity and encounters a type-II singularity later on.

Finally, we observe that $\det\mathbf{N}=0$ may be regarded as a $(d-1)$-degree polynomial in the constants c_k which distinguish different gravity theories. It is then easy to see that there is a large class of theories admitting type-II solutions. In even dimensions we can always fix the value of one of the c_k in terms of the others in order to satisfy the vanishing determinant condition. This leaves a $k_{\max}$-dimensional class of gravity theories with type-II solutions. In odd dimensions, it may be necessary to fix the values of more than one of the c_k. In either case, a nonisolated class of Gauss-Bonnet extended theories contain the new class of singularity.

III. PERTURBATIVE SOLUTION NEAR A SINGULARITY

We now investigate the perturbative behavior of a solution near a type-II singular point. Let α_0 satisfy $\det\mathbf{N}(\alpha_0)=0$ and Eq. (7a) and expand

$$\alpha_i(t)=\alpha_{0i}+\varepsilon_i(t)\ , \tag{12}$$

where α_{0i} is the $i$the component of α_0. To linear order in ε_i, Eq. (7b) reduces to

$$\left[\sum_n \varepsilon_n E_n\right]\frac{d}{dt}\varepsilon_i=\sum_m (F_i)_m\varepsilon_m+G_i\ , \tag{13}$$

where

$$E_m\equiv\frac{\partial}{\partial\alpha_m}[\det\mathbf{N}]\Bigg|_{\alpha_0}\ , \tag{14a}$$

$$G_i\equiv\left[\sum_j D_{ji}\beta_j\right]\Bigg|_{\alpha_0}\ , \tag{14b}$$

$$(F_i)_m\equiv\frac{\partial}{\partial\alpha_m}\left[\sum_j D_{ji}\beta_j\right]\Bigg|_{\alpha_0}\ . \tag{14c}$$

We notice that E_m are just the components of the gradient of S_d. Solving to lowest nonvanishing order gives

$$\varepsilon_i(t)=K_i\sqrt{st}\ , \tag{15}$$

where K_i is a constant determined in terms of the variables of Eqs. (14) by

$$K_i=\pm\frac{G_i}{\left[(s/2)\sum_j E_j G_j\right]^{1/2}}\ . \tag{16}$$

$s=\pm1$ is chosen so that the quantity under the square-root sign in Eq. (16) is always positive, and we have chosen the initial condition $\varepsilon_i(t=0)=0$.

These are the solutions we require. Since ε_i is parallel to G_i, and E_i, being the gradient of S_d, is perpendicular to I, we conclude that whenever $G\cdot E\neq 0$ the solution curve moves away from I. Since G_i and E_i are independent, this is the generic case. Whenever the solution of Eq. (15) exists, it represents a universe emerging from or moving toward a type-II singularity.

Finally, we consider the consistency of the signs to be chosen in Eqs. (15) and (16). Together with Eq. (13), Eqs. (15) and (16) imply

$$\sum E_i G_i=\left[\sum E_n\varepsilon_n\right]\left[\sum E_m\frac{d\varepsilon_m}{dt}\right]\ . \tag{17}$$

Then, differentiating Eq. (15), we see that ε and $d\varepsilon/dt$ are either parallel or antiparallel depending on the sign of s. When they are parallel, $\sum E_n\varepsilon_n$ and $\sum E_i d\varepsilon_i/dt$ have the same sign and Eq. (17) shows that $\sum E_i G_i$ is positive. This happens when the solution is moving away from α_0. In this case, $s=+1$. Similarly, when ε and $d\varepsilon/dt$ are antiparallel, $\sum E_i G_i<0$, and we find $s=-1$. This gives a solution that approaches α_0 as t evolves from a small negative value to zero, which is consistent with the fact that ε and $d\varepsilon/dt$ point in opposite directions. Finally, the choice of overall sign in the expression, Eq. (16), for K_i determines whether the system is moving in the region of Ω with $\det\mathbf{N}$ positive or negative. To summarize, we have the following results, to lowest order in t, near a type-II singularity:

$$\alpha_i(t)=\alpha_{i0}\pm\frac{2G_i}{(2\sum E_j G_j)^{1/2}}\sqrt{t}\ \begin{cases}\text{motion away from }\alpha_0\ ,\\ (+)\text{ if }\det\mathbf{N}>0\ ,\\ (-)\text{ if }\det\mathbf{N}<0\ ,\end{cases} \tag{18a}$$

$$\alpha_i(t)=\alpha_{i0}\pm\frac{2G_i}{(-2\sum E_j G_j)^{1/2}}\sqrt{-t}\ \begin{cases}\text{motion toward }\alpha_0\ ,\\ (-)\text{ if }\det\mathbf{N}>0\ ,\\ (+)\text{ if }\det\mathbf{N}<0\ .\end{cases} \tag{18b}$$

IV. METRIC COMPONENTS NEAR A SINGULARITY AND GEODESIC DEVIATION

In this section, we look at the metric components near a singular point, and show that the Universe undergoes an inflationary state. We then investigate the behavior of two nearby geodesics near a singularity by solving the equation of geodesic deviation.

Consider a solution moving away from the singular point α_0 into the $\det\mathbf{N}>0$ region, where the solution can be written as

$$\alpha_i(t)=\alpha_{0i}+K_i\sqrt{t}\ . \tag{19}$$

The metric components become

$$A_i(t)=\exp\left[\int\alpha_i(t)dt\right]=\exp(\alpha_{0i}t+\tfrac{2}{3}K_i t^{3/2}) \tag{20}$$

and we note once again that these values remain finite as t approaches zero. When $\alpha_{0i}t+\frac{2}{3}K_i t^{3/2}>0$, the Universe undergoes an inflationary expansion whereas, if $\alpha_{0i}t+\frac{2}{3}K_i t^{3/2}<0$, it undergoes a contraction. The exception to this is when it so happens that

$$\alpha_{0i}t_1+\tfrac{2}{3}K_i t_1^{3/2}=0\ , \tag{21}$$

for some t_1 in the range of our approximation,

$$\sqrt{t_1}=-\frac{3\alpha_{i0}K^i}{2K_jK^j}\ . \tag{22}$$

If $\alpha_{i0}K^i$ is sufficiently small compared with K_iK^i and if these products have opposite signs, there is a value t_1, where the Universe reverses its direction of expansion.

We conclude with a proof that distinct comoving geodesics remain distinct as the singularity is approached. In terms of the α's, the only nonvanishing components of the curvature tensor are

$$R^{i0}{}_{i0}=\frac{d\alpha_i}{dt}+\alpha_i^2\quad(\text{no sum})\ , \tag{23a}$$

$$R^{ij}{}_{ij}=\alpha_i\alpha_j\quad(\text{no sum})\ . \tag{23b}$$

Then in comoving coordinates the equations of geodesic deviation reduce to

$$\frac{d^2\xi^0}{dt^2}=0\ , \tag{24a}$$

$$\frac{d^2\xi^i}{dt^2}+\left[\frac{d\alpha_i}{dt}+\alpha_i^2\right]\xi^i=0\ ,\quad 1\le i\le d-1\ , \tag{24b}$$

where we use the fact that a geodesic in comoving coordinates is given by $x^i=$ const so that $dx^i/d\tau=0$ and $\tau=t$. Clearly, the singularity is reached in a finite proper time. In Eqs. (24), ξ^0 represents a temporal distance between two geodesics whereas ξ^i measures the spatial distance.

Let

$$\alpha_i=\alpha_{0i}+K_i\sqrt{st} \tag{25}$$

as before and choose $\xi^0=0=$ const. Then Eq. (24a) is satisfied and Eq. (24b) reduces to

$$\frac{d^2\xi^i}{dt^2}-\left[\alpha_{i0}^2+2\alpha_{i0}K_i\sqrt{st}+\frac{sK_i}{2\sqrt{st}}\right]\xi^i=0\quad(\text{no sum}) \tag{26}$$

neglecting the second-order term in $\sqrt{t}$. Solving this by a power expansion, we get

$$\xi^i=a_{i0}+a_{i1}st+\tfrac{2}{3}sa_{i0}K_i(st)^{3/2}+\dots\ , \tag{27}$$

where a_{i0} and a_{i1} are arbitrary constants.

Clearly, ξ^i and its first derivative are well behaved and need not vanish as t approaches zero, which means that at a singular point the comoving coordinate system does not break down. Only the acceleration diverges at the singularity. We also observe that since $\xi^i=a_{i0}$ and $d\xi^i/dt=sa_{i1}$ at $t=0$, if we choose a_{i1} to be positive, then ξ^i tends to increase moving away ($s>0$) from the singularity, while ξ^i tends to decrease if a geodesic is approaching ($s<0$) the singularity. If $a_{i1}=0$ for all i, then the geodesics are parallel (up to the next order term) at the singularity.

There are some situations where the comoving coordinates can break down. When the third term can be neglected in Eq. (26), $\xi^i=0$ has a real root if a_{i0} and a_{i1} have opposite signs and if the value a_{i0}/a_{i1} is the same for all i, so the coordinates become degenerate. But a_{i0}/a_{i1} is the same for all i only on a set of measure zero.

V. TYPE-II COSMOLOGICAL SINGULARITIES IN EXTENDED GRAVITY

While our principal intent is the study of Kasner solutions, it should be noted that type-II singularities occur in other cosmological solutions to extended gravity. In this section we examine two such solutions: the geometrically free solutions [7] and extended Robertson-Walker spacetimes.

Geometrically free solutions, first described by Wheeler [7] and further characterized by Müller-Hoissen [29], leave parts of the metric arbitrarily specifiable. Such solutions clearly allow arbitrary pathologies to be introduced. If one or more of the free metric components is taken to vary as $\exp[(t-t_0)^\alpha]$, with $0<\alpha<2$, then the curvature will, in general, develop singular parts at time t_0 even though the metric component will still be well-defined. However, since there is no dynamical principle determining this case of type-II behavior, it is of minor interest.

Of considerably more interest is the specific occurrence of type-II singularities in extended Robertson-Walker spacetimes [11,30]. As an example, we now give the full classification of the spatially flat Robertson-Walker solutions to $k_{\max}=2$ extended gravity, including the singularity type.

We follow the notation of [30]. Starting from the line element

$$ds^2=-dt^2+a^2(t)d\sigma^2 \tag{28}$$

with $d\sigma^2$ describing a flat $(d-1)$-dimensional space, we let

$$x\equiv\frac{\lambda\dot{a}}{a}\ , \tag{29}$$

where $\dot{a}$ is the time derivative of a and λ is a positive constant. One of the two field equations is solved by replacing the suitably normalized matter density by a power of the scale factor. In terms of x, the remaining field equation may be written as

$$\alpha\lambda\dot{x}(\beta_1+2\beta_2x^2)+\beta_0+\beta_1x^2+\beta_2x^4=0 , \tag{30}$$

where β_0, β_1, and β_2 are normalized versions of the coupling constants in the original Lagrangian [Eq. (1)] and $\alpha>0$ characterizes the equation of state. Our results follow from Eq. (30). For convenience we also define

$$\begin{aligned}\gamma_0&=\frac{\beta_0}{\beta_2} ,\\ \gamma_1&=\frac{\beta_1}{\beta_2} ,\\ \Delta^2&=\left[\frac{\gamma_1}{2}\right]^2-\gamma_0 ,\\ \delta&=\left[-\frac{\gamma_1}{2}\right]^{1/2} .\end{aligned} \tag{31}$$

We assume throughout that none of these variables vanish.

We will require three theorems. For the first two, we note that the curvature components depend only on x and $\dot{x}$, and, in particular, that the scalar curvature is a linear combination of x^2 and $\dot{x}$. Examining possible divergences of x and $\dot{x}$ proves the results. A study of the remaining critical points of Eq. (30) yields the third theorem.

Theorem 1. Type-I curvature singularities occur if and only if $x^2\to\infty$. In a neighborhood of the singular points the matter density ρ diverges and

$$a=a_0(t-t_0)^{2\alpha} ,$$

$$x=\frac{2\alpha\lambda}{(t-t_0)} .$$

Theorem 2. Type-II curvature singularities occur if and only if x remains finite while $\dot{x}^2\to\infty$. In a neighborhood of the singular points

$$\rho\to-\beta_2\Delta^2 , \quad \dot{\rho}\to 0 ,$$

$$a=a_0\left\{1\pm\frac{\delta}{\lambda}\left[(t-t_0)\pm\frac{3}{4\alpha\lambda}\left[\frac{-\Delta^2}{\gamma_1}(t-t_0)^3\right]^{1/2}\right]\right\} ,$$

$$x\to\pm\delta .$$

Notice that, at the type-II singularity, the matter density and scale factor remain regular. The vanishing rate of density change $\dot{\rho}\to 0$ provides a necessary and sufficient condition for the existence of this type-II divergence; $x\to\pm\delta$ also provides a necessary and sufficient condition. In fact, if we expand Eq. (30) in a neighborhood of $x=\pm\delta$, it reduces to

$$4\alpha\lambda\delta\varepsilon\dot{\varepsilon}=\Delta^2 ,$$

which is of a form similar to Eq. (13) for the extended Kasner solution. Therefore, the singularities are not only both type II, but actually have the same $(t-t_0)^{-1/2}$ rate of divergence [see Eq. (15)]. The extended Kasner case, of course, is complicated by the anisotropy.

Finally, we have the following.

Theorem 3. The following conditions are equivalent:

(1) $x^2\to\alpha_\pm$, where $\alpha_\pm$ are the roots for x^2 of $\gamma_0+\gamma_1x^2+x^4=0$.

(2) $|t|\to\infty$.

(3) $\rho\to 0$.

Any one of the conditions (1)–(3) implies a scale factor dependence of $a\sim\exp[\pm\sqrt{\alpha_\pm}(t-t_0)/\lambda]$ and finite curvature as $\pm\sqrt{\alpha_\pm}(t-t_0)$ tends to infinity.

Based on these theorems, and choosing β_2 so that ρ is positive, we find that the qualitative history of any solution may be determined once we know (1) the range of x

TABLE I. Classification of spatially flat, $k_{max}=2$, extended Robertson-Walker cosmologies.

Region	Range of x	γ_0	γ_1	Δ^2	$\dot{x}$	Description
I	$-\infty\le x\le\infty$	+	+	±	−	Expands from type I to maximum radius then recontracts to type I singularity
IIa	$-\sqrt{\alpha_+}\le x\le\sqrt{\alpha_+}$	−	+	+	−	Contracts from infinite radius at $t=-\infty$, bounces and expands indefinitely
IIb	$\sqrt{\alpha_+}\le x\le\infty$	−	+	+	−	Expands from type I indefinitely to $x=\sqrt{\alpha_+}$
IIIa	$-\delta\le x\le\delta$	−	−	+	−	Expands from type II then recontracts to another type II
IIIb	$\delta\le x\le\sqrt{\alpha_+}$	−	−	+	+	Expands from type II indefinitely to $x=\sqrt{\alpha_+}$
IIIc	$\sqrt{\alpha_+}\le x\le\infty$	−	−	+	−	Expands from type I indefinitely to $x=\sqrt{\alpha_+}$
IVa	$-\sqrt{\alpha_-}\le x\le\sqrt{\alpha_-}$	+	−	+	+	Nonsingular with bounce
IVb	$\sqrt{\alpha_-}\le x\le\delta$	+	−	+	−	Expands from type II indefinitely to $x=\sqrt{\alpha_-}$
IVc	$\delta\le x\le\sqrt{\alpha_+}$	+	−	+	+	Expands from type II indefinitely to $x=\sqrt{\alpha_+}$
IVd	$\sqrt{\alpha_+}\le x\le\infty$	+	−	+	−	Expands from type I indefinitely to $x=\sqrt{\alpha_+}$
Va	$-\delta\le x\le\delta$	+	−	−	+	Contracts from type II, bounces and expands to type II
Vb	$\delta\le x\le\infty$	+	+	−	−	Contracts from type I to type II

and (2) the signs of γ_1, γ_2, Δ^2, and $\dot{x}$. We tabulate the possibilities in Table I. Regions I–V correspond to the regions depicted in Fig. 1 of [30]. Deruelle [33] has pointed out an error in Ref. [30] occurring in the discussion of region-V solutions, and our results confirm this observation. The initial and final states of all type-V cosmologies are curvature singularities. We also note from Eqs. (29) and (30) that the time reverse of any solution in Table I is also a solution.

VI. SUMMARY

In an arbitrary number of dimensions greater than four, we demonstrate the existence of a new class of curvature singularity in anisotropic, time-dependent solutions to maximally Gauss-Bonnet extended gravity. It is shown that, in addition to the usual Kasner-like (type-I) singularities, there are solutions for which certain curvature invariants diverge while the metric components remain finite (type II). The singularity behavior of these type-II solutions therefore differs from the classical Kasner solution. We find both solutions emerging from and solutions evolving toward a type-II singularity and show that the Universe undergoes an inflationary expansion or a deflationary contraction near such singularities. From the equation of geodesic deviation it is shown generically that the comoving coordinate system does not break down at a singularity. A brief classification of the corresponding singularity types in Robertson-Walker cosmologies is also provided.

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