# Interacting Einstein-conformal scalar waves

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A large class of solutions of the Einstein-conformal scalar equations is identified. They describe the interacting asymptotic conformal scalar waves and are generated from Einstein minimally coupled scalar spacetimes via the Bekenstein transformation. Particular emphasis is given to the study of the global properties and the singularity structure of the obtained solutions. It is shown that, in the case of the absence of pure gravitational radiation in the initial data, the formation of the final singularity is not only generic, but is even inevitable.

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## I. INTRODUCTION

Colliding gravitational waves have attracted a lot of interest in the last two decades [1–10]. Apart from the character of the nonlinearities of the gravitational interaction, the interest was probably caused by characteristic curvature singularities occurring as the result of the collision of two waves. Much work has been done on the structure of the singularities [6–9] in the case of collision of either sourceless or various source waves, with the result that the final singularity formation is, in fact, generic.

A relevant contribution concerning the singularity formation was made by Hayward [11], who formulated the criterion of "incoming" regularity. In other words, he proposed to make a clear distinction whether the singularity formation occurs for the interaction of waves which are initially regular or singular. Then the problem reads: Under what conditions may the initially regular waves avoid the singularity formation? For the case of the purely gravitational (sourceless) waves, Hayward himself found that the regular waves generically produce the curvature singularities. However, there were also exceptional cases where the singularities were avoided.

In $D = 2+1$ dimensions the present authors found that in the interactions of regular asymptotic scalar waves the singularity is always formed [12]. We considered the minimally coupled scalar field. A similar conclusion was then obtained in $D = 3+1$ by Hayward [13], who has shown that if the pure gravitational radiation is absent in the initial data then the interactions of the minimally coupled scalar waves always end up in a singularity.

In this paper, we wish to study a source field of a different type and find whether similar conclusions about the inevitability of the singularity formation can be reached. We shall work with the conformal scalar field with the field equation (in the $D$-dimensional spacetime) [15,16]

$$\nabla_\alpha \nabla^\alpha \phi - \frac{D-2}{4(D-1)} R\phi = 0, \tag{1.1}$$

following from the action

$$S = \int \left[ \left( \frac{1}{2\kappa} - \frac{D-2}{8(D-1)}\phi^2 \right) R - \frac{1}{2}(\nabla\phi)^2 \right] \sqrt{-g}\, d^D x.$$

Unlike the other massless field equations (i.e., Maxwell, Dirac, or Weyl), the minimally coupled massless scalar equation is not conformally invariant. The coupling according to (1.1) cures this "deficiency" and, in any case, it is a reasonable alternative for gravitational coupling of the scalar field. The stress tensor for the conformal scalar field is quite different from the ordinary one, and we may therefore test the singularity formation problem in a different setting to previously.

From a technical point of view, it is not difficult to generate solutions of the Einstein-conformal scalar equations from the minimally coupled Einstein-scalar solutions via the generalized Bekenstein transformation [14–16]. However, the structure of singularities requires independent analysis, since the Bekenstein transformation multiplies the original metric by a nontrivial conformal factor. This operation, in general, *may* change the asymptotic behavior of the Riemann tensor components. Indeed, consider, for instance, the dilaton gravity in $D = 2+1$ with the action

$$S = \int d^3x \sqrt{-g}\, e^{\phi} R.$$

There is the following (black hole) solution of the corresponding field equations [17]

$$ds^2 = -\frac{r-2m}{r}\,dt^2 + \frac{r}{r-2m}\,dr^2 + r^2 d\vartheta^2,$$

$$\phi = \ln|r \sin\vartheta|. \tag{1.2}$$

On the other hand, under the transformation

$$\widetilde{g}_{\alpha\beta} = e^{2\phi} g_{\alpha\beta},\ \widetilde{\phi} = \phi,$$

the action $S$ changes into the action of the minimally

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coupled scalar field

$$S = \int d^3x\sqrt{-\widetilde{g}}\,(\widetilde{R} - 2\widetilde{\phi}_\gamma\widetilde{\phi}^\gamma).$$

The metric corresponding to (1.2) becomes

$$ds^2 = r^2\sin^2\vartheta\left(-\frac{r-2m}{r}\,dt^2 + \frac{r}{r-2m}\,dr^2 + r^2 d\vartheta^2\right).$$

It is not difficult to demonstrate that the singularity structures of both metrics differ considerably from each other.

In Sec. II we study the singularity structure of the Einstein-conformal scalar spacetimes. We arrive again at the same conclusions as for the minimally coupled scalar field, namely, for initially regular waves (without sourceless part) the singularity formation is inevitable.

## II. CONFORMAL SCALAR WAVES

We recall first some known facts we shall use later. The metric of the general interacting plane waves spacetimes is of the form [11]

$$ds^2 = -2e^{-\widetilde{M}}du\,dv + e^{-\widetilde{P}+\widetilde{Q}}\cosh\widetilde{W}dx^2 + e^{-\widetilde{P}-\widetilde{Q}}\cosh\widetilde{W}dy^2 - \sinh\widetilde{W}e^{-\widetilde{P}}dx\,dy,$$

with all functions $\widetilde{M}$, $\widetilde{P}$, $\widetilde{Q}$, and $\widetilde{W}$ depending on $u$ and $v$ only. So-called asymptotic waves introduced by Hayward as a generalization of the colliding waves case satisfy [11]

$$(\widetilde{M},\ \widetilde{P},\ \widetilde{Q},\ \widetilde{W})(u\to-\infty,\ v\to-\infty) = 0,$$

$$(\widetilde{M},\ \widetilde{P},\ \widetilde{Q},\ \widetilde{W})(u,\ v\to-\infty) = (\widetilde{M},\ \widetilde{P},\ \widetilde{Q},\ \widetilde{W})(u), \tag{2.1}$$

$$(\widetilde{M},\ \widetilde{P},\ \widetilde{Q},\ \widetilde{W})(u\to-\infty,\ v) = (\widetilde{M},\ \widetilde{P},\ \widetilde{Q},\widetilde{W})(v).$$

Taking the minimally coupled scalar field $\widetilde{\phi}$ as the source, i.e., putting on the right-hand side (RHS) of Einstein equations the stress tensor

$$\widetilde{T}_{\mu\nu} = \widetilde{\phi}_\mu\widetilde{\phi}_\nu - \tfrac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{\phi}_\sigma\widetilde{\phi}_\rho\widetilde{g}^{\sigma\rho},$$

[where $\widetilde{\phi}$ fulfills the same asymptotic conditions as (2.1)] Hayward has found [13] the solutions for the collinear waves[1] (with $\widetilde{W}=0$). They can be expressed in the form[2]

$$\widetilde{P} = -\ln[1-f(u)-g(v)],$$

$$\widetilde{Q} = k\ln(1-f-g) + p\cosh^{-1}\left[\frac{1+f-g}{1-f-g}\right] + q\cosh^{-1}\left[\frac{1-f+g}{1-f-g}\right]$$
$$+ \int_0^\infty [A(\omega)J_0(\omega(1-f-g)) + B(\omega)N_0(\omega(1-f-g))]\sin(\omega(f-g))d\omega$$
$$+ \int_0^\infty [C(\omega)J_0(\omega(1-f-g)) + D(\omega)N_0(\omega(1-f-g))]\cos(\omega(f-g))d\omega,$$

$$\widetilde{\phi} = \lambda\ln(1-f-g) + \pi\cosh^{-1}\left[\frac{1+f-g}{1-f-g}\right] + \chi\cosh^{-1}\left[\frac{1-f+g}{1-f-g}\right]$$
$$+ \int_0^\infty [\mathcal{A}(\omega)J_0(\omega(1-f-g)) + \mathcal{B}(\omega)N_0(\omega(1-f-g))]\sin(\omega(f-g))d\omega$$
$$+ \int_0^\infty [\mathcal{C}(\omega)J_0(\omega(1-f-g)) + \mathcal{D}(\omega)N_0(\omega(1-f-g))]\cos(\omega(f-g))d\omega,$$

subject to the constraints

$$\int_0^\infty [C(\omega)J_0(\omega) + D(\omega)N_0(\omega)]d\omega = 0,$$
$$\int_0^\infty [\mathcal{C}(\omega)J_0(\omega) + \mathcal{D}(\omega)N_0(\omega)]d\omega = 0,$$

[1]Unfortunately, the exact solutions with scalar sources and noncollinear waves are not known, though it can be shown that the nondiagonality of the metric is always due to the presence of the noncollinear *purely gravitational* waves in the initial data [18].

[2]These solutions are *locally* equivalent to some Gowdy cosmological models (see [19]).

where $f(u)$ and $g(v)$ are functions, $k$, $p$, $q$, $\lambda$, $\pi$, and $\chi$ are real numbers, $A(\omega)$, $B(\omega)$, $C(\omega)$, $D(\omega)$, $\mathcal{A}(\omega)$, $\mathcal{B}(\omega)$, $\mathcal{C}(\omega)$, and $\mathcal{D}(\omega)$ may be integrable functions or distributions and $J_0$ and $N_0$ are zero-order Bessel and Neumann functions. The last—unexpressed—metric function $\widetilde{M}$ is given by direct integration of the relevant Einstein equations

$$\begin{aligned} 2\widetilde{P}_{uu} - \widetilde{P}_u^2 + 2\widetilde{P}_u\widetilde{M}_u &= \widetilde{Q}_u^2 + 2\kappa\widetilde{\phi}_u^2, \\ 2\widetilde{P}_{vv} - \widetilde{P}_v^2 + 2\widetilde{P}_v\widetilde{M}_v &= \widetilde{Q}_v^2 + 2\kappa\widetilde{\phi}_v^2, \\ 2\widetilde{M}_{uv} &= \widetilde{Q}_u\widetilde{Q}_v - \widetilde{P}_u\widetilde{P}_v + 2\kappa\widetilde{\phi}_u\widetilde{\phi}_v. \end{aligned} \tag{2.2}$$

The asymptotic conditions (2.1), requiring certain asymptotic behavior of the functions $f(u)$ and $g(v)$, are met by the choice

$$f(u) = [-a(u-u_s)]^{2/(2-p^2-2\kappa\pi^2)} \quad \text{for } p^2 + 2\kappa\pi^2 > 2,$$

$$f(u) = \exp[a(u-u_s)] \quad \text{for } p^2 + 2\kappa\pi^2 = 2,$$

and

$$g(v) = [-b(v-v_s)]^{2/(2-q^2-2\kappa\chi^2)} \quad \text{for } q^2 + 2\kappa\chi^2 > 2,$$

$$g(v) = \exp[b(v-v_s)] \quad \text{for } q^2 + 2\kappa\chi^2 = 2.$$

This choice obviously describes strictly asymptotic waves, where the flat part of the spacetime "before the collision" exists only asymptotically and the waves are interacting all the time. The true colliding waves[3] can be obtained by taking

$$\begin{aligned} f(u) &= \theta(u)\,(au)^{2/(2-p^2-2\kappa\pi^2)}, \\ g(v) &= \theta(v)\,(bv)^{2/(2-q^2-2\kappa\chi^2)}. \end{aligned}$$

If, moreover,

$$\begin{aligned} \frac{3}{2} &< p^2 + 2\kappa\pi^2 < 2, \\ \frac{3}{2} &< q^2 + 2\kappa\chi^2 < 2, \end{aligned} \tag{2.3}$$

the junction conditions on $u = 0$ and $v = 0$ are automatically fulfilled. The requirement (2.3) means that only continuous waves (on $u = 0$ and $v = 0$) are considered [18]. All the results obtained below are valid for both strictly asymptotic waves and continuous colliding waves.

Now, the solutions for the self-gravitating conformal scalar field can be easily obtained from the self-gravitating minimal scalar spacetimes via a generalized Bekenstein transformation [14–16], linking the $D$-dimensional scalar field $\widetilde{\phi}$ and metric $\widetilde{g}_{\alpha\beta}$ to the $D$-dimensional conformal scalar field $\phi$ and metric $g_{\alpha\beta}$ as follows

$$\begin{aligned} \phi &= \left(\kappa\frac{D-2}{4(D-1)}\right)^{-1/2} \tanh\left[\left(\kappa\frac{D-2}{4(D-1)}\right)^{-1/2}\widetilde{\phi}\right], \\ g_{\alpha\beta} &= \left(\cosh\left[\left(\kappa\frac{D-2}{4(D-1)}\right)^{-1/2}\widetilde{\phi}\right]\right)^{4/(D-2)}\widetilde{g}_{\alpha\beta}. \end{aligned} \tag{2.4}$$

In terms of metric functions $\widetilde{P}$, $\widetilde{M}$, $\widetilde{Q}$ and analogously defined $P$, $M$, and $Q$ we have the following transformation rules

$$\begin{aligned} \phi &= \sqrt{\frac{6}{\kappa}}\tanh\left[\sqrt{\frac{\kappa}{6}}\widetilde{\phi}\right], \\ P &= \widetilde{P} - 2\ln\cosh\left[\sqrt{\frac{\kappa}{6}}\widetilde{\phi}\right], \\ M &= \widetilde{M} - 2\ln\cosh\left[\sqrt{\frac{\kappa}{6}}\widetilde{\phi}\right], \\ Q &= \widetilde{Q}. \end{aligned} \tag{2.5}$$

We see that (2.1) imply fulfillment of the same asymptotic conditions for the new metric functions and the new (conformal scalar) field. It means that the metric $g_{\alpha\beta}$ also describes the interacting asymptotic waves spacetimes and we can study the conditions for the regularity of initial data as well as the consequent creation of singularities in this interaction.[4]

We turn to the study of the singularity structure of the new spacetimes (2.5). Our first question is about the regularity of initial data. For this purpose Hayward [11] has postulated the criterion for initial regularity. It says that at the asymptotic caustics (i.e., $f = 0$, $g = 1$ or $f = 1$, $g = 0$) there are no curvature singularities in the sense of Ellis–Schmidt classification [22].

Near the asymptotic caustic, e.g. ($g = 0$, $f = 1$), where the metric is $v$ independent, it is convenient to take a new null coordinate $u'$ instead of $u$ such that

$$du' = e^{-M}du.$$

Then the following vierbein is orthonormal and parallelly propagated along the incomplete geodesics respecting $x$ and $y$ symmetry and hitting the asymptotic caustic:

$$\begin{aligned} e^{\alpha}_{(0)} &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \\ e^{\alpha}_{(1)} &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right), \\ e^{\alpha}_{(2)} &= (0, 0, e^{\frac{1}{2}(P-Q)(u')}, 0), \\ e^{\alpha}_{(3)} &= (0, 0, 0, e^{\frac{1}{2}(P+Q)(u')}). \end{aligned}$$

[3]Colliding waves with minimally coupled scalar source have been considered by Wu [20] and Halilsoy [21].

[4]Actually, there is one more branch of the generalized Bekenstein transformation (see [16]) which, however, does not preserve the asymptotic conditions, so we shall not deal with it anymore.

It is straightforward to compute the only two (mutually independent) nonzero vierbein components of the Riemann curvature tensor as

$$R_{2020} = -\tfrac{1}{8}[2Q_{u'u'} - 2P_{u'u'} + (Q_{u'} - P_{u'})^2],$$
$$R_{3030} = \tfrac{1}{8}[2Q_{u'u'} + 2P_{u'u'} - (Q_{u'} + P_{u'})^2].$$

Going back to the original coordinates we have

$$R_{2020} = -\tfrac{1}{8}e^{2M}[\,2Q_{uu} + 2Q_uM_u - 2P_{uu} - 2P_uM_u + (Q_u - P_u)^2],$$
$$R_{3030} = \tfrac{1}{8}e^{2M}[\,2Q_{uu} + 2Q_uM_u + 2P_{uu} + 2P_uM_u - (Q_u + P_u)^2].$$

The incoming regularity now requires the boundedness of both components. Using (2.5), we can write

$${}^uR^- \equiv R_{2020} - R_{3030} = -\tfrac{1}{2}\mathrm{Ch}^{-4}\, e^{2\widetilde{M}}[\widetilde{Q}_{uu} + \widetilde{Q}_u\widetilde{M}_u - \widetilde{Q}_u\widetilde{P}_u],$$

$${}^uR^+ \equiv R_{2020} + R_{3030} = \frac{1}{4}\mathrm{Ch}^{-4}\, e^{2\widetilde{M}}\left[2\widetilde{P}_{uu} - 4\sqrt{\frac{\kappa}{6}}\widetilde{\phi}_{uu}\mathrm{Th} - \frac{2}{3}\kappa\widetilde{\phi}_u^2\mathrm{Ch} + \left(\widetilde{P}_u - 2\sqrt{\frac{\kappa}{6}}\widetilde{\phi}_u\mathrm{Th}\right)\left(2\widetilde{M}_u - \widetilde{P}_u - 2\sqrt{\frac{\kappa}{6}}\widetilde{\phi}_u\mathrm{Th}\right) - \widetilde{Q}_u^2\right],$$

with Ch and Th standing instead of $\cosh(\widetilde{\phi}\sqrt{\kappa/6})$ and $\tanh(\widetilde{\phi}\sqrt{\kappa/6})$, respectively.

We have to identify the behavior of ${}^uR^{\pm}$ for $f \sim 1$. Taking into account the asymptotic behavior of the Bessel and Neumann functions

$$J_0(w) \sim 1 - \frac{w^2}{4} + \cdots,$$
$$J_0'(w) \sim -\frac{w}{2} + \cdots,$$
$$N_0(w) \sim \left(1 - \frac{w^2}{4}\right)\ln w + \cdots,$$
$$N_0'(w) \sim \frac{1}{w} - \frac{w}{2}\ln w + \cdots,$$

we obtain (for $f \sim 1$)

$$\widetilde{Q}_u \sim f_u\left[\frac{c}{1-f} + d\ln(1-f) + e + h(1-f)\ln(1-f) + \cdots\right],$$

$$\widetilde{\phi}_u \sim f_u\left[\frac{c^*}{1-f} + d^*\ln(1-f) + e^* + h^*(1-f)\ln(1-f) + \cdots\right],$$

where

$$c = p - k - \int_0^\infty [B(\omega)\sin(\omega) + D(\omega)\cos(\omega)]d\omega,$$

$$d = \int_0^\infty [\omega B(\omega)\cos(\omega) - \omega D(\omega)\sin(\omega)]d\omega,$$

$$e = \frac{p}{2} + \int_0^\infty \omega[A(\omega) + B(\omega)\ln\omega]\cos(\omega)d\omega - \int_0^\infty \omega[C(\omega) + D(\omega)\ln\omega]\sin(\omega)d\omega,$$

$$h = \tfrac{1}{2}\int_0^\infty \omega^2[B(\omega)\sin(\omega) + D(\omega)\cos(\omega)]d\omega,$$

$$c^* = \pi - \lambda - \int_0^\infty [\mathcal{B}(\omega)\sin(\omega) + \mathcal{D}(\omega)\cos(\omega)]d\omega,$$

$$d^* = \int_0^\infty [\omega\mathcal{B}(\omega)\cos(\omega) - \omega\mathcal{D}(\omega)\sin(\omega)]d\omega, \tag{2.6}$$

$$e^* = \frac{\pi}{2} + \int_0^\infty \omega[\mathcal{A}(\omega) + \mathcal{B}(\omega)\ln\omega]\cos(\omega)d\omega - \int_0^\infty \omega[\mathcal{C}(\omega) + \mathcal{D}(\omega)\ln\omega]\sin(\omega)d\omega,$$

$$h^* = \tfrac{1}{2}\int_0^\infty \omega^2[\mathcal{B}(\omega)\sin(\omega) + \mathcal{D}(\omega)\cos(\omega)]d\omega.$$

From (2.2) we have

$$\widetilde{M}_u = f_u\left[\frac{c^2 + 2\kappa c^*2 - 1}{2}\frac{1}{1-f} + (cd + 2\kappa c^*d^*)\ln(1-f) + \cdots\right],$$

hence

$$\widetilde{M} = -\frac{c^2 + 2\kappa c^{*2} - 1}{2}\ln(1-f) + \text{bounded}.$$

Therefore, the leading singular terms in ${}^uR^-$ and ${}^uR^+$ read

$$-8f_u^2\,\frac{c^2+2\kappa c^{*2}-1}{2}c\,(1-f)^{4\sqrt{\kappa/6}|c^*|-(c^2+2\kappa c^{*2}-1)-2},$$

$$4f_u^2\,|c^*|\,\left[\frac{8\kappa}{3}|c^*|-2\sqrt{\frac{\kappa}{6}}(c^2+2\kappa c^{*2}+1)\right]\times(1-f)^{4\sqrt{\kappa/6}|c^*|-(c^2+2\kappa c^{*2}-1)-2}.$$

Both coefficients of proportionality vanish if

$$c=0,\ |c^*|=(2\pm1)/\sqrt{6\kappa},$$

$$|c|=\tfrac{1}{2},\ |c^*|=\sqrt{\frac{3}{8\kappa}},$$

$$c=c^*=0,$$

$$c^*=0,\ |c|=1.$$

In the first two cases there are subleading singular terms, which cannot be eliminated, but in the remaining cases we can exclude them by fitting some other coefficients in the expansions of $\widetilde{\phi}$ and $\widetilde{Q}$. Hence, the initial data are free of curvature singularities if $c=c^*=d=0$ or if $c^2=1$, $c^*=d=d^*=h=0$. The analogous conditions have to be satisfied at the other asymptotic caustic.

Now, we wish to study the components of the curvature tensor at the caustic $1-f-g=0$, $|f-g|\neq0$. It is sufficient [13] to consider the scalar curvature $R$ given by

$$R=-e^M(P_uP_v+2M_{uv}-Q_uQ_v),$$

and the component $\Psi_2$ of the Weyl spinor in the null spin frame [13]

$$\Psi_2=\tfrac{1}{3}e^M(Q_uQ_v-P_uP_v+M_{uv}).$$

If one of them is unbounded, then there is a (final) curvature singularity at the caustic. We take the suitable combinations of $\Psi_2$ and $R$:

$$V_1=e^MM_{uv},$$

$$V_2=e^M(Q_uQ_v-P_uP_v).$$

Using (2.5) we have

$$V_1=\mathrm{Ch}^{-2}e^{\widetilde{M}}\left[\widetilde{M}_{uv}-2\sqrt{\frac{\kappa}{6}}\widetilde{\phi}_{uv}\mathrm{Th}-\frac{1}{3}\kappa\widetilde{\phi}_u\widetilde{\phi}_v\mathrm{Ch}^{-2}\right],$$

$$V_2=\mathrm{Ch}^{-2}e^{\widetilde{M}}\left[\widetilde{Q}_u\widetilde{Q}_v-\left(\widetilde{P}_u-2\sqrt{\frac{\kappa}{6}}\widetilde{\phi}_u\mathrm{Th}\right)\times\left(\widetilde{P}_v-2\sqrt{\frac{\kappa}{6}}\widetilde{\phi}_v\mathrm{Th}\right)\right].$$

In the neighborhood of the caustic it is convenient to introduce other functions $t$ and $z$ instead of $f$ and $g$, given by

$$t=1-f-g,\qquad z=f-g.$$

Then the caustic is formed by the points $t=0, |z|\neq1$. The asymptotic behavior of $\widetilde{\phi}$ and $\widetilde{Q}$ near $t=0$ is

$$\widetilde{\phi}\sim\mathcal{E}(z)\ln t+\mathcal{F}(z)t^2,$$

$$\widetilde{Q}\sim E(z)\ln t+F(z)t^2,$$

where

$$\mathcal{E}(z)=\lambda-\pi-\chi+\int_o^\infty[\mathcal{B}(\omega)\sin(\omega z)+\mathcal{D}(\omega)\cos(\omega z)]d\omega,$$

$$E(z)=k-p-q+\int_0^\infty[B(\omega)\sin(\omega z)+D(\omega)\cos(\omega z)]d\omega,$$

and the forms of $F(z)$ and $\mathcal{F}(z)$ are not important. Then the leading singular terms of $V_1$, $V_2$ are proportional to

$$\left[E^2(z)+2\kappa\mathcal{E}^2(z)-1-4\sqrt{\frac{\kappa}{6}}|\mathcal{E}(z)|\right]t^{2\sqrt{\kappa/6}|\mathcal{E}(z)|+\frac{1}{2}[1-E^2(z)-2\kappa\mathcal{E}^2(z)]-2},$$

$$\left[E^2(z)-\frac{2}{3}\kappa\mathcal{E}^2(z)-1+4\sqrt{\frac{\kappa}{6}}|\mathcal{E}(z)|\right]t^{2\sqrt{\kappa/6}|\mathcal{E}(z)|+\frac{1}{2}[1-E^2(z)-2\kappa\mathcal{E}^2(z)]-2},$$

respectively. Both coefficients of proportionality vanish if

$$\mathcal{E}=0,\ |E|=1,$$

$$E=0, |\mathcal{E}|=\sqrt{\frac{3}{2\kappa}}.$$

In the second case, $V_1$ is unbounded due to the subleading term. Hence the only way to keep both $V_1$ and $V_2$ bounded is to set $\mathcal{E}(z)\equiv0$ and $|E(z)|\equiv1$.

Therefore both the criterion for the incoming regularity and the necessary condition for the avoiding of the final singularity are in the case of conformal scalar waves the same as in the case of minimal scalar waves [13]. Hence the conclusions have to be the same, too. In particular, the formation of final singularities in the interaction of regular asymptotical conformal scalar waves is generic. Moreover, if the pure gravitational radiation is absent in the initial data, i.e.,

$$(-Q_{uu}+P_uQ_u-M_uQ_u)=0,\quad v\to-\infty,$$

$$(-Q_{vv}+P_vQ_v-M_vQ_v)=0,\quad u\to-\infty,$$

the final singularities are even inevitable[5].

[5]We have obtained a similar result also in $D = 2 + 1$: in the interaction of *regular* asymptotic conformal scalar waves the scalar curvature singularity is *always* produced.

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