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Temperature dependence of the induced Chern-Simons term in 2+1 dimensions

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We study the temperature dependence of the radiatively generated parity-breaking (Chern-Simons) term for massive fermions coupled to an external Abelian gauge field in 2+1 dimensions, using the imaginary time formalism. We allow the gauge field to be rapidly varying in space (as in a vortex, for example), and we also consider arbitrary discrete frequency components of the field. We thus go beyond the usual "low momentum" and "adiabatic" approximations. We give a simple analytical expression that is exact at both large and small momenta and provides a good interpolation at intermediate values. We also show under which conditions the usual adiabatic approximation is accurate, providing a simple physical picture for its prediction. Finally, we consider the same problem for massless fermions coupled to a non-Abelian gauge field.

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I. INTRODUCTION

The calculation of the induced Chern-Simons term at finite temperature has become a subject of increasing interest [1–6], because of its relevance for anyon models of superconductivity [7], and also because of the intrinsic theoretical interest of learning how the passage to finite temperature affects nonperturbative aspects of quantum field theory which are understood at zero temperature [8]. In this respect the Chern-Simons term is in a somewhat similar position to the chiral anomaly (in even dimensional spacetimes), which was studied by many authors [9–14]. Their common result is that whichever calculational scheme is used, the chiral anomaly is temperature independent. Further, this temperature independence can be demonstrated without any of the usual particular techniques (chiral Jacobians, anomalous diagrams, etc.), but just by studying the corresponding index theorem for the relevant Dirac operator in the appropriate (time periodic) spacetime [14]. The usual physical explanation of this fact is that the chiral anomaly is a large momentum phenomenon (the existence of the anomaly is a consequence of the necessity of introducing some form of ultraviolet cutoff), and so the result is insensitive to the difference between an infinite or finite time dimension. That is plausible, but of course subject to the condition that the fermions should be *massless*. Otherwise, the explicit chiral symmetry breaking produced by a mass term will induce a temperature-dependent contribution to the divergence of the axial-vector current, because the mass sets up a finite scale which will be important when the temperature is of the order of the mass or bigger.

In the case of massive fermions in 2+1 dimensions we are in the odd dimensional analogue of the latter situation, because the explicit parity violation is due to a finite mass (we will consider also the case of massless non-Abelian fermions). So it is reasonable on physical grounds to expect a temperature-dependent Chern-Simons term. For masses going to infinity one should recover the zero-temperature limit, because the scale introduced by the mass is never seen by the finite (imaginary) time interval, and so it will produce the same answer as in the zero-temperature case.

In the present paper we study the radiatively generated parity-violating term at finite temperature, for the case of massive fermions coupled to an external Abelian gauge field in 2+1 dimensions. An important aim is to obtain useful results when the gauge field is allowed to be rapidly varying in space (as in a vortex, for example), and when arbitrary $n \neq 0$ (Matsubara) frequency components are important. This means that we go beyond the usual "low momentum" and "adiabatic" approximations.

Indeed, there are many calculations of the induced Chern-Simons term at finite temperature [1–4] which contain the result that the effective Chern-Simons action at $T \neq 0$ is just the $T = 0$ action multiplied by a temperature-dependent factor [15]:

$$W_{T\neq 0} = \tanh\left(\frac{\beta|M|}{2}\right) W_{T=0} , \tag{1}$$

where M is the fermion mass and $\beta = \frac{1}{kT}$. Equation (1) is frequently derived by making use of low momentum and adiabatic approximations, which essentially amount to retaining the lowest-order term in an expansion of the Lagrangian in powers of derivatives. While this certainly gives correctly the coefficient of what might be called "the" Chern-Simons term (i.e., the one of the form $A\,dA$), it will not in general provide an adequate description of the full induced parity-violating effect which is of second order in A. The latter will be a nonlocal quantity, which we shall call the "full" Chern-Simons term.

An indication of the inadequacy of Eq. (1) can be seen from the following consideration. In the imaginary time

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formalism [16], the gauge field has Euclidean-space momentum components $(\frac{2\pi n}{\beta}, \mathbf{p})$, so that the condition for the validity of such a combined low momentum and adiabatic expansion is expected to be

$$\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2\right]^{1/2} \ll |M| \,, \tag{2}$$

as we shall explicitly confirm in what follows. Equation (2) is of course equivalent to the *simultaneous* conditions

$$\frac{|\mathbf{p}|}{|M|} \ll 1, \quad \frac{2\pi n}{\beta} \ll |M| \,. \tag{3}$$

The first is just the statement that the associated bosonic field should be spatially smooth on the scale M^{-1}, irrespective of the value of the temperature, and is clearly a low momentum condition; but the second requires

$$\beta|M| \gg 2\pi n \,, \tag{4}$$

which is essentially a condition for adiabaticity. Now, only if $n = 0$ (the static case) will (4), which is essentially a condition for a kind of adiabaticity (in periodic imaginary time), be automatically satisfied, and in this case Eq. (1) will provide the leading finite temperature behavior, for spatially smooth fields. But it must be emphasized that in considering the equilibrium thermodynamics of our system all the discrete frequencies n will have to be considered, not just the $n = 0$ one. Now for $n \neq 0$, Eq. (4) requires $\beta|M|$ to be large, so that (1) would then be expected to be relevant at sufficiently low temperatures. The trouble, as we shall see below, is that the higher time derivatives, which arise in the presence of $n \neq 0$ components [ignored in obtaining (1)] lead to power law terms in $(\beta|M|)^{-1}$ which are certainly important in comparison with the exponentially small temperature correction (at low temperatures) in Eq. (1). Hence, even for spatially smooth configurations, Eq. (1) is not reliable for predicting the finite temperature behavior (as was pointed out in a related context by Niemi and Semenoff [17]). Certainly, improved expansions can be found for $n \neq 0$; but what is really required is an expression which may be used when conditions (3) and (4) do not hold, so that arbitrary spatial configurations and temperatures can be handled.

It is of course straightforward to calculate the full induced Chern-Simons action at $T \neq 0$ without making the above approximations (see Sec. III below). But this leaves us with a somewhat awkward nonlocal expression, which is inconvenient for applications. However, the form factor embodying the nonlocality is in fact very simple, being given by (the parity-violating part of) the fermion loop contribution to the vacuum polarization. Just such nonlocal expressions arise in a number of contexts and recently a useful "pole approximation" has been developed to simplify them [18]. This approximation has the virtue that it is a very simple algebraic function, yet it gives the exact loop integral result for external momenta tending to zero or to infinity, while in the intermediate region it provides a good approximation to the exact result. Thus it goes well beyond the low momentum approximation, providing a useful interpolation formula across the whole range of momenta. It is this idea which we shall use and adapt here to give a practical approximate expression for the general induced Chern-Simons term at finite temperature.

In Sec. II we consider the zero-temperature case as a warm-up excercise. We briefly calculate the well-known [19] parity-violating-induced term, and give a simpler algebraic expression that agrees with the exact solution when the momentum tends to zero or infinity. In the intermediate region it is an excellent approximation to the exact result.

In Sec. III we present a calculation of the induced Chern-Simons term at finite temperature, keeping the full momentum dependence until reaching the final expression as a Feynman parameter integral with a smooth integrand, which is suitable for analyzing the behavior of the integral in different regimes. We also generalize the algebraic approximation which we introduced at zero temperature to the case of nonzero temperature, obtaining a formula that describes the behavior of the exact result in a much simpler but still quite accurate way.

In Sec. IV we discuss the conditions under which the approximations used to derive (1) are acceptable, and provide a justification of the result previously obtained [1–4] within a much simpler framework. We also compare our approximate formula, evaluated for low values of the momenta, with the standard low momentum expansion of the exact result.

In Sec. V we consider the case of massless fermions coupled to a non-Abelian gauge field. This case is qualitatively different from the Abelian one in that (for appropriate groups) there is an induced Chern-Simons term even when the fermions are massless [20], the parity-violating term appearing when one wants to define the theory in a gauge invariant way. We shall show, however, that in this case the induced Chern-Simons term is independent of temperature. Finally, in Sec. VI we present our conclusions.

II. THE INDUCED CHERN-SIMONS TERM AT $T = 0$

A. Exact calculation

We will calculate here the induced Chern-Simons term for massive fermions in 2+1 dimensions, coupled to an external Abelian gauge field, at zero temperature. We will not dwell on the dynamics of the vector field itself, so we do not include a bare Lagrangian for it. The theory is defined by the generating functional

$$\mathcal{Z} = \exp\left(iW[A]\right) = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\left(i\int d^3x \mathcal{L}\right) \,, \tag{5}$$

where

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - e\not{A} - M)\psi, \quad \gamma^0 = \sigma_3, \gamma^1 = i\sigma_1, \gamma^2 = i\sigma_2 \,, \tag{6}$$

where $\sigma_j, j = 1,2,3$ are the familiar Pauli matrices. $W[A]$ is the full effective action for A_μ, generated by integrating out the fermions. So, except for a field-independent constant, the effective action will be given by

$$W[A] = -i \ln \det(\not\partial + ie\not A + iM) = -i\mathrm{Tr} \ln(\not\partial + ie\not A + iM) , \quad (7)$$

where the determinant and the trace must be understood both in spin indices and in the functional sense.

As we are looking for the Chern-Simons-like term, we can expand the logarithm in Eq. (7) up to order 2 in e, so as to get in $W[A]$ a term with two A's.[1] Then, the effective action of Eq. (7) will contain up to this order the three terms

$$W = W^{(0)} + W^{(1)} + W^{(2)} + O(e^3) , \quad (8)$$

where

$$W^{(0)} = -i \ln \det(\not\partial + iM) , \quad (9)$$

$$W^{(1)} = e\mathrm{Tr}[(\not\partial + iM)^{-1}\not A] , \quad (10)$$

$$W^{(2)} = \frac{e^2}{2i}\mathrm{Tr}[(\not\partial + iM)^{-1}\not A(\not\partial + iM)^{-1}\not A] . \quad (11)$$

$W^{(0)}$ is just the free energy of uncoupled fermions, and $W^{(1)}$ gives zero when we evaluate the trace, as can be easily seen in the momentum representation. So we are left with $W^{(2)}$ only. We can give an expression for $W^{(2)}$ in the momentum representation as

$$W^{(2)} = \frac{ie^2}{2}\int \frac{d^3p}{(2\pi)^3}\, \tilde A_\mu(-p)\tilde\Gamma^{\mu\nu}(p)\tilde A_\nu(p) , \quad (12)$$

where the Fourier transforms are defined as

$$\tilde A_\mu(p) = \int d^3x\, e^{ip\cdot x} A_\mu(x) , \quad (13)$$

and $\tilde\Gamma_{\mu\nu}$ is given by

$$\tilde\Gamma_{\mu\nu}(p) = \int \frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\left(\gamma_\mu \frac{1}{\not p + \not k - M}\gamma_\nu \frac{1}{\not k - M}\right) , \quad (14)$$

where tr means over the Dirac indices only. Evaluating the trace in Eq. (14), we observe that there appear both parity-conserving and parity-violating terms. In what follows we keep only the latter, but we still use the same symbol $\tilde\Gamma$ to denote the parity-violating contribution (as we will not consider hereafter the parity-conserving part, there is no risk of confusion). So, with this remark in mind, after a straightforward calculation we obtain

$$\tilde\Gamma^{\mu\nu}(p) = -2F(p)\,\epsilon^{\mu\nu\lambda}p_\lambda , \quad (15)$$

where

$$F(p) = -iM\int \frac{d^3k}{(2\pi)^3}\frac{1}{[(p+k)^2 - M^2](k^2 - M^2)} \quad (16)$$

is the loop integral in dimensionless form. As indicated in the Introduction, it is F which controls the nonlocality in (12). To calculate the integral, we first perform a Wick rotation of the momenta: $k_0 \to ik_E$, $p_0 \to ip_E$. We exponentiate the denominators in Eq. (16), using the familiar introduction of the corresponding Schwinger parameters; next we complete the square in the exponent, shift the momentum k by a constant, and perform the momentum integration to obtain

$$F(p) = \frac{M}{8\pi^{3/2}}\int_0^1 d\alpha \int_0^\infty d\rho \rho^{-1/2} \times \exp\{-\rho[\alpha(1-\alpha)|p^2| + M^2]\} , \quad (17)$$

where $p^2 = -(p_E^2 + \mathbf{p}^2)$. The two remaining integrations are elementary, and so one finally obtains [21]

$$F(p) = \frac{M}{4\pi|p|}\arcsin\left[\left(1 + \frac{4M^2}{|p^2|}\right)^{-1/2}\right] , \quad (18)$$

where $|p| = (|p^2|)^{1/2}$. This result, introduced into Eqs. (15) and (12), therefore gives the effective parity-violating term for any external momentum.

B. Algebraic approximation of $F(p)$

Now we construct an expression that provides a very simple algebraic approximate formula for the exact value of the loop integral $F(p)$ [Eq. (18)], and so also of the effective parity-violating term. Of course we do not really need any such approximate formula in this case, since we have the exact analytic form (18). However, the introduction of the approximation at this stage will serve to motivate its much more necessary use in the $T \neq 0$ case which follows. The technique has been applied to (3+1)-dimensional loop integrals [18] in another context, and in that case the approach is called the "pole approximation." The essential idea is to find a simple algebraic expression which gives the exact loop integral result for momenta tending to zero or infinity, and which provides a reasonably good approximation to the exact result for intermediate momenta. We note that, as $\frac{|p^2|}{M^2} \to 0$, the exact loop integral $F(p)$ of Eq. (18) behaves as

$$F(p) \to \frac{M}{8\pi|M|} \quad \left(\frac{|p^2|}{M^2} \to 0\right) , \quad (19)$$

while, when $\frac{|p^2|}{M^2} \to \infty$,

$$F(p) \to \frac{M}{8|p|} \quad \left(\frac{|p^2|}{M^2} \to \infty\right) . \quad (20)$$

[1] In the non-Abelian case, one should consider also the terms with three A's, but in fact knowing only the term with two A's and using gauge invariance one can predict the form of the term with three A's.

We see that a simple algebraic formula that has the same limits as (18), and is smooth in the intermediate region [as $F(p)$ is], is given by

$$F(p) \simeq f(p) = \frac{M}{8(|p^2| + \pi^2 M^2)^{1/2}} \,. \tag{21}$$

In Fig. 1 we show the exact $[F(p)]$ and the approximate $[f(p)]$ functions in a given range of the dimensionless variable $\frac{|p|}{M}$. Note that the agreement between them is very good for small or large momenta, and the most appreciable departure (still only about 10% at most) occurs at $|p| \sim M$. The approximation of F by f will emerge as the zero-temperature limit of the corresponding approximation in the finite temperature case, discussed in the next section.

III. THE INDUCED CHERN-SIMONS TERM AT $T > 0$

A. Exact calculation

We will use the imaginary time formalism [16] in what follows; this amounts to doing the following replacements in the respective $T = 0$ expressions:

$$\begin{aligned} \int d^3x &\longrightarrow -i \int_0^\beta d\tau \int d^2x \,, \\ \int \frac{d^3k}{(2\pi)^3} &\longrightarrow \beta^{-1} \sum_{m=-\infty}^{m=+\infty} \int \frac{d^2k}{(2\pi)^2} \,, \\ g_{\mu\nu} &\longrightarrow \mathrm{diag}(-1,-1,-1) \end{aligned} \tag{22}$$

[so, by the last rule every scalar product of two three-vectors a_μ, b_μ means $a \cdot b = -(a_0 b_0 + \mathbf{a} \cdot \mathbf{b})$]. We see that the effective action to order A^2, in the momentum representation, takes the form

$$W^{(2)}[A] = \frac{ie^2}{2\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2p}{(2\pi)^2} \, \tilde{A}_\mu(-p_n) \, \tilde{\Gamma}^{\mu\nu}(p_n) \, \tilde{A}_\nu(p_n) , \tag{23}$$

where

$$p_\mu^n = (\rho_n, \mathbf{p}) \,, \; \rho_n = \frac{2\pi n}{\beta} \tag{24}$$

and where

$$\begin{aligned} \tilde{\Gamma}_{\mu\nu}(p_n) = \beta^{-1} \sum_{m=-\infty}^{m=+\infty} \int \frac{d^2k}{(2\pi)^2} \\ \times \mathrm{tr} \left(\gamma_\mu \frac{1}{\not{p}_n + \not{k}_m - M} \gamma_\nu \frac{1}{\not{k}_m - M} \right) \,, \end{aligned} \tag{25}$$

with

$$k_\mu^m = (\omega_m, \mathbf{k}), \quad \omega_m = \frac{(2m+1)\pi}{\beta} \tag{26}$$

[note that p_n should have an even Matsubara frequency, so that $p_n + k_m$ has an odd frequency in Eq. (25), as it should].

Evaluating the trace in Eq. (25), and keeping only the parity-violating part, as in the zero-temperature case (but now in Euclidean space throughout), we obtain

$$\tilde{\Gamma}^{\mu\nu}(p_n) \;=\; 2i \; F(p_n) \; \epsilon^{\mu\nu\lambda} p_\lambda^n \tag{27}$$

($\epsilon^{012} = +1$), where again the nonlocality is contained in the loop integral

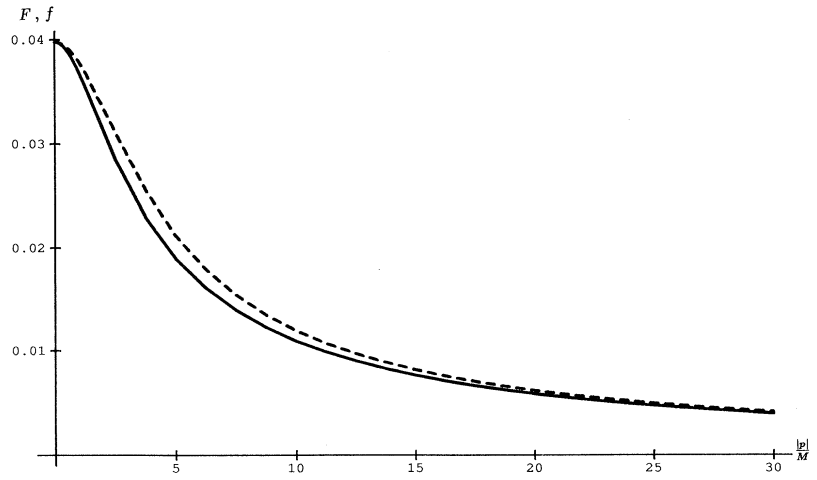

FIG. 1. The exact (F, solid curve) and approximate (f, dashed curve) integral $F(p)$ at $T = 0$ as functions of $\frac{|p|}{M}$.

$$F(p_n) = M\ \beta^{-1} \sum_{m=-\infty}^{+\infty} \int \frac{d^2k}{(2\pi)^2} \frac{1}{[(p_n + k_m)^2 - M^2](k_m{}^2 - M^2)} . \tag{28}$$

The effective action to order A^2 is then

$$W^{(2)}[A] = -\frac{e^2}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2p}{(2\pi)^2} \tilde{A}_\mu(-p_n) F(p_n) \epsilon^{\mu\nu\lambda} p_{\nu n} \tilde{A}_\lambda(p_n) . \tag{29}$$

All that remains to do now is to calculate $F(p_n)$ in order to obtain the parity-violating term in the efffective action. We introduce a Feynman parametric integral in Eq. (28) in order to obtain a more manageable expression

$$\begin{aligned} F(p_n) &= \frac{M}{\beta} \sum_{m=-\infty}^{+\infty} \int_0^1 d\alpha \int \frac{d^2k}{(2\pi)^2} \frac{1}{\{\alpha[(p_n + k_m)^2 - M^2] + (1-\alpha)(k_m^2 - M^2)\}^2} \\ &= \frac{M}{\beta} \sum_{m=-\infty}^{+\infty} \int_0^1 d\alpha \int \frac{d^2k}{(2\pi)^2} \frac{1}{[(\mathbf{k} + \alpha\mathbf{p})^2 + (\omega_m + \alpha\rho_n)^2 + \alpha(1-\alpha)(\rho_n{}^2 + \mathbf{p}^2) + M^2]^2} . \end{aligned} \tag{30}$$

After a shift in $\mathbf{k}$, the $\mathbf{k}$ integration can be done easily, yielding

$$F(p_n) = \frac{M}{4\pi\beta} \sum_{m=-\infty}^{+\infty} \int_0^1 d\alpha \frac{1}{(\omega_m + \alpha\rho_n)^2 + \alpha(1-\alpha)(\rho_n^2 + \mathbf{p}^2) + M^2} . \tag{31}$$

Now, remembering the definitions of the corresponding Matsubara frequencies,

$$F(p_n) = \frac{M\beta}{4\pi(2\pi)^2} \int_0^1 d\alpha \sum_{m=-\infty}^{+\infty} \frac{1}{(m + \frac{1}{2} + \alpha n)^2 + \alpha(1-\alpha)[n^2 + (\frac{\beta}{2\pi})^2 \mathbf{p}^2] + (\frac{\beta M}{2\pi})^2} . \tag{32}$$

The series in Eq. (32) is a particular case of the more general formula

$$S = \sum_{m=-\infty}^{+\infty} \frac{1}{(m - z_1)(m - z_2)} , \tag{33}$$

where z_1 and z_2 are the roots of the quadratic form in the denominator of Eq. (32), which in our case are complex conjugate. Provided the roots have a nonzero imaginary part, S can be easily evaluated, for example, with the Regge trick [22]. The result for S is

$$S = \frac{\pi}{z_2 - z_1} [\mathrm{cotg}(\pi z_1) - \mathrm{cotg}(\pi z_2)] , \tag{34}$$

and introducing the explicit form of the roots z_1 and z_2 derived from Eq. (32), we find

$$S = \frac{\pi}{r(\alpha)} \frac{\sinh[2\pi r(\alpha)]}{\cosh[2\pi r(\alpha)] + \cos[2\pi\alpha n]} , \tag{35}$$

where $r(\alpha)$ is defined as

$$r(\alpha) = \frac{\beta}{2\pi} [\alpha(1-\alpha)|p_n^2| + M^2]^{1/2} \tag{36}$$

with $p_n^2 = -(\rho_n^2 + \mathbf{p}^2)$. So the loop integral $F(p_n)$ which goes into Eq. (29) is given by

$$F(p_n) = \frac{\beta M}{2(2\pi)^3} \int_0^1 d\alpha \frac{\pi}{r(\alpha)} \frac{\sinh[2\pi r(\alpha)]}{\cosh[2\pi r(\alpha)] + \cos(2\pi\alpha n)} . \tag{37}$$

B. Algebraic approximation for $F(p_n)$

We now seek to generalize the algebraic approximation of the $T = 0$ case to the $T > 0$ case, providing an approximation for the loop integral of Eq. (37).

Although for brevity we have shown F as depending only on p_n, we can see from (36) and (37) that it is in fact a function of the dimensionless quantities $\beta|p_n|$ (where $|p_n| = |p_n^2|^{1/2}$), βM, and n. The βM dependence is to be expected from the $|p_n| \to 0$ limit, Eq. (1), but some discussion is in order concerning the separate dependence on $\beta|p_n|$ and n. This is associated, of course, with the loss of (2+1)-dimensional covariance in passing to the finite temperature case via Eqs. (22) and (24), for example, with the result that ρ_n and $\rho_n^2 + \mathbf{p}^2$ enter the right-hand side of (37) separately and asymmetrically. This separate dependence of F on n and $\beta|p_n|$ does present a difficulty, in principle, for our approximate approach, since it is not at all clear how to set about finding a simple function which, say, correctly reproduces (37) both for large and small $\beta|p_n|$, and for large and small n. A (related) problem is that, rather than n and $\beta|p_n|$, we might be more interested physically in the temperature variable T, and particularly in the low- and high-temperature limits of (37); but although these limits are simple enough in the special case $n = 0 = |\mathbf{p}|$, in the general case the limiting regimes will depend on n and $|p_n|$ as well as on M.

We have adopted what seems to be the most straightforward approach, which is to ignore the separate n dependence of (37), which arises via the term $\cos(2\pi\alpha n)$ for

$n \neq 0$, and to look for a simple function which matches the leading behaviors of (37) both for small and large $|p_n|$, recognizing that this may lead to some loss of accuracy in certain regimes. It is simple to verify that, for $|p_n| \to \infty$, the function $F(p_n)$ behaves asymptotically as

$$F(p_n) \sim 1/|p_n| \,, \tag{38}$$

and that it goes to a finite value when $|p_n| \longrightarrow 0$ (both limits are taken at any fixed, but arbitrary, β). Both behaviors *are independent of the way we take the limit*, i.e., on the relation between the spatial and temporal components of the momentum which, as noted above, now appear in an asymmetrical way in the loop integral. This fact and the $T = 0$ result encourage us to choose a simple profile for the approximation to F:

$$F(p_n) \simeq f(p_n) = \frac{a}{(|p_n^2| + b^2)^{1/2}} \,, \tag{39}$$

where a and b are β-dependent parameters.

In order to fix the values of the parameters a and b, we need to impose two conditions on f. We match the values of F and its approximant f at $|p_n| = 0$ and also equate the coefficients of their $\frac{1}{|p_n|}$ terms. Both values of F can be computed exactly at any temperature, and so after a little algebra we obtain the approximant f:

$$f(p_n) = \frac{M}{8} \frac{|\sinh(\beta M/2)|}{[\sinh^2(\beta M/2)|p_n^2| + \pi^2 \cosh^2(\beta M/2) M^2]^{1/2}} \,. \tag{40}$$

Equation (40) provides the desired interpolation formula for F, and is the main result of this paper. We note that f depends on βM and $\beta|p_n|$, but not on n separately; also, that (1) is recovered in the $|p_n| \to 0$ limit.

We now want to give an idea of the accuracy of (40). Our procedure of matching the large and small $|p_n|$ behavior should presumably mean that (40) is a good approximation to (37) in the case $n = 0$, since then n does not appear separately and $|p_n| = |\mathbf{p}|$ for all T. Figure 2 shows a plot of F [Eq. (37)] and f [Eq. (40)] for $\beta M = \pi$, as functions of $\frac{|\mathbf{p}|}{M}$ when $n = 0$. We note that the accuracy is very similar to that in the $T = 0$ case, Fig. 1—indeed the curves themselves are very similar in shape, the temperature affecting mainly the normalization. This behavior remains at any temperature, as expected.

Figure 3 compares F and f in the other extreme situation, when $|\mathbf{p}| = 0$ and $n \neq 0$, so that $|p_n| = |\rho_n| = \frac{2\pi n}{\beta}$. We choose $\beta M = \pi$, so that $\frac{\rho_n}{M} = 2n$. Again the agreement is very good, despite the appearance of a small ripple at low $\frac{|\rho_n|}{M}$ in the exact result, and which is associated with the $\cos(2\pi\alpha n)$ term in (37) which the interpolation formula smooths out. Note, however, that we have plotted the curve as if the discrete variable in the horizontal axis were continuous, but it is not. Moreover, the peak of the ripple appears exactly at $\frac{\rho_n}{M} = 1$, which is outside the set of admissible values of $\frac{\rho_n}{M}$, as the physical values of this ratio are (for this example) $2 \times$ integer. In fact, for the first nonzero momentum, the exact and approximate curves almost coincide. So, this ripple appears only for values of the external momentum which correspond to the gauge fields being considered as fermionic, which is not the case here.

Anyway, the smoothing introduced by the approximation is a pleasant feature if one wants to regard the discrete momentum as "approximately continuous," especially for higher temperatures, where the ripple becomes much sharper but is always concentrated on unphysical values of the discrete momentum. Figure 4 shows the case $\beta M = \frac{\pi}{100}$, for which $\frac{\rho_n}{M} = 200n$.

So far we have presented two particular paths for the variation of the momentum, either purely spatial or purely temporal. For the sake of completeness we show in Fig. 5 the case of varying the discrete momentum but including a constant spatial part for the external momentum, for the same temperature as in Fig. 4. Again the agreement is very good, and besides the obvious global decrease in the values of F and f, we see that the ripple smooths out [this is due to the fact that the cos in the denominator of (37) is relatively small compared with the other term in the denominator]. In summary, we may

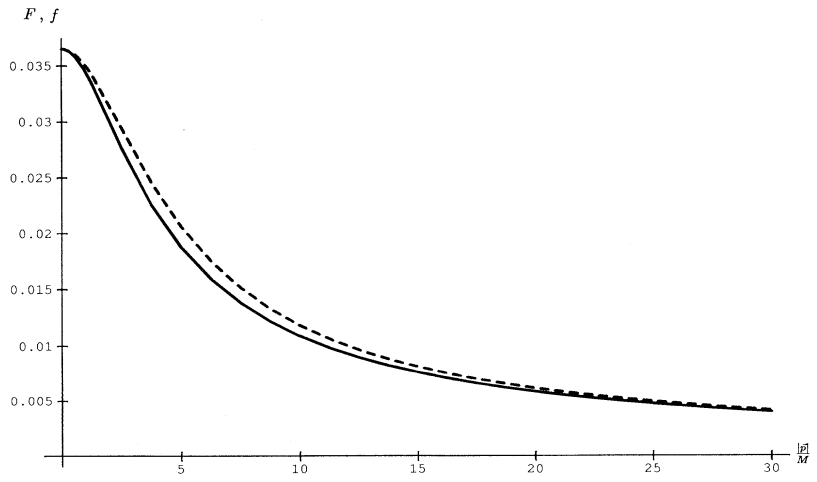

FIG. 2. The exact (F, solid curve) and approximate (f, dashed curve) integral $F(p_n)$ for $\beta M = \pi$ and $n = 0$, as a function of $\frac{|\mathbf{p}|}{M}$.

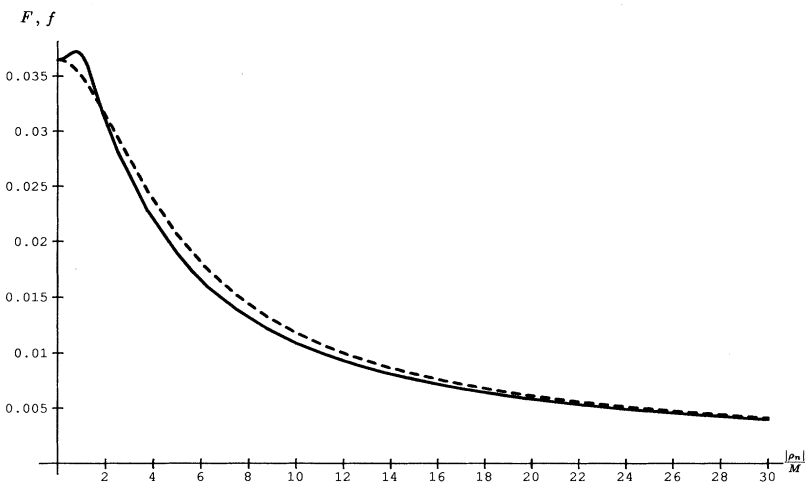

FIG. 3. The exact (F, solid curve) and approximate (f, dashed curve) integrals $F(p_n)$ for $\beta M = \pi$ and $\mathbf{p} = 0$, as a function of $\frac{|\rho_n|}{M}$. Physical values of $\frac{|\rho_n|}{M}$ are multiples of 2.

conclude from Figs. 3–5 that expression (40) deals very well with the $n \neq 0$ (nonstatic) case, for which the adiabatic approximation is valueless.

It is easy to see that the zero-temperature limit of the approximation (40) is the algebraic approximation (21) for the zero-temperature case, which itself contains the correct low and high p limits. It is also worth remarking that we can safely take the $M \to 0$ limit in (37) or (40) to confirm [21] that there is no induced Chern-Simons term at all in the massless Abelian case; this limit cannot, of course, be discussed sensibly within the framework of an expansion in powers of $\frac{|p_n|}{M}$.

We conclude that our simple expression (40) does provide a useful approximation to (37), and can be inserted in the summed or integrated form (23). As one possible application, suppose that the field A_μ had a singular part, let us say a static vortex. Then the strong spatial variation of A_μ in the vicinity of the center of the vortex makes the low momentum approximation useless, but we can still compute its contribution to the effective action to a good approximation by using (40). More importantly, expression (40) provides a simple and accurate formula capable of dealing with the nonstatic case, for arbitrary spatial variation.

IV. EQUATION (1) REVISITED

The approximate result (1) for the induced Chern-Simons term at finite temperature [1,5] can be obtained from our equation (37), or its approximant of Eq. (40) (which in this limit becomes exact) under the conditions

$$\frac{|\mathbf{p}|}{M} \ll 1, \quad \frac{2\pi n}{\beta M} \ll 1 \tag{41}$$

as stated in the Introduction. In particular, even if one assumes that A_μ has nonzero Fourier components in the discrete variable only for $n \leq 1$, say, one still must assume $\beta M \gg 2\pi$, as is clearly implied by Eq. (4) when one puts $n = 1$. Of course, if the mass were infinite, that condition would be satisfied, but for finite M and finite β it is their dimensionless product which must be large. The upshot is that the adiabatic approximation is implicitly also, in

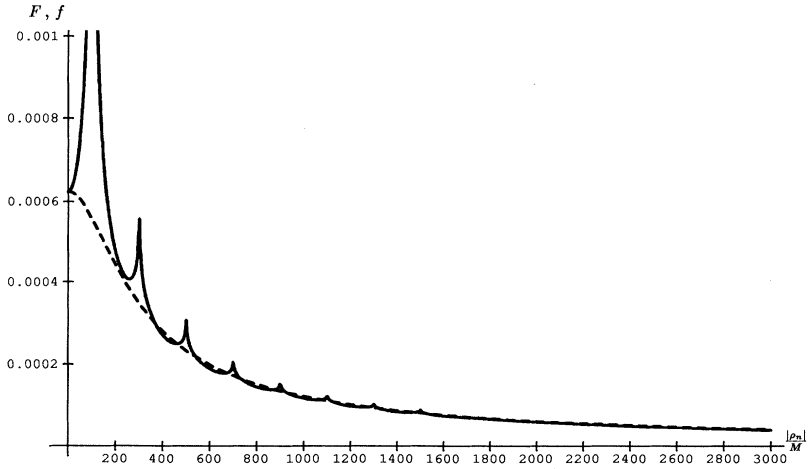

FIG. 4. The exact (F, solid curve) and approximate (f, dashed curve) integrals $F(p_n)$ for $\beta M = \frac{\pi}{100}$ and $\mathbf{p} = 0$, as a function of $\frac{|\rho_n|}{M}$. Physical values of $\frac{|\rho_n|}{M}$ are multiples of 200.

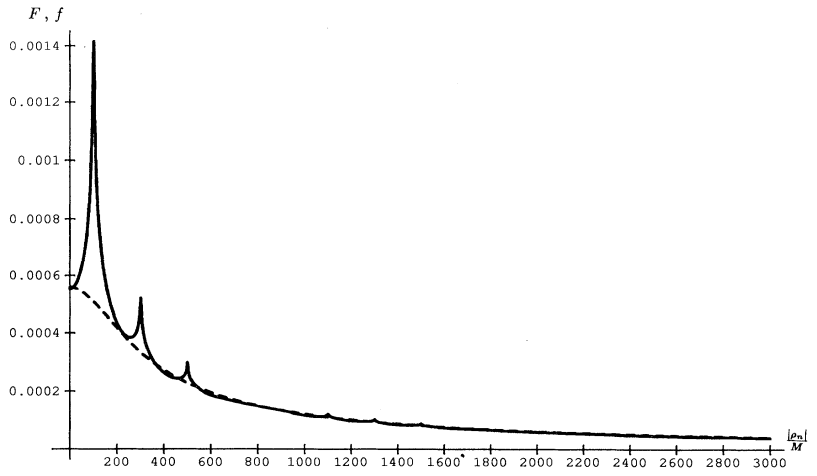

FIG. 5. The exact (F, solid curve) and approximate (f, dashed curve) integrals $F(p_n)$ for $\beta M = \frac{\pi}{100}$ and $\frac{|\mathbf{p}|}{M} = 100$, as a function of $\frac{|\rho_n|}{M}$. Physical values of $\frac{|\rho_n|}{M}$ are multiples of 200.

this case, a low-temperature ($T \ll M$) approximation. If the conditions of Eq. (41) hold, one obtains the usual result given in Eq. (1)—but, as already anticipated in the Introduction, this is only reliable for n strictly equal to zero; for $n \neq 0$, corrections involving powers of $(\beta|M|)^{-1}$ will enter, in comparison with which the deviation of tanh from unity is both exponentially small and unreliable. To obtain a quantitative criterion for the reliability of the tanh factor, we consider the ratio (r) between our formula [Eq. (40)] and the tanh factor [Eq. (1)]. Clearly, this is equal to the ratio between $f(p_n)$ and $f(0)$:

$$r(p_n) = \frac{f(p_n)}{f(0)} . \tag{42}$$

For the sake of simplicity, we assume $\mathbf{p} = 0$ and require r to be equal to one within an error less than 10%. Then one easily obtains a relation between n and βM,

$$n \leq \frac{\beta M}{2\sqrt{5}} , \tag{43}$$

which clearly says that the lower the temperature, the larger the range of discrete frequencies where Eqs. (1) and (40) agree within a 10% error. For the case of Fig. 3, where $\beta M = \pi$, we get from Eq. (43) that n can only be equal to zero, as is evident from the curve. For very low temperatures, n can be larger (for example, if $\beta M = 100\pi$, n could be between 0 and ~ 50).

Now we show a comparison of the next term in the momentum expansions of the exact [Eq. (37)] and approximate [Eq. (40)] results. As an example, we will set $n = 0$ for the external momentum, so that the expansion parameter will be $\frac{\mathbf{p}^2}{M^2}$ (we do not consider the opposite case, i.e., expanding in $\frac{\rho_n}{M}$, because the physical values of ρ_n are discrete, and if they were regarded as continuous, the behavior of the expansion of the exact solution would be nonanalytic, as is seen in Fig. 4). The expansion of both results up to $\frac{\mathbf{p}^2}{M^2}$ order is indicated by

$$F(\mathbf{p}) \simeq F_0 + F_1 \frac{\mathbf{p}^2}{M^2}, \tag{44}$$

$$f(\mathbf{p}) \simeq f_0 + f_1 \frac{\mathbf{p}^2}{M^2}, \tag{45}$$

where, of course, F_0 and f_0 are equal. The ratio F_1/f_1 can be calculated after a little algebra:

$$\frac{F_1}{f_1} = \frac{\pi^2}{6} \frac{\tanh(\mu) + \mu[\tanh^2(\mu) - 1]}{\tanh^3(\mu)}, \tag{46}$$

where $\mu = \frac{\beta M}{2}$. Then it is easy to see that the ratio of Eq. (46) tends to the *same* limit when μ tends to 0 or ∞, this limit being equal to $\frac{\pi^2}{6} \simeq 1.5$. For finite values of μ, this ratio remains almost constant around $\frac{\pi^2}{6}$. So we see that our approximation does not fit exactly the terms of the low momentum expansion, because we have only adjusted our analytical approximation of Eq. (40) with the exact one of Eq. (37) at zero and infinite momentum. Nevertheless, the expansions of both expressions are not very different, and this property is almost temperature independent. The same can be seen *mutatis mutandis* in the expansion around $n = 1$ for the external momentum.

We shall now provide a physical explanation of the factor $\tanh(\frac{\beta|M|}{2})$ that appears in front of the $T = 0$ Chern-Simons term, in the crudest approximation.

We recall that the term that induces the parity breaking is the mass term. In the corresponding Dirac Hamiltonian,

$$H_D = \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \gamma_0 M + eA_0 , \tag{47}$$

the operator responsible for the parity breaking is the matrix γ_0. If we calculate the thermal average of that matrix with H_D as the Hamiltonian, and under the assumption $\beta M \gg 1$, we see that

$$\langle \gamma_0 \rangle = \frac{\mathrm{tr}(\gamma_0 e^{-\beta H_D})}{\mathrm{tr}(e^{-\beta H_D})} \simeq \frac{\mathrm{tr}(\gamma_0 e^{-\beta M \gamma_0})}{\mathrm{tr}(e^{-\beta M \gamma_0})} \tag{48}$$

(note that we dropped from H_D all the terms but the mass term; it is the crudest approximation one can do). Now we should be careful about the fact that in the sum over states one must consider the Dirac vacuum as the lowest state. This is equivalent to replacing $\gamma_0 \to \frac{\gamma_0 + 1}{2}$ in Eq. (48) *in the Hamiltonian only.* Then,

$$\langle\gamma_0\rangle = \frac{\mathrm{tr}[\gamma_0 e^{-\beta M(\frac{\gamma_0+1}{2})}]}{\mathrm{tr}[e^{-\beta M(\frac{\gamma_0+1}{2})}]}$$
$$= \frac{\mathrm{tr}\{\gamma_0[\cosh(\frac{\beta M}{2}) - \gamma_0 \sinh(\frac{\beta M}{2})]\}}{\mathrm{tr}[\cosh(\frac{\beta M}{2}) - \gamma_0 \sinh(\frac{\beta M}{2})]}$$
$$= -\tanh(\frac{\beta M}{2}) = -\frac{M}{|M|}\tanh\left(\frac{\beta|M|}{2}\right) . \quad (49)$$

So we can interpret this result as explaining that, only in the crude approximation we are considering here, the induced Chern-Simons term at finite temperature is of the form

$$W^{(2)}_{T\neq 0} = \langle\gamma_0\rangle_{T\neq 0}\, W^{(2)}_{T=0} . \quad (50)$$

This expression can be compared with the induced mass term for a vector field in presence of a nonzero vacuum expectation of the corresponding Higgs field ϕ:

$$\mathcal{L}_M = \frac{1}{2}\langle\phi\rangle_{T\neq 0}\, A_\mu A^\mu , \quad (51)$$

where the β dependence of $\langle\phi\rangle_\beta$ comes from the β dependence of the ϕ average due to the temperature dependence of the effective potential. As in Eq. (50), ϕ is the "breaking operator," and $\frac{1}{2}A_\mu A^\mu$ is the symmetry-breaking Lagrangian.

V. MASSLESS FERMIONS IN A NON-ABELIAN BACKGROUND FIELD

As was shown in Ref. [20], a kind of "parity anomaly" appears in the $T = 0$ effective action for a massless fermion field (or an odd number of them) coupled to a non-Abelian gauge field whose group enjoys the property that its Π_3 is the additive group of the integers. It is manifested as a parity nonconservation in the effective action when one uses a gauge invariant regularization procedure,[2] and conversely as a gauge noninvariance of the effective action (under large gauge transformations) when one uses a parity-conserving regularization.

The way in which gauge invariant regulators destroy parity symmetry is easily understood in the example of the Pauli-Villars procedure, which introduces an explicit (and eventually infinite) mass term into the action of the regulated theory, and thus breaks parity. As there are no subtractions to be made, the "renormalized" fermion determinant is defined simply as

$$(\det\not{D})_R = \lim_{M\to\infty}\frac{\det\not{D}}{\det(\not{D}+iM)}$$
$$= \lim_{M\to\infty}\det[\not{D}(\not{D}+iM)^{-1}] , \quad (52)$$

where $\not{D}$ is the Dirac operator,

$$\not{D} = \not{\partial} + \not{A} ,\ A_\mu = gA^a_\mu\lambda_a ,\ [\lambda^a,\lambda^b] = f^{abc}\lambda^c , \quad (53)$$

and $\not{D}+iM$ comes from the action for the regulator field. Clearly, when $M\to\infty$ we will obtain a gauge invariant and parity-conserving contribution, plus a Chern-Simons term (whose sign is that of the regulator mass).

The converse phenomenon is also explained in Ref. [20], through index theorems for Dirac operators. It turns out that

$$\det\not{D} \longrightarrow \det\not{D}\ (-1)^{|n|} , \quad (54)$$

when a large gauge transformation of winding number n is performed on the gauge fields, and $\det\not{D}$ is defined in a parity-conserving way. Turning now to our interest in finite temperature effects, as long as we require the model to be invariant under large gauge transformations at $T = 0$, we get the infinite fermion mass in the regulator, and so we are in the region of applicability of the "adiabatic" approximation for the non-Abelian theory [1], i.e., $\beta M \gg 1$. This approximation for the non-Abelian case gives the same result as in the Abelian case, the $T > 0$ Chern-Simons term being just the $T = 0$ one multiplied by the same factor as in the Abelian case. The limit $M\to\infty$ can then be taken within this approximation, obtaining the result that the induced Chern-Simons term at finite temperature is temperature dependent, because the thermal factor goes to 1 in the limit. This is the main conclusion of this section.

It is worth remarking that in the Abelian case there is no similar phenomenon at $T = 0$, because of the triviality of the homotopy group $\Pi^3(S_1) = 1$. So one can regulate the integrals in a parity-conserving and gauge invariant way.

VI. CONCLUSIONS

We have shown that the temperature and momentum dependence of the full (nonlocal) induced Chern-Simons term in the effective action for massive fermions coupled to an Abelian vector field at $T \geq 0$ can be described through a simple analytical formula. The accuracy of this approximation does not rest on a low momentum, adiabatic, or small temperature assumption, and so enables one to calculate different objects of interest in a wide range of parameters (for example, temperature dependence of the induced topological mass). Of course, one should limit oneself to calculations that describe phenomena of a purely parity-violating character; otherwise one should include the rest of the effective action to one-loop order.

[2]The ultraviolet behavior of the integrals that appear to lowest order in the fermionic determinant is superficially divergent, although it happens that when one adds a regulator field in order to cure their apparent ill definedness, one finds that after the calculation of the regulated expression the cutoff can be taken to infinity and no divergence appears. Then there are no subtractions to be done: in a sense, the results are finite but regularization dependent. A similar fact occurs when one calculates the anomalous divergence of the axial-vector current in (3+1)-dimensional massless QED; it is always finite, but regularization dependent. The ill definedness is of the type $0\times\infty$, and not of the more usual ∞ type.

We want to remark that the calculations we have presented in this paper have been restricted in two, possibly important, respects: first, we have not considered gauge field configurations of nontrivial topology; and second, we have worked to second order only in powers of the gauge field, and have not considered possible non-perturbative effects (due to a large external field, for instance). As regards the first point, the case of non-Abelian massless fermions, which we discussed in the previous section, shows that the possibility of nontrivial large gauge transformations has indeed a very profound effect on the temperature dependence of the induced Chern-Simons term, namely, the latter does not depend on the temperature at all. Of course, our argument was based on the requirement of gauge invariance at $T = 0$, and from this followed the above-mentioned consequence; we have not needed to consider index theorems concerning Dirac operators on $S^1 \times S^2$, the calculation on S^3 of Ref. [20] being enough for our purposes. We also note that the calculation of the induced Chern-Simons term might well give a different result if performed in the presence of a topologically nontrivial background (which could in principle be done in the same way as when one computes the fermion determinant in the presence on instantons, for example). As far as we are aware, the impact of these kinds of complications on the Chern-Simons term at finite temperature remain generally unexplored, although Ref. [4] gives some results for the case of a topologically nontrivial space (viz., a torus).

As regards the second point (i.e., nonperturbative effects), we note that in the case of a constant external magnetic field was considered in Ref. [2], with the result that the *local* part of the induced Chern-Simons term was independent of the field. There seems to be no reason why the method of the present paper could not also be applied to such a case, to study the full parity-violating term in such a background.

Note added in proof. The calculations described in Sec. III are correct for discrete imaginary values of the energy variables. However, expression (37) does not, as it stands, provide the correct analytic continuation to real energy values [for an analogous case see H. A. Weldon, Phys. Rev. D **47**, 594 (1993), Sec. III]. In particular, the correct continuation is not analytic at $p_0 = |\mathbf{p}| = 0$ when $T \neq 0$, as was pointed out in Ref. [6]. We shall give a complete discussion of this issue elsewhere.

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[1] K. S. Babu, A. Das, and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987).
[2] E. R. Poppitz, Phys. Lett. B **252**, 417 (1990).
[3] L. Moriconi, Phys. Rev. D **44**, R2950 (1991).
[4] M. Burgess, Phys. Rev. D **44**, 2552 (1991).
[5] W. T. Kim, Y. J. Park, K. Y. Kim, and Y. Kim, Phys. Rev. D **46**, 3674 (1993).
[6] Y. C. Kao and M. F. Yang, Phys. Rev. D **47**, 730 (1993).
[7] X. G. Wen and A. Zee, Phys. Rev. Lett. **62**, 2873 (1989).
[8] L. Alvarez-Gaume, L. Della Pietra, and G. Moore, Ann. Phys. (N.Y.) **163**, 288 (1985).
[9] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[10] M. Reuter and W. Dittrich, Phys. Rev. D **32**, 513 (1985).
[11] A. Das and A. Karev, Phys. Rev. D **36**, 623 (1987).
[12] A. Das and A. Karev, Phys. Rev. D **36**, 2591 (1987).
[13] Y. L. Liu and G. J. Ni, Phys. Rev. D **38**, 3840 (1988).
[14] A. A. Rawlinson, D. Jackson, and R. J. Crewther, Z. Phys. C **56**, 679 (1992).
[15] K. Ishikawa and T. Matsuyama, Nucl. Phys. **B280** [F518], 523 (1987).
[16] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).
[17] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135**, 99 (1986).
[18] I. Adjali, I. J. R. Aitchison, and J. A. Zuk, Nucl. Phys. **A573**, 457 (1992).
[19] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[20] A. N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); see also S. Rau and R. Yahalom, Phys. Lett. B **172**, 227 (1988).
[21] R. D. Pisarski, Phys. Rev. D **29**, 2423 (1984); H. Banerjee, G. Bhattacharya, and J. S. Bhattacharya, *ibid.* **37**, 1706 (1988).
[22] C. W. Bernard, Phys. Rev. D **9**, 3312 (1974).