

Effects of nongauge potentials on the spin- $\frac{1}{2}$ Aharonov-Bohm problem

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Some recent work has attempted to show that the singular solutions which are known to occur in the Dirac description of spin- $\frac{1}{2}$ Aharonov-Bohm scattering can be eliminated by the inclusion of strongly repulsive potentials inside the flux tube. It is shown here that these calculations are generally unreliable since they necessarily require potentials which lead to the occurrence of Klein's paradox. To avoid that difficulty the problem is solved within the framework of the Galilean spin- $\frac{1}{2}$ wave equation which is free of that particular complication. It is then found that the singular solutions can be eliminated provided that the nongauge potential is made energy dependent. The effect of the inclusion of a Coulomb potential is also considered with the result being that the range of flux parameter for which singular solutions are allowed is only one-half as great as in the pure Aharonov-Bohm limit. Expressions are also obtained for the binding energies which can occur in the combined Aharonov-Bohm-Coulomb system.

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I. INTRODUCTION

In classical physics it is a trivial fact that the absence of a force necessarily implies zero scattering. On the other hand, the very remarkable Aharonov-Bohm (AB) effect [1] shows that this does not apply in the realm of quantum mechanics and that potentials (as opposed to fields themselves) can indeed have observable consequences. Thus charged particles are found to be scattered by a thin magnetized filament even though it is possible, by shielding the flux tube or filament, to establish that penetration into the region of nonvanishing magnetic field cannot occur.

For the scattering of a nonrelativistic particle of mass M by the potential

$$eA_i = \alpha \epsilon_{ij} r_j / r^2, \tag{1}$$

where r_i is the radius vector in two dimensions and α is the flux parameter one needs to solve the Schrödinger equation

$$\frac{1}{2M} \left[\frac{1}{i} \nabla - e \mathbf{A} \right]^2 \psi = E \psi. \tag{2}$$

Upon writing

$$\psi(r, \phi) = \sum_{-\infty}^{\infty} e^{im\phi} f_m(r) \tag{3}$$

Eq. (2) reduces to the Bessel equation

$$\left[\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + k^2 - (m + \alpha)^2 / r^2 \right] f_m(r) = 0. \tag{4}$$

Since (4) has both a regular and irregular solution, it is necessary to give a boundary condition which allows a unique result to be obtained. One could, of course, simply require that $f_m(r)$ be finite at $r=0$ and thereby eliminate *ab initio* the irregular solution. This, in fact, gives

the well-known AB solution. Since, however, a resolution of this issue by fiat is totally unsuccessful when spin is included, a more physical approach would clearly be preferable. This is accomplished [2] by replacing (1) with [3]

$$eA_i = \begin{cases} \alpha \epsilon_{ij} r_j / r^2, & r > R, \\ 0, & r < R, \end{cases} \tag{5}$$

and taking the limit $R \rightarrow 0$ after matching boundary conditions at $r=R$. Clearly, the vector potential (5) mathematically effects the replacement of an idealized zero thickness filament by one of finite radius R which has a surface distribution of magnetic field given by

$$eH = -\frac{\alpha}{R} \delta(r - R).$$

Actually, the specific details of the model (5) can be shown to be irrelevant [3] provided only that the flux distribution is independent of angle and has no δ -function contribution at the origin. It is thus straightforward to establish [2] that the irregular solution is absent and that the usual AB solution obtains in the $R=0$ limit.

When spin is included, the situation becomes quite different. Here one is concerned with the (two-component) Dirac equation

$$E\psi = (M\beta + \beta\gamma\Pi)\psi, \tag{6}$$

where $\Pi_i = -i\partial_i - eA_i$ with A_i as in (1). A convenient choice for the matrices in (6) is

$$\beta = \sigma_3, \quad \beta\gamma_i = (\sigma_1, s\sigma_2),$$

where the σ 's are the usual Pauli matrices and $s = \pm 1$ for spin "up" and spin "down," respectively. Upon reduction to a second-order form one obtains from (6) the result

$$(E^2 - M^2)\psi = \left[\Pi^2 + \alpha s \sigma_3 \frac{1}{r} \delta(r) \right] \psi,$$

which, upon using a partial-wave decomposition of the form (3), becomes

$$\left[\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + k^2 - (m + \alpha)^2 / r^2 - \alpha s \sigma_3 \frac{1}{r} \delta(r) \right] f_m = 0,$$

where

$$k^2 = E^2 - m^2.$$

It is significant that the idealization to a zero radius flux tube has given rise to the complication that the delta function occurs at a singular point of the differential equation.

Different approaches have been attempted in dealing with this difficulty. Alford and co-workers [4] simply required the upper component to be regular at the origin for the case of a repulsive δ function. Clearly, one cannot do this for the attractive case since that would make the spin term trivial and imply an amplitude which coincides with the spinless AB result. This would contradict the helicity conservation which is known to be valid [5] for the system described by (6). On the other hand, Gerbert [6] has taken an approach which essentially states that an arbitrary linear combination of the two solutions $J_{|m+\alpha|}(kr)$ and $J_{-|m+\alpha|}(kr)$ in the region $r > R$ can be taken so long as it is normalizable for $R \rightarrow 0$. Thus when $|m + \alpha| < 1$ an arbitrary parameter θ appears which describes the relative contribution of these two functions. This somewhat mathematical approach consequently suffers from the appearance of a parameter with no clear physical origin.

In Ref. [3] an approach was taken which was based on the physically reasonable modification (5) of the vector potential. Upon matching boundary conditions at $r = R$ and letting $R \rightarrow 0$ it was found that the correct solution was always the regular one except when $\alpha s < 0$ and $|m + \alpha| < 1$. In that case there would always be one and only one allowed irregular solution. This occurred for

$$m = -N, \quad N \geq 0 \quad (7)$$

when $s = -1$ and

$$m = -N - 1, \quad N + 1 \leq 0 \quad (8)$$

for $s = +1$ with the integer N defined by

$$\alpha = N + \beta$$

where

$$0 \leq \beta < 1.$$

It is interesting to note that in the cases (7) and (8) for which irregular solutions are allowed there is *no* contribution from the regular solution $J_{|m+\alpha|}(kr)$. Thus the solution obtained in Ref. [3] does correspond to a solution of the type obtained by Gerbert without, however, the introduction of his mixing parameter. Consequently, the solution is unique and can be shown [5] to be consistent with helicity conservation. A point worth mentioning is that this approach was motivated by a desire to formulate the problem from the outset in a physically meaningful way and *not* (as recently stated [7]) to provide

(after the fact) a physical motivation by which to determine Gerbert's θ parameter.

It is clear simply from the $\alpha s < 0$ condition that the solution of Ref. [3] is not anyonic. This has led to some serious concerns by those who consider the anyonic properties of the spinless AB system to be of fundamental significance. Thus, for example, Ref. [7] has taken the introduction of a boundary at $r = R$ one step further by including a nonvanishing nongauge potential for $r < R$ and examining its effects when it diverges in certain ways. In the following section this work is examined and it is shown that Klein's paradox makes that approach unreliable. If, on the other hand, a Galilean spin- $\frac{1}{2}$ wave equation is used in place of the Dirac equation which it closely resembles, then the appearance of Klein's paradox can be avoided. This leads to the determination of an "inside" potential which allows one to force both regular and irregular solutions to occur. However, it implies an energy-dependent potential, a fact which is easily seen to follow from dimensional considerations. Section III considers the case in which a ξ/r potential is also included. Remarkably, it is found that for arbitrarily small ξ the domain of α for which singular solutions can be obtained now shrinks to $|m + \alpha| < \frac{1}{2}$. This contrasts sharply with the claims of Ref. [7], illustrating again the pitfalls of dealing with equations in which Klein's paradox is known to occur.

II. KLEIN'S PARADOX AND HOW TO AVOID IT

It was proposed in Ref. [7] that one modify the spin- $\frac{1}{2}$ AB problem as treated in Ref. [3] by including for $r < R$ a constant potential u_R . Since there is no flux for $r < R$, one has to consider in that domain the equation

$$\left[\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + [(E - u_R)^2 - M^2] - m^2 / r^2 \right] f_m(r) = 0,$$

which has the allowed solution $J_m(k_0 r)$ where

$$k_0^2 = (E - u_R)^2 - M^2.$$

Since all the results of Ref. [7] depend upon the limit of $u_R \rightarrow \infty$, it is of interest to note the behavior of k_0 as a function of u_R . As u_R increases from zero, k_0 is a real number ($E > M$) which eventually vanishes at $u_R = E - M$. At this point k_0 becomes imaginary and the function $J_m(k_0 r)$ goes from sinusoidal oscillation to a real exponential [i.e., $\exp(|k_0| r)$]. As u_R continues to increase, k_0 vanishes again at $u_R = E + M$ and the function $J_m(k_0 r)$ is again oscillatory and remains so as u_R increases without limit. All of these features are reminiscent of the phenomenon of Klein's paradox [8] and one concludes that the Dirac equation is not an adequate framework for this problem in the limit of arbitrarily large u_R .

On the other hand, the type of questions raised in Ref. [7] are amenable to treatment provided that one substitutes for the Dirac equation its Galilean limit. This has the advantage of preserving the magnetic moment interaction while at the same time eliminating the trouble-

some kinematic relationships which characterize the Dirac equation in the presence of unbounded potentials. It is the former which is of interest to this study, whereas the latter can only be completely resolved (in principle, if not in practice) by a full field-theoretical approach. Thus one takes as an appropriate model the Galilean equation

$$[\mathcal{E}\frac{1}{2}(1+\beta)+M(1-\beta)-\beta\gamma\Pi]\psi=0$$

where $\mathcal{E}=E-M$ is the "nonrelativistic" energy. The above was derived in Ref. [2] from the Dirac equation and is simply the 2+1 space version of results obtained by Lévy-Leblond [9]. It is easily seen to imply for ψ_1 (the upper component of ψ) the result

$$\left[\mathcal{E}-\frac{1}{2M}\Pi^2-\frac{\alpha s}{2M}\frac{1}{R}\delta(r-R)\right]\psi_1=0$$

provided that one takes a vector potential of the form (5). Again, carrying out a partial-wave expansion, one obtains the radial equation

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+k^2-[m+\alpha\theta(r-R)]^2/r^2-\alpha s\frac{1}{R}\delta(r-R)\right]f_m(r)=0, \quad (9)$$

where $k^2=2M\mathcal{E}$ and the step function

$$\theta(x)\equiv\frac{1}{2}\left[1+\frac{x}{|x|}\right]$$

has been introduced for conciseness.

One can now include the effect of a repulsive potential as considered in Ref. [7]. Upon letting $\mathcal{E}\rightarrow\mathcal{E}-u_R$ one sees that the solution for $r < R$ is $J_m(k_0r)$ with

$$k_0^2=2M(\mathcal{E}-u_R).$$

At $u_R=0$ this implies an oscillatory $J_m(k_0r)$ which becomes a real exponential for all $u_R > \mathcal{E}$. In other words, the wave function is exponentially damped in the expected quantum-mechanical fashion as r decreases from R . The crucial point is that there is no subsequent transition back to oscillatory behavior as u_R increases without limit and consequently Klein's paradox has been eliminated. Thus Eq. (9) is seen to be an appropriate vehicle for carrying out the program of Ref. [7] which seeks to determine whether a suitably repulsive u_R can allow the simultaneous occurrence of both $J_{|m+\alpha|}(kr)$ and $J_{-|m+\alpha|}(kr)$ for $r > R$.

To carry out this study one writes

$$\begin{aligned} f_m(r < R) &= I_m(k_0r), \\ f_m(r > R) &= AJ_{|m+\alpha|}(kr) + BJ_{-|m+\alpha|}(kr), \end{aligned}$$

where I_m is the usual Bessel function of imaginary argument and it has been assumed that $u_R > \mathcal{E}$. The boundary conditions

$$\begin{aligned} I_m(|k_0|R) &= AJ_{|m+\alpha|}(kR) + BJ_{-|m+\alpha|}(kR), \\ R\frac{\partial}{\partial r}[AJ_{|m+\alpha|}(kr) + BJ_{-|m+\alpha|}(kr) - I_m(|k_0|r)]_{r=R} &= \alpha s I_m(|k_0|R) \end{aligned}$$

can be solved to yield for A/B the result

$$\frac{A}{B} = \frac{-|m| + g(|k_0|R)}{2|m+\alpha| + |m| - g(|k_0|R)} (kR)^{-2|m+\alpha|}, \quad (10)$$

where

$$g(x) = x \frac{\partial}{\partial x} \ln I_m(x)$$

and use has been made of the relation

$$|m| + |m+\alpha| = -\alpha s$$

since, as shown in Ref. [3], only in this partial wave is a singular solution possible.

One can now answer the question posed in Ref. [7] whether a finite nonzero A/B is possible. It is, however, clear that since $u_R \rightarrow \infty$ must yield $B/A=0$ just as $u_R=0$ gives $A/B=0$, there *must* exist a value for u_R which implies a finite A/B . The real issue it would seem is whether a u_R can be found which is independent of the energy \mathcal{E} . It is not difficult to show (most trivially, by dimensional considerations) that no such energy independent solutions exist. This is in marked contrast to the results claimed in Ref. [7]. In that work the scale for the potential u_R is determined by the mass M , or, in other words, by a factor which unavoidably requires special relativity. Since Klein's paradox has been seen to make that approach unreliable, the scale for u_R is determined necessarily by the nonrelativistic energy \mathcal{E} .

The (energy-dependent) solution of (10) given by

$$|k_0|R = \lambda(kR)^{|m+\alpha|}, \quad (11)$$

where λ is arbitrary, implies for $R \rightarrow 0$ that

$$A/B = \frac{1}{4} \frac{\lambda^2}{|m+\alpha|} \frac{1}{|m|+1}.$$

Thus any (positive) value of this ratio can be obtained by appropriate choice of λ . Somewhat curiously, negative values of A/B can also be generated provided that u_R is attractive but diverges according to the same power law as in (11). Thus an affirmative answer has been found for the issue of fine-tuning raised in Ref. [7]. The fact that this tuning requires an intricate dependence of the interior potential on the energy seems to imply, however, that it can have little, if any, utility.

III. COULOMB MODIFICATIONS

It is known that partial wave functions for the AB problem can be obtained exactly even when a $1/r$ potential is included. In particular, such solutions have been obtained by Law *et al.* [10] for the spinless case. This generalization has also been included in Ref. [7] in the context of their interior repulsive potential u_R . Since it has been remarked already that Klein's paradox adverse-

ly affects such calculations, it is of interest to describe the results obtained when the Galilean spin- $\frac{1}{2}$ equation is employed for such an analysis.

Upon taking the potential to be

$$V(r) = \begin{cases} u_R, & r < R \\ \xi/r, & r > R \end{cases},$$

one finds that the appropriate wave equations for the individual partial waves in the expansion of the upper component of ψ are

$$\left[\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + k_0^2 - \frac{m^2}{r^2} \right] f_m(r) = 0$$

for $r < R$, and

$$\left[\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + k^2 - 2M\xi/r - \frac{(m+\alpha)^2}{r^2} \right] f_m(r) = 0$$

for $r > R$. It is straightforward to obtain the solution

$$f_m(r) = J_m(k_0 r)$$

for $r < R$, while, for $r > R$,

$$f_m(r) = A_m e^{ikr} (-2ikr)^{|m+\alpha|} F(|m+\alpha| + \frac{1}{2} + iM\xi/k |2|m+\alpha| + 1| - 2ikr) + B_m e^{ikr} (-2ikr)^{-|m+\alpha|} F(-|m+\alpha| + \frac{1}{2} + iM\xi/k |1 - 2|m+\alpha|| - 2ikr),$$

where $F(a|c|z)$ is the usual confluent hypergeometric function. Note that A_m and B_m are the coefficients of the regular and irregular solutions, respectively. It is worth remarking that because of possible modifications which could be induced by the Coulomb term no assumptions have been made as to which partial waves can have irregular solutions.

Upon applying the boundary conditions at $r = R$ there obtains

$$\begin{aligned} \frac{A_m}{B_m} = & \left[J_{|m|}(k_0 R) R \frac{\partial}{\partial R} e^{ikR} (-2ikR)^{-|m+\alpha|} F(-|m+\alpha| + \frac{1}{2} + iM\xi/k |1 - 2|m+\alpha|| - 2ikr) \right. \\ & - e^{ikR} (-2ikR)^{-|m+\alpha|} F(-|m+\alpha| + \frac{1}{2} + iM\xi/k |1 - 2|m+\alpha|| - 2ikR) \left. \left[\alpha s + R \frac{\partial}{\partial R} \right] J_{|m|}(k_0 R) \right] \\ & \times \left[e^{ikR} (-2ikR)^{|m+\alpha|} F(|m+\alpha| + \frac{1}{2} + iM\xi/k |2|m+\alpha| + 1| - 2ikR) \left[\alpha s + R \frac{\partial}{\partial R} \right] J_{|m|}(k_0 R) \right. \\ & \left. - J_{|m|}(k_0 R) R \frac{\partial}{\partial R} e^{ikR} (-2ikR)^{|m+\alpha|} F(|m+\alpha| + \frac{1}{2} + iM\xi/k |2|m+\alpha| + 1| - 2ikR) \right]^{-1}. \end{aligned} \tag{12}$$

One now takes the $R \rightarrow 0$ limit and finds (as in the $\xi = 0$ case) that B_m must vanish unless $\alpha s < 0$, and

$$|m| + |m + \alpha| = -\alpha s. \tag{13}$$

However, Eq. (13) is necessary but has not been shown to be sufficient. Nor can one merely assume on the basis of the analysis of Refs. [3, 6] that $|m + \alpha| < 1$. As stressed in the former work a second condition emerges when one considers the next-to-leading term in powers of R . In the $\xi = 0$ case things are considerably simpler since the solutions for both $r < R$ and $r > R$ are Bessel functions whose expansions are characterized by the fact that only alternate powers of the argument occur. This is *not* true for the confluent hypergeometric function and one finds from (12) that the expansion in powers of R yields

$$A_m/B_m \sim \xi R^{1-2|m+\alpha|} \tag{14}$$

for small R . Thus singular solutions are possible for $\xi \neq 0$ only when $|m + \alpha| < \frac{1}{2}$ rather than the full range $|m + \alpha| < 1$ assumed in Ref. [7]. For the case $\xi = 0$ the result (14) is replaced by

$$A_m/B_m \sim R^{2-2|m+\alpha|} \tag{15}$$

because of the noted property of the Bessel function for small argument. Clearly, the condition $|m + \alpha| < 1$ follows from (15) whereas the considerably stronger condition $|m + \alpha| < \frac{1}{2}$ is required when a Coulomb term is present. It should be remarked that when this condition is satisfied, the result (11) again obtains to determine the dependence of u_R on k and R in the case that both regular and irregular solutions are required to occur.

The energies \mathcal{E}_n of the bound states which occur for $\xi < 0$ are readily determined from the series expansions of the relevant confluent hypergeometric functions. These yield for $u_R = 0$ the results

$$\mathcal{E}_n = -\frac{1}{2} \frac{M\xi^2}{(n - \frac{1}{2} \pm |m + \alpha|)^2}, \quad n = 1, 2, \dots,$$

where the upper and lower signs refer, respectively, to the case of regular and irregular solutions. Of particular interest is the fact that the binding energies of the irregular solutions become arbitrarily large as $|m + \alpha|$ approaches $\frac{1}{2}$. This illustrates the crucial role played by the condition $|m + \alpha| < \frac{1}{2}$ and lends added credence to the derivation presented above.

IV. CONCLUSION

This paper has explored the possibility that more general solutions to the spin- $\frac{1}{2}$ AB problem can be found. As in the earlier work of Ref. [7] this has been done by introducing a very short range, but singular, repulsive force. Not surprisingly, it has been found (at least in the unambiguous Galilean case) that such solutions can, in fact, exist. This is physically reasonable since the effect of such a potential is to reduce significantly the interaction of the magnetic moment with the singular magnetic field at the origin. On the other hand, such a potential must be required to be energy dependent, and it also has the disadvantage of violating helicity conservation, a property which would otherwise be satisfied. Since the solution [3] of this problem without the additional nongauge potential is known to be at variance with anyon features, it is clear that those properties can be restored only with considerable difficulty. If one elects to do this, the effect is to negate the full dynamical participation of the spin in the interaction.

One of the most interesting results obtained here has to do with the modifications associated with the inclusion of a Coulomb term. It was found that the condition $|m + \alpha| < 1$ which is generally thought to follow from a

condition of normalizability of the solution is not sufficient. A more careful analysis shows that only half that range is, in fact, allowed for singular solutions. This has as an immediate consequence that if one considers a gas of such particles then the discontinuities known to characterize the second virial coefficient [11] $B_2(\alpha, T)$ are shifted from integer values of α to half-integer values. This is particularly noteworthy because of the fact that the transition point has no dependence on the strength of the Coulomb potential. Thus one can imagine this parameter to be continuously decreased to zero and find that the discontinuities in $B_2(\alpha, T)$ generally occur at half integers but at integral values when the Coulomb term exactly vanishes. This provides a most remarkable example of a system in which a point of discontinuity of a variable which has a macroscopic discontinuity has itself a discontinuous dependence on a microscopic parameter. This is a subject which clearly merits additional study.

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