Continuum light-cone quantization of Gross-Neveu models

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The Gross-Neveu models are quantized on the light cone, using an equation of motion approach tailored to the large N limit. Vacuum properties are exhibited and the fermion-antifermion integral equation of Tamm-Dancoff type is derived and solved, with particular attention paid to the issue of renormalization. The main results are verified in equal time quantization, by means of the infinite momentum frame.

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I. INTRODUCTION

"Gross-Neveu models" refer to a family of self-interacting fermion field theories in one space dimension which were studied extensively during the second half of the 1970's [1–8]. The simplest version consists of massless fermions with an internal degree of freedom (N "colors") and quartic scalar-scalar interactions. In addition, a chirally invariant model (the two-dimensional "Nambu–Jona-Lasinio model" [9]) and massive theories have also been considered. These models exhibit a number of interesting phenomena, such as dynamical mass generation, spontaneous symmetry breaking, dimensional transmutation, fermion-antifermion and multifermion bound states. They are asymptotically free, renormalizable, and soluble in the $1/N$ expansion [1] or, in the case of the chiral Gross-Neveu model, by Bethe ansatz techniques even for arbitrary N [7,8]. Recently, interest in Gross-Neveu models has been revived, both in the context of relativistic many-body theories [10] and as a testing ground for lattice calculations of hadron properties [11].

In this paper, we shall reconsider the Gross-Neveu models for yet another reason. During the last years, a great deal of effort has been devoted to better understanding and developing techniques of light-cone quantization in field theory [12,13]. These efforts are triggered by the necessity of eventually solving relativistic bound-state problems in strong interaction physics. In particular, in $1+1$ dimensions and in the Hamiltonian framework, the technical advantage of the light-cone quantization over ordinary equal-time quantization can be quite striking [14,18]. Conceptual difficulties initially posed by nonperturbative phenomena are by now fairly well understood in the particular case of two-dimensional gauge theories, QED_2 and QCD_2 [18]. Here, the severe infrared divergences can be controlled by quantizing in a finite interval. If one insists on a spacelike separation between the end points of the interval (where the boundary conditions are imposed), one is led to introduce coordinates slightly "rotated" with respect to the ordinary light-cone variables. Thereby, one can avoid the difficulties encountered in the naive application of discrete lightcone quantization (e.g., with the fermion condensate or the axial anomaly in QED_2). Nevertheless, most of the technical advantages of the light cone can be recovered in a well-defined limiting procedure.

Among many other unrealistic features, one special property of two-dimensional gauge theories is the fact that they are super-renormalizable. As soon as one starts to think about more realistic, four-dimensional field theories, one has to face the difficult problem of how to deal with renormalization in light-cone quantization [19,20]. Therefore, it is of some interest to study first two-dimensional models which need UV renormalization, while still being soluble. One model where such a study has been attempted is scalar ϕ^4 field theory [17]. However, since no systematic approximation scheme is known and since the distinction between light-cone and ordinary coordinates was not strictly respected in Ref. [17], the results are not yet conclusive. Here, we propose to treat the Gross-Neveu models in the large-N limit on the light cone. These models are well behaved in the infrared, so that there is no particular incentive to use a finite interval formulation; rather, we shall work from the outset in the continuum. We shall employ relativistic many-body methods which have already proven useful in solving the Gross-Neveu models in ordinary coordinates [10], as well as in the context of large-N QCD_2 [18]. We are fully aware that the Gross-Neveu models are far too simple to be representative for the whole spectrum of renormalizable field theories. Nevertheless, the light-cone formulation has not yet been given, to the best of our knowledge,[1] and turns out to be of interest in its own right, being somewhat more involved than in the case of the super-renormalizable large-N QCD_2. By way of example, since

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[1]Reference [21] addresses the Nambu–Jona-Lasinio model on the light cone. As is well known, this model is not renormalizable in $3+1$ dimensions; hence the thrust of this paper is necessarily very different from ours, in spite of some common aspects.

the fermion mass in the Gross-Neveu model is generated dynamically, reflecting directly nontrivial vacuum properties, it is clear from the beginning that the conventional folklore about the "triviality of the vacuum" cannot be entirely correct. Once the light-cone Tamm-Dancoff equation has been derived, the gain in simplicity for various applications will be shown to be again substantial.

The paper is organized as follows: In Sec. II, we apply a systematic $1/N$ expansion to the light-cone Gross-Neveu model and point out where the issue of renormalization enters. Our main aim is to derive the Tamm-Dancoff equation for the fermion-antifermion system corresponding to the well-known 't Hooft equation of two-dimensional QCD [14]. In Sec. III, we apply this equation to bound-state and scattering problems using several variants of the Gross-Neveu model. In Sec. IV, we exhibit the connection to the "infinite momentum frame" limit in equal-time quantization, and Sec. V contains our summary and conclusions.

II. FORMALISM

The Lagrangian of the Gross-Neveu model "family" is given by

$$\mathcal{L}=\bar{\chi}i\gamma^{\mu}\partial_{\mu}\chi+\tfrac{1}{2}g^2[(\bar{\chi}\chi)^2-\lambda(\bar{\chi}\gamma^5\chi)^2]-m_0\bar{\chi}\chi \ . \tag{2.1}$$

Here, we have suppressed the color indices, i.e., used the notation

$$\bar{\chi}\chi=\sum_{i=1}^{N}\bar{\chi}_i\chi_i \ . \tag{2.2}$$

The original model of Ref. [1] with scalar interaction and massless fermions is obtained by choosing $\lambda=0$, $m_0=0$; the chirally invariant model corresponds to $\lambda=1$, $m_0=0$, and occasionally a mass term has been considered ($m_0\neq0$). We use the following "chiral" representation for the γ matrices:

$$\gamma^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma^1=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\qquad \gamma^5=\gamma^0\gamma^1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \ . \tag{2.3}$$

The metric tensor is chosen as

$$g^{\mu\nu}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{2.4}$$

and the fermion field in terms of right- and left-handed components will be denoted by

$$\chi=\begin{bmatrix}\phi\\ \psi\end{bmatrix} \ . \tag{2.5}$$

Introducing light-cone coordinates

$$x^{\pm}=\frac{1}{\sqrt{2}}(x^0\pm x^1),\quad \partial_{\pm}=\frac{1}{\sqrt{2}}(\partial_0\pm\partial_1) \ , \tag{2.6}$$

with x^+ the (light-cone) time, x^- the spatial coordinate, the Lagrangian (2.1) is converted into

$$\mathcal{L}=i\sqrt{2}(\phi^{\dagger}\partial_+\phi+\psi^{\dagger}\partial_-\psi)-m_0(\phi^{\dagger}\psi+\psi^{\dagger}\phi) +\tfrac{1}{2}g^2[(\phi^{\dagger}\psi+\psi^{\dagger}\phi)^2-\lambda(\phi^{\dagger}\psi-\psi^{\dagger}\phi)^2] \ . \tag{2.7}$$

Varying the action with respect to $\phi^{\dagger}$ and $\psi^{\dagger}$, we obtain the Euler-Lagrange equations

$$\begin{aligned} i\sqrt{2}\partial_+\phi&=\{m_0-g^2[(1-\lambda)\phi^{\dagger}\psi+(1+\lambda)\psi^{\dagger}\phi]\}\psi \ ,\\ i\sqrt{2}\partial_-\psi&=\{m_0-g^2[(1+\lambda)\phi^{\dagger}\psi+(1-\lambda)\psi^{\dagger}\phi]\}\phi \ . \end{aligned} \tag{2.8}$$

As expected in light-cone quantization, only the right-handed fermion field ϕ is dynamical, whereas the left-handed field ψ obeys an equation of constraint. The canonical momentum conjugate to ϕ is given by

$$\pi(x)=\frac{\delta\mathcal{L}}{\delta\partial_+\phi(x)}=i\sqrt{2}\phi^{\dagger}(x) \ , \tag{2.9}$$

and we impose the standard equal light-cone time anticommutation relations:

$$\{\phi^{\dagger}(x),\phi(y)\}=\frac{1}{\sqrt{2}}\delta(x-y),\quad \{\phi(x),\phi(y)\}=0 \ . \tag{2.10}$$

As in Ref. [18], we shall develop a systematic large-N expansion around operators bilinear in the fermion fields, using an equations of motion approach. Since (unlike in QCD_2) we cannot solve the constraint and eliminate ψ at this stage, we have to introduce two types of bilinears:

$$\begin{aligned} Q(x^+;x^-,y^-)&:=\frac{1}{N}\sum_i\phi_i^{\dagger}(x^+,y^-)\phi_i(x^+,x^-) \ ,\\ A(x^+;x^-,y^-)&:=\frac{1}{N}\sum_i\phi_i^{\dagger}(x^+,y^-)\psi_i(x^+,x^-) \ . \end{aligned} \tag{2.11}$$

For $Q(x^+;x^-,y^-)$, the following equation of motion is readily derived from Eq. (2.8) (suppressing the x^+ variable to ease the notation):

$$\begin{aligned} i\sqrt{2}\partial_+Q(x^-,y^-)=&\{m_0-Ng^2[(1-\lambda)A(x^-,x^-)+(1+\lambda)A^{\dagger}(x^-,x^-)]\}A(x^-,y^-)\\ &-A^{\dagger}(y^-,x^-)\{m_0-Ng^2[(1+\lambda)A(y^-,y^-)+(1-\lambda)A^{\dagger}(y^-,y^-)]\} \ . \end{aligned} \tag{2.12}$$

$A(x^+;x^-,y^-)$, on the other hand, satisfies the constraint

$$i\sqrt{2}\frac{\partial}{\partial x^-}A(x^-,y^-)=\{m_0-Ng^2[(1+\lambda)A(x^-,x^-)+(1-\lambda)A^{\dagger}(x^-,x^-)]\}Q(x^-,y^-) \ . \tag{2.13}$$

Following Ref. [18], we now expand Q and A around their "classical" (c-number) parts, which can be identified with the corresponding vacuum expectation values:

$$Q(x^+;x^-,y^-)=\rho(x^- - y^-) + \frac{1}{\sqrt{N}}\tilde{Q}(x^+;x^-,y^-)+\cdots , \tag{2.14}$$

$$A(x^+;x^-,y^-)=a(x^- - y^-) + \frac{1}{\sqrt{N}}\tilde{A}(x^+;x^-,y^-)+\cdots .$$

The assumption that the leading fluctuating part is of order $1/\sqrt{N}$ will be verified in the course of consistently solving Eqs. (2.12) and (2.13). Inserting this expansion into Eqs. (2.12) and (2.13), we find in zeroth order

$$a(x^- - y^-)\{m_0 - Ng^2[(1-\lambda)a(0)+(1+\lambda)a^*(0)]\} = a^*(y^- - x^-)\{m_0 - Ng^2[(1+\lambda)a(0) +(1-\lambda)a^*(0)]\} , \tag{2.15}$$

$$i\sqrt{2}\frac{\partial}{\partial x^-}a(x^- - y^-) = \{m_0 - Ng^2[(1+\lambda)a(0)+(1-\lambda)a^*(0)]\} \times\rho(x^- - y^-) . \tag{2.16}$$

These equations determine the vacuum properties in a Hartree approximation. Their implications are somewhat different, depending on whether or not we have a situation with chiral symmetry. Therefore, let us discuss these two cases separately.

In all cases where the chiral symmetry is explicitly broken (either by the interaction or the mass term), Eq. (2.15) for $x^- = y^-$ implies that $a(0)$ is real; the value of $a(0)$ is related to the fermion condensate via

$$\langle\bar{\psi}\psi\rangle = 2Na(0) . \tag{2.17}$$

More generally, Eq. (2.15) yields

$$a(x^- - y^-)=a^*(y^- - x^-) . \tag{2.18}$$

Defining the physical fermion mass through

$$m := m_0 - g^2\langle\bar{\psi}\psi\rangle = m_0 - 2Ng^2a(0) , \tag{2.19}$$

Eq. (2.16) can be rewritten as

$$i\sqrt{2}\frac{\partial}{\partial x^-}a(x^-)=m\rho(x^-) . \tag{2.20}$$

Upon Fourier transformation using the convention

$$f(k)=\int dx^- e^{ikx^-}f(x^-) \quad (f=\rho,a) , \tag{2.21}$$

we get

$$a(k)=a^*(k)=\frac{m}{\sqrt{2}}\frac{\rho(k)}{k} . \tag{2.22}$$

The self-consistency condition following from Eqs. (2.19) and (2.22) (called in the following the Hartree equation) is

$$1-\frac{m_0}{m} = -\sqrt{2}Ng^2\int\frac{dk}{2\pi}\frac{\rho(k)}{k} . \tag{2.23}$$

For the chirally symmetric model ($\lambda=1, m_0=0$), we introduce the modulus and the phase of the complex number $a(0)$:

$$a(0)=|a(0)|e^{i\varphi} . \tag{2.24}$$

Here, both $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$ condensates are nonvanishing, in general, and related via

$$\langle\bar{\psi}\psi\rangle = 2N|a(0)|\cos\varphi , \quad \langle\bar{\psi}\gamma^5\psi\rangle = 2N|a(0)|i\sin\varphi . \tag{2.25}$$

Relation (2.22) still holds in the form

$$a(k)=\frac{m}{\sqrt{2}}\frac{\rho(k)}{k} , \tag{2.26}$$

but $m=-2Ng^2a(0)$ is now complex, and Rem plays the role of the physical fermion mass. The self-consistency relation can be obtained from Eq. (2.23) above by setting $m_0=0$.

Of the discrete symmetries, charge conjugation C is the most important one in light-cone quantization because, unlike parity or time reversal, it is still manifest. The right-handed fermion field transforms as

$$C\phi_i(x^-)C^{-1}=\phi_i^\dagger(x^-) . \tag{2.27}$$

If the vacuum is C invariant, we can deduce the following symmetry property of the density matrix:

$$\langle 0|\phi_i^\dagger(y^-)\phi_i(x^-)|0\rangle = \langle 0|\phi_i(y^-)\phi_i^\dagger(x^-)|0\rangle . \tag{2.28}$$

Using the anticommutation relations (2.10), this implies that

$$\rho(x^- - y^-)+\rho(y^- - x^-)=\frac{1}{\sqrt{2}}\delta(x^- - y^-) \tag{2.29}$$

or, in momentum space,

$$\rho(k)+\rho(-k)=\frac{1}{\sqrt{2}} . \tag{2.30}$$

This relation will frequently be used below.

So far, we have done all the formal manipulations without worrying about possible divergences and regularization. In the free theory, the density matrix $\rho(k)$ is simply given by

$$\rho(k)=\frac{1}{\sqrt{2}}\Theta(-k) \tag{2.31}$$

since all the negative momentum states of the right-handed fermions are filled (see, e.g., Ref. [18]). In general, one assumes that the interacting vacuum in light-cone quantization is the same as the free one. Therefore, the integral appearing in the Hartree equation (2.23) needs both UV and IR regularization. By contrast, in normal coordinates, only UV regularization is necessary [10]. In equal-time quantization, the renormalizability of the Gross-Neveu models guarantees that the coupling constant g^2 will always be accompanied by the same divergent integral if one calculates physical quantities, so that the final results do not depend on the cutoff [1]. As we shall see below, the same phenomenon occurs in light-cone quantization. We can introduce IR and UV cutoffs to define the integral in Eq. (2.23) properly; eventually, however, no physical results will depend on these cutoffs, so that we do not have to specify them in more detail at this point. In particular, there will be no need to relate UV and IR cutoffs, as has sometimes been done in light-cone quantization in order to reproduce known results from ordinary coordinates.

We now proceed to the next order in the large-N expansion (2.14). To order $1/\sqrt{N}$, we encounter the linearized equation of motion and constraint corresponding to the Tamm-Dancoff approximation (TDA) in many-body theory. Without serious loss of generality, we can assume a nonchiral model or the chiral model with the special choice $\varphi=0$. Equations (2.12) and (2.13) then yield the two equations

$$i\sqrt{2}\partial_+\tilde{Q}(x^-,y^-)=m[\tilde{A}(x^-,y^-)-\tilde{A}^\dagger(y^-,x^-)] -Ng^2a(x^--y^-)[(1-\lambda)\tilde{A}(x^-,x^-)+(1+\lambda)\tilde{A}^\dagger(x^-,x^-)-(1+\lambda)\tilde{A}(y^-,y^-) -(1-\lambda)\tilde{A}^\dagger(y^-,y^-)]\ , \tag{2.32}$$

$$i\sqrt{2}\frac{\partial}{\partial x^-}\tilde{A}(x^-,y^-)=m\tilde{Q}(x^-,y^-)-Ng^2\rho(x^--y^-)[(1+\lambda)\tilde{A}(x^-,x^-)+(1-\lambda)\tilde{A}^\dagger(x^-,x^-)]\ . \tag{2.33}$$

In order to obtain the usual light-cone formulation in terms of right-handed (dynamical) fermion fields only, we have to eliminate $\tilde{A},\tilde{A}^\dagger$ from Eq. (2.32), using the constraint (2.33). This is again most conveniently done in k space. The Fourier transform of the fluctuation operator $\tilde{Q}(x^-,y^-)$ will be defined by

$$\tilde{Q}(k',k)=\int dx^-dy^-e^{i(k'x^--ky^-)}\tilde{Q}(x^-,y^-)\ , \tag{2.34}$$

and similarly for $\tilde{A}$. Then, Eqs. (2.32) and (2.33) go over into

$$i\sqrt{2}\partial_+\tilde{Q}(k',k)=m[\tilde{A}(k',k)-\tilde{A}^\dagger(k,k')] -Ng^2\left\{[(1-\lambda)a(k)-(1+\lambda)a(k')]\int\frac{dq}{2\pi}\tilde{A}(q-k,q-k')+[(1+\lambda)a(k)-(1-\lambda)a(k')] \times\int\frac{dq}{2\pi}\tilde{A}^\dagger(q-k',q-k)\right\}\ , \tag{2.35}$$

$$\sqrt{2}k'\tilde{A}(k',k)=m\tilde{Q}(k',k)-Ng^2\rho(k)\left\{(1+\lambda)\int\frac{dq}{2\pi}\tilde{A}(q-k,q-k')+(1-\lambda)\int\frac{dq}{2\pi}\tilde{A}^\dagger(q-k',q-k)\right\}\ . \tag{2.36}$$

Let us sandwich these operator equations between the vacuum on the right and eigenstates of P and H with momentum P and energy $\mathcal{E}$ on the left (mesons, $q\bar{q}$ scattering states):

$$\langle\alpha,P|\begin{bmatrix}\tilde{Q}(k',k)\\ \tilde{A}(k',k)\\ \tilde{A}^\dagger(k,k')\end{bmatrix}|0\rangle=2\pi\delta(P-k+k')\begin{bmatrix}\phi(P,k)\\ \chi(P,k)\\ \chi'(P,k)\end{bmatrix}\ . \tag{2.37}$$

α denotes all other quantum numbers needed to specify the state completely. Using

$$i\partial_+\langle\alpha,P|\tilde{Q}(k',k)|0\rangle=\langle\alpha,P|[\tilde{Q}(k',k),H]|0\rangle=-\mathcal{E}\phi(P,k)\ , \tag{2.38}$$

the equation of motion (2.35) for $\tilde{Q}(k',k)$ becomes an eigenvalue equation for $\phi(P,k)$,

$$-\sqrt{2}\mathcal{E}\phi(P,k)=m[\chi(P,k)-\chi'(P,k)] -Ng^2\{[(1-\lambda)a(k)-(1+\lambda)a(k-P)]F(P)+[(1+\lambda)a(k)-(1-\lambda)a(k-P)]F'(P)\}\ , \tag{2.39}$$

where we have introduced the notation

$$F(P)=\int \frac{dq}{2\pi}\chi(P,q),\quad F'(P)=\int \frac{dq}{2\pi}\chi'(P,q)\ . \tag{2.40}$$

The constraint (2.36) (and its Hermitian conjugate) yield

$$\sqrt{2}(k-P)\chi(P,k)=m\phi(P,k)-Ng^2\rho(k)[(1+\lambda)F(P)+(1-\lambda)F'(P)]\ , \tag{2.41}$$

$$\sqrt{2}k\chi'(P,k)=m\phi(P,k)-Ng^2\rho(k-P)[(1+\lambda)F'(P)+(1-\lambda)F(P)]\ . \tag{2.42}$$

These last two equations can serve to eliminate $F(P)$ and $F'(P)$ and subsequently derive the desired equation for ϕ which contains only right-handed fermion fields. Since the case for general λ is rather nontransparent (and not necessary), we now specialize to $\lambda=0,1$.

(a) $\lambda=0$. Here, only the combination $F+F'$ enters Eqs. (2.39)–(2.42), and we find

$$F(P)+F'(P)=\frac{m}{\sqrt{2}}\int\frac{dk}{2\pi}\left[\frac{1}{k}-\frac{1}{P-k}\right]\phi(P,k)-\frac{Ng^2}{\sqrt{2}}\int\frac{dk}{2\pi}\left[\frac{\rho(k)}{k-P}+\frac{\rho(k-P)}{k}\right]\times[F(P)+F'(P)]\ . \tag{2.43}$$

Using charge conjugation in the form of Eq. (2.30), we can show that

$$\int\frac{dk}{2\pi}\frac{\rho(k-P)}{k}=\int\frac{dk}{2\pi}\frac{\rho(k)}{k-P} \tag{2.44}$$

(here, one has to use $\int dk/k=0$, i.e., to interpret this integral as a principal value integral). Therefore,

$$F(P)+F'(P)=Z(P)\frac{m}{\sqrt{2}}\int\frac{dk}{2\pi}\left[\frac{1}{k}-\frac{1}{P-k}\right]\phi(P,k) \tag{2.45}$$

with the definition

$$Z^{-1}(P):=1+\sqrt{2}Ng^2\int\frac{dk}{2\pi}\frac{\rho(k-P)}{k}\ . \tag{2.46}$$

Later, we shall see that $Z(P)$ is in fact a constant independent of P, but we do not wish to put in any prejudice at this stage.

(b) $\lambda=1$. In this case, the equations for F and F' derived from (2.41) and (2.42) decouple, and

$$F(P)=-Z(P)\frac{m}{\sqrt{2}}\int\frac{dk}{2\pi}\frac{1}{P-k}\phi(P,k)\ ,\qquad F'(P)=Z(P)\frac{m}{\sqrt{2}}\int\frac{dk}{2\pi}\frac{1}{k}\phi(P,k)\ . \tag{2.47}$$

Combining Eqs. (2.39)–(2.47), we get after some algebra the following eigenvalue equations for ϕ.

$\lambda=0$:

$$\mathcal{E}\phi(P,k)=\frac{m^2}{2}\left[\frac{1}{k}+\frac{1}{P-k}\right]\phi(P,k)-\frac{Ng^2m^2}{2\sqrt{2}}Z(P)[\rho(k-P)-\rho(k)]\left[\frac{1}{k}-\frac{1}{P-k}\right]\times\int\frac{dk'}{2\pi}\left[\frac{1}{k'}-\frac{1}{P-k'}\right]\phi(P,k')\ ; \tag{2.48}$$

$\lambda=1$:

$$\mathcal{E}\phi(P,k)=\frac{m^2}{2}\left[\frac{1}{k}+\frac{1}{P-k}\right]\phi(P,k)-\frac{Ng^2m^2}{\sqrt{2}}Z(P)[\rho(k-P)-\rho(k)]\times\int\frac{dk'}{2\pi}\left[\frac{1}{kk'}+\frac{1}{(P-k)(P-k')}\right]\phi(P,k')\ . \tag{2.49}$$

In either case, we recognize in the first term on the right-hand side the standard kinetic energy in light-cone coordinates. The interaction term is separable, consisting of one term for $\lambda=0$ and two terms for $\lambda=1$. Notice that both cases can again be recombined by writing the integral as

$$\int\frac{dk'}{2\pi}\left[\left[\frac{1}{k}-\frac{1}{P-k}\right]\left[\frac{1}{k'}-\frac{1}{P-k'}\right]+\lambda\left[\frac{1}{k}+\frac{1}{P-k}\right]\left[\frac{1}{k'}+\frac{1}{P-k'}\right]\right]\phi(P,k')\ . \tag{2.50}$$

This form exhibits in a more transparent way the symmetry properties under charge conjugation. Using Eqs. (2.14), (2.27), (2.34), and (2.37), we find

$$\langle\tilde{\alpha},P|\tilde{Q}(k',k)|0\rangle=-\langle\alpha,P|\tilde{Q}(-k,-k')|0\rangle\ , \tag{2.51}$$

where

$$|\tilde{\alpha},P\rangle=C|\alpha,P\rangle \tag{2.52}$$

is the charge conjugate state. Thus, charge conjugation corresponds (up to a phase) to exchanging k and $P-k$. H conserves the symmetry property of the wave function $\phi(P,k)$ under this transformation, and the first term in (2.50) acts only in odd, the second only in even states.

Equations (2.48) and (2.49) are the main result of this section. Together with the definition of the "renormalization constant" Z, Eq. (2.46), and the Hartree condition (2.23), they fully define the Tamm-Dancoff equation in light-cone coordinates from which properties of $q\bar{q}$ bound and scattering states can be derived. We wish to emphasize that so far we have neither replaced $\rho(k)$ by the anticipated step function of the free vacuum, nor have we introduced the usual rescaled variable $x=k/P$ running from 0 to 1. The reason is that due to the renormalization, these manipulations are somewhat delicate in the present case and may lead to ill-defined expressions. In the following section, where we consider some specific applications of this formalism, we shall see that it is nevertheless possible to avoid all ambiguities, provided one consistently uses the Hartree equation. This will also shed some light on the physical role of $Z(P)$, Eq. (2.46), which we have introduced for purely notational reasons above.

III. APPLICATIONS TO BOUND-STATE AND SCATTERING PROBLEMS

In the present section, we shall demonstrate the soundness of the light-cone formalism developed above by applying it to a few specific cases. Let us start with the nonchiral, massless Gross-Neveu model ($\lambda=0, m_0=0$). The Tamm-Dancoff equation (2.48) for $\phi(P,k)$ has the form of a Schrödinger equation with separable potential, and can be dealt with in exactly the same way as the analogue nonrelativistic problem. Before doing this, however, it is useful to introduce an effective coupling constant as follows:

$$g^2_{\rm eff}(P) := g^2 Z(P) \tag{3.1}$$

[$Z(P)$ has been previously defined in Eq. (2.46); its P dependence will eventually turn out to be spurious, so that the name "coupling constant" for $g_{\rm eff}$ is justified.] Using the Hartree condition (2.23) and charge conjugation, one can derive the following relation for $g^2_{\rm eff}$:

$$1=\frac{Ng^2_{\rm eff}(P)}{\sqrt{2}}\int\frac{dk}{2\pi}[\rho(k-P)-\rho(k)]\left[\frac{1}{k}+\frac{1}{P-k}\right] . \tag{3.2}$$

This equation can be regarded as renormalization prescription for the effective coupling constant $g^2_{\rm eff}(P)$, in the same way as the Hartree equation is used to renormalize the bare coupling constant g^2. Now let us exploit the similarity of Eq. (2.48) with the Schrödinger equation and use standard time-independent scattering theory to derive the $q\bar{q}$ scattering matrix. We write the Hamiltonian underlying Eq. (2.48) as

$$H=H_0+V \tag{3.3}$$

with the (light-cone) kinetic energy H_0,

$$\langle k|H_0(P)|k'\rangle = 2\pi\delta(k-k')\frac{m^2}{2}\left[\frac{1}{k}+\frac{1}{P-k}\right] \tag{3.4}$$

and the separable potential V,

$$\langle k|V(P)|k'\rangle=\langle k|\tilde{g}_P\rangle\lambda(P)\langle g_P|k'\rangle . \tag{3.5}$$

The "form factors" of this potential are identified as

$$\begin{aligned}\langle k|g_P\rangle&=\frac{1}{k}-\frac{1}{P-k} ,\\ \langle k|\tilde{g}_P\rangle&=[\rho(k-P)-\rho(k)]\langle k|g_P\rangle ,\end{aligned} \tag{3.6}$$

and the "coupling constant" is given by

$$\lambda(P)=-\frac{Ng^2_{\rm eff}(P)m^2}{2\sqrt{2}} . \tag{3.7}$$

Both H_0 and V are diagonal in the total momentum P as required by translational invariance; however, unlike in the nonrelativistic case, V has a nontrivial P dependence. Like V, the T matrix will have a separable form with the same form factors:

$$\langle k|T(\mathcal{E},P)|k'\rangle=\langle k|\tilde{g}_P\rangle\tau(\mathcal{E},P)\langle g_P|k'\rangle . \tag{3.8}$$

The Lippmann-Schwinger equation relates $\tau(\mathcal{E},P)$ to $\lambda(P)$ via

$$\tau(\mathcal{E},P)=\frac{\lambda(P)}{1-\lambda(P)\langle g_P|G_0(\mathcal{E},P)|\tilde{g}_P\rangle} , \tag{3.9}$$

where G_0 denotes the free Green's function

$$G_0(\mathcal{E},P)=[\mathcal{E}-H_0(P)+i\epsilon]^{-1} . \tag{3.10}$$

Using Eqs. (3.2) and (3.7), we can transform Eq. (3.9) into

$$-\frac{1}{\tau(\mathcal{E},P)}=\frac{2P}{m^2}\int\frac{dk}{2\pi}\frac{\rho(k-P)-\rho(k)}{k(P-k)} +\langle g_P|G_0(\mathcal{E},P)|\tilde{g}_P\rangle . \tag{3.11}$$

The first term on the right-hand side is potentially dangerous. If we would try to replace ρ by a step function in here, we would encounter ill-defined expressions. However, this term is precisely cancelled by a corresponding term in the matrix element of G_0, so that the sum of the two terms is well behaved. Indeed, we have

$$\begin{aligned}\langle g_P|G_0(\mathcal{E},P)|\tilde{g}_P\rangle&=\int\frac{dk}{2\pi}[\rho(k-P)-\rho(k)]\frac{\left[\frac{1}{k}-\frac{1}{P-k}\right]^2}{\mathcal{E}-\frac{m^2}{2}\left[\frac{1}{k}+\frac{1}{P-k}\right]+i\epsilon}\\ &=-\frac{2P}{m^2}\int\frac{dk}{2\pi}[\rho(k-P)-\rho(k)]\left[\frac{1}{k(P-k)}-\frac{2P\mathcal{E}-4m^2}{2P\mathcal{E}k(P-k)-P^2m^2+i\epsilon}\right] .\end{aligned} \tag{3.12}$$

The announced cancellation takes place and the net result, using the standard notation

$$s=2P\mathcal{E} \tag{3.13}$$

for the square of the invariant mass of the $q\bar{q}$ system, becomes

$$-\frac{1}{\tau(s,P)}=\frac{2P}{m^2}(s-4m^2)\int\frac{dk}{2\pi}\frac{\rho(k-P)-\rho(k)}{sk(P-k)-m^2P^2+i\epsilon} . \tag{3.14}$$

At this point, the integral does not contain unwanted singularities any more, and we can safely replace $\rho(k)$ by the free light-cone vacuum expression (2.31). Rescaling variables in the usual way via $k=xP$, we find that τ depends only on s and has the simple, Lorentz invariant form

$$-\frac{1}{\tau(s)}=\frac{\sqrt{2}(s-4m^2)}{m^2}\int_0^1\frac{dx}{2\pi}\frac{1}{sx(1-x)-m^2+i\epsilon}\ . \tag{3.15}$$

$\tau(s)$ has a pole at $s=4m^2$, exhibiting the well-known $q\bar{q}$ bound state at threshold in the large-N limit of the Gross-Neveu model [1]. The final result (3.15) is expressed in terms of the physical fermion mass, whereas the bare coupling constant has disappeared, in agreement with the "dimensional transmutation" discussed in Ref. [1]. The fermion loop integral on the right-hand side of Eq. (3.15) is already in the simple form usually obtained in covariant perturbation theory, using the Feynman parameter, and can now be trivially evaluated:

$$-\frac{1}{\tau(s)}=\begin{cases}-\dfrac{\sqrt{2}}{\pi m^2}\sqrt{1-\eta}\left[\ln\dfrac{1-\sqrt{1-\eta}}{1+\sqrt{1-\eta}}+i\pi\right] & \text{for } \eta<1\ ,\\ \dfrac{2\sqrt{2}}{\pi m^2}\sqrt{\eta-1}\arctan\dfrac{1}{\sqrt{\eta-1}} & \text{for } \eta>1\ .\end{cases} \tag{3.16}$$

Here we have defined

$$\eta:=4m^2/s\ , \tag{3.17}$$

i.e., $\eta<1$ corresponds to scattering kinematics. This simple calculation should be contrasted with the corresponding calculation in normal coordinates which leads to much more "ugly" integrals, and where the Lorentz invariance is deeply hidden (see the following section).

As a second application, we consider the chirally invariant, massless model ($\lambda=1$, $m_0=0$). The Hamiltonian which can be inferred from Eq. (2.49) now involves a two-term separable interaction:

$$\langle k|V(P)|k'\rangle=\lambda(P)(\langle k|\tilde{g}_P\rangle\langle g_P|k'\rangle+\langle k|\tilde{f}_P\rangle\langle f_P|k'\rangle)\ . \tag{3.18}$$

Here, $\lambda(P)$, g_P, and $\tilde{g}_P$ are as defined in Eqs. (3.6) and (3.7), whereas the new form factors f_P and $\tilde{f}_P$ differ by a relative sign from the previous ones:

$$\begin{aligned}\langle k|f_P\rangle&=\frac{1}{k}+\frac{1}{P-k}\ ,\\ \langle k|\tilde{f}_P\rangle&=[\rho(k-P)-\rho(k)]\langle k|f_P\rangle\ .\end{aligned} \tag{3.19}$$

Since H preserves C parity as pointed out above, we can discuss separately the even and odd states under interchange of k and $P-k$. In the odd sector, only the term $\sim\langle k|\tilde{g}_P\rangle\langle g_P|k'\rangle$ in the interaction acts and everything is identical to the previous case ($\lambda=0$). In the even sector, only the second term $\sim\langle k|\tilde{f}_P\rangle\langle f_P|k'\rangle$ contributes and Eq. (2.49) reduces to

$$\mathcal{E}\phi(P,k)=\frac{m^2}{2}\left[\frac{1}{k}+\frac{1}{P-k}\right]\left[\phi(P,k)-\frac{Ng^2_{\text{eff}}(P)}{\sqrt{2}}[\rho(k-P)-\rho(k)]\int\frac{dk'}{2\pi}\left[\frac{1}{k'}+\frac{1}{P-k'}\right]\phi(P,k')\right]\ . \tag{3.20}$$

Here, the expected massless meson ("pion") appears, signaling the breaking of chiral symmetry. Indeed, the ansatz

$$\phi(P,k)=\rho(k-P)-\rho(k) \tag{3.21}$$

solves Eq. (3.20) with the eigenvalue $\mathcal{E}=0$ [it just converts the expression inside the large square brackets into the renormalization condition for $g^2_{\text{eff}}(P)$, Eq. (3.2)]. Repeating the steps performed in the $\lambda=0$ case, the T matrix in the even channel can be derived as

$$\begin{aligned}-\frac{1}{\tau(s)}&=\frac{2P}{m^2}\int\frac{dk}{2\pi}\frac{\rho(k-P)-\rho(k)}{k(P-k)}+\int\frac{dk}{2\pi}[\rho(k-P)-\rho(k)]\frac{[1/k+1/(P-k)]^2}{\mathcal{E}-(m^2/2)[1/k+1/(P-k)]+i\epsilon}\\ &=\frac{\sqrt{2}s}{m^2}\int_0^1\frac{dx}{2\pi}\frac{1}{sx(1-x)-m^2+i\epsilon}\ .\end{aligned} \tag{3.22}$$

Except for the replacement of an overall factor $s-4m^2$ by s, the result coincides with the one above. Again, the bare coupling constant has disappeared in favor of the physical fermion mass.

Summarizing the procedure followed so far, we emphasize the crucial role played by the renormalization condition. If we had attempted to set $\rho(k)=\Theta(-k)/\sqrt{2}$ before exploiting this condition, we would invariably have run into ill-defined distributions like $\Theta(x)/x$. On the level of the renormalized, finite expressions, however, there is no such problem.

Once we are aware of this subtlety, we can further simplify the TDA equations and cast them into the standard light-cone form. Although this does not yield any new results, it is instructive and may be of interest when comparing different field theories in light-cone quantization. Using rescaled momenta and step functions for $\rho(k)$, we simply rewrite Eqs. (2.48) and (2.49) as

$$s\phi(x)=m^2\left[\frac{1}{x}+\frac{1}{1-x}\right]\phi(x)-\frac{Ng^2_{\text{eff}}m^2}{4\pi}\int_0^1dy\left[\left[\frac{1}{x}-\frac{1}{1-x}\right]\left[\frac{1}{y}-\frac{1}{1-y}\right]+\lambda\left[\frac{1}{x}+\frac{1}{1-x}\right]\left[\frac{1}{y}+\frac{1}{1-y}\right]\right]\phi(y) \tag{3.23}$$

and the renormalization condition in the form

$$1=\frac{Ng_{\rm eff}^2}{2\pi}\int_0^1\frac{dx}{x}=\frac{Ng_{\rm eff}^2}{2\pi}\int_0^1\frac{dx}{1-x} . \tag{3.24}$$

The integrals appearing in (3.23) and (3.24) should be understood as shorthand notation for

$$\lim_{\delta\to 0}\int_\delta^{1-\delta}dx(\cdots) ,$$

so that all the expressions are well defined; x is now restricted to the interval [0,1]. Equation (3.23) is the most compact form of the 't Hooft equation for the massless Gross-Neveu models. From this equation and the renormalization condition (3.24), one can now easily rederive the T matrix or the bound state properties discussed in the present section. In particular, the "pion" wave function is simply given by $\phi(x)=\text{const}$ $(0\le x<1)$, exactly as in the chiral limit of large-N QCD_2.

The transition from g^2 to $g_{\rm eff}^2$ can now be interpreted as follows. In the rescaled variables, the Hartree equation (2.23) reads

$$\begin{aligned}1&=-\frac{Ng^2}{2\pi}\int_{-\infty}^0\frac{dx}{x}\\ &=\frac{Ng^2}{2\pi}\left[\int_0^1\frac{dx}{x}+\int_1^\infty\frac{dx}{x}\right] .\end{aligned}\tag{3.25}$$

Taking the second integral to the left-hand side and dividing through, this can be written in the equivalent form

$$1=\frac{Ng_{\rm eff}^2}{2\pi}\int_0^1\frac{dx}{x} , \tag{3.26}$$

with

$$g_{\rm eff}^2=\left[1-\frac{Ng^2}{2\pi}\int_1^\infty\frac{dx}{x}\right]^{-1}g^2 . \tag{3.27}$$

On the other hand, using our original definition of $Z(P)$, Eq. (2.46), we have

$$\begin{aligned}Z^{-1}(P)&=1+\sqrt{2}Ng^2\int\frac{dk}{2\pi}\frac{\rho(k-P)}{k}\\ &=1+\frac{Ng^2}{2\pi}\int\frac{dk}{k}\Theta(P-k)\\ &=1-\frac{Ng^2}{2\pi}\int_1^\infty\frac{dx}{x} .\end{aligned}\tag{3.28}$$

Thus, $g_{\rm eff}^2$ of Eq. (3.27) agrees with $g_{\rm eff}^2(P)$ introduced above in Eq. (3.1), and we see that Z is indeed independent of P. g^2 is the bare coupling constant, whereas $g_{\rm eff}^2=Zg^2$ can be interpreted as an effective coupling constant to be used when working only in the interval [0,1]. The remainder of the k axis has simply been projected out. It is plausible that the quark momenta for a $q\bar{q}$ state of momentum P are not strictly confined to $0<k<P$. Otherwise, it would indeed be hard to understand dynamical mass generation which requires some (residual) vacuum dynamics. However, everything which goes on outside the interval [0,1] can apparently be absorbed in a redefinition of the coupling constant. These remarks will be further clarified in the following section, where we rederive the same results in ordinary coordinates, using the infinite momentum frame.

As third and last application, we consider the massive Gross-Neveu models. The Hartree condition (2.23) now contains the ratio m_0/m; this implies the following condition for $g_{\rm eff}^2$:

$$1-\frac{m_0}{m}\frac{g_{\rm eff}^2(P)}{g^2}=\frac{Ng_{\rm eff}^2(P)}{\sqrt{2}}\int\frac{dk}{2\pi}[\rho(k-P)-\rho(k)]\times\left[\frac{1}{k}+\frac{1}{P-k}\right] . \tag{3.29}$$

The Tamm-Dancoff equation, on the other hand, is unchanged, so that the bare mass enters only through this renormalization condition.

By way of example, for $\lambda=0$ (nonchiral Gross-Neveu model), the $q\bar{q}$ problem can again be phrased in terms of $\tau(\mathcal{E},P)$ of Eq. (3.11), which still holds. Imposing the modified Hartree condition (2.23) for $m_0\neq 0$ and using the notation

$$\lambda(P)=\lambda_0 Z(P),\quad \lambda_0=-\frac{Ng^2m^2}{2\sqrt{2}} , \tag{3.30}$$

we find the simple result

$$\frac{1}{\tau(s;m,m_0)}=\frac{m_0}{m\lambda_0}+\frac{1}{\tau(s;m,0)} , \tag{3.31}$$

where $\tau(s;m,m_0=0)$ stands for $\tau(s)$ as used above. Since both terms on the right-hand side of Eq. (3.31) are negative, there is no bound-state pole any more; the marginally bound meson disappears for $m_0\neq 0$. In the chiral model with a symmetry-breaking mass term, we can discuss again the even and odd channels separately. For the even states, we find the following eigenvalue condition for a bound state of mass μ:

$$\frac{1}{\sqrt{\eta-1}}\arctan\frac{1}{\sqrt{\eta-1}}=\frac{\pi m_0}{Ng^2m},\quad \eta=\frac{4m^2}{\mu} . \tag{3.32}$$

In the limit $m_0\to 0$, we can apply mass perturbation theory, exploiting the fact that $\eta\to\infty$:

$$\frac{\mu^2}{4m^2}=\frac{1}{\eta}\simeq\frac{\pi m_0}{Ng^2m} . \tag{3.33}$$

We recover the standard relation between pion mass, quark mass, and condensate, well known from QCD_2:

$$\mu^2=\frac{4\pi m_0 m}{Ng^2}=-\frac{4\pi}{N}m_0\langle\bar{\psi}\psi\rangle . \tag{3.34}$$

Once again, the corresponding calculation in normal coordinates requires significantly more effort; see, e.g., Ref. [11] for both QCD_2 and the Gross-Neveu model. Finally, we note that the 't Hooft equation in the shorthand notation of Eq. (3.23) is still valid for the massive case. The dependence on m_0 appears only in the renormalization condition (3.24), which has to be replaced [cf. Eq. (3.29)] by

$$1-\frac{m_0}{m}\frac{g_{\rm eff}^2}{g^2}=\frac{Ng_{\rm eff}^2}{2\pi}\int_0^1\frac{dx}{x} . \tag{3.35}$$

IV. INFINITE MOMENTUM FRAME

Here, we shall rederive the main results of Secs. II and III, using standard equal time quantization together with the old idea of the infinite momentum frame [13]. First, this will serve to prove the equivalence of the two approaches for the particular case of the Gross-Neveu models. More importantly, it provides a framework in which the light-cone formulation emerges more naturally as a "limit" than in straight light-cone quantization, and which is therefore conceptually simpler. In particular, the relationship between renormalization and nontrivial vacuum structure and the role of the effective coupling constant $g^2_{\rm eff}$ can be understood more easily.

The nonchiral, massless Gross-Neveu model [i.e., $\lambda=0$, $m_0=0$ in the notation of Eq. (2.1)] has recently been studied in ordinary coordinates, using the same type of "equations of motion approach" as here [10]. Since this particular example is sufficient for our present purpose, we do not have to start all over again but can use many of the results of Ref. [10]. The reader is referred to this paper for further details of the formalism in ordinary coordinates.

In equal-time quantization, the fermion-antifermion Tamm-Dancoff equation assumes the standard matrix form of the "random phase approximation" (RPA):

$$\begin{bmatrix} A & B \\ -B & -A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \mathcal{E} \begin{bmatrix} X \\ Y \end{bmatrix} . \tag{4.1}$$

Here, X and Y are matrix elements of particle-hole creation and annihilation operators, respectively, between vacuum and $q\bar{q}$ states, reflecting the possibilities of forward and backward propagating fermions. The integral operators A and B have been derived in Ref. [10]. Interpreting Eq. (4.1) as a two-channel Schrödinger equation and using the explicit form of A and B given in Ref. [10], we can identify the following Hamiltonian: The kinetic energy part is given by

$$\langle k|H_0(P)|k'\rangle = 2\pi\delta(k-k')E(k-P,k) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.2}$$

with

$$E(k-P,k)=E(k-P)+E(k) , \quad E(k)=\sqrt{m^2+k^2} \tag{4.3}$$

the (Hartree) single-particle energies. The potential has the separable form

$$\langle k|V(P)|k'\rangle = Ng^2\bar{v}(k-P)u(k)\bar{v}(k')u(k'-P) \times \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} . \tag{4.4}$$

Here, $u(k)$ and $v(k)$ are the positive and negative energy Hartree spinors which in the present case are just free, massive spinors. The physical fermion mass m is defined by the Hartree condition

$$1=\frac{Ng^2}{2\pi}\int \frac{dk}{E(k)} . \tag{4.5}$$

Clearly, the right-hand side needs an UV cutoff, and we interpret Eq. (4.5) in the usual way as renormalization prescription for the bare coupling constant g^2. We now proceed as in light-cone quantization, i.e., derive the T matrix corresponding to the Hamiltonian $H=H_0+V$, Eqs. (4.2)–(4.4). V is separable and it is easy to see that the ansatz

$$\langle k|T(\mathcal{E},P)|k'\rangle = \tau(\mathcal{E},P)\bar{v}(k-P)u(k)\bar{v}(k')u(k'-P) \times \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \tag{4.6}$$

satisfies the Lippmann-Schwinger equation, provided we choose

$$\tau(\mathcal{E},P)=\frac{Ng^2}{1-Ng^2I(\mathcal{E},P)} . \tag{4.7}$$

The integral $I(\mathcal{E},P)$ is defined as the matrix element of the free Green's function between the form factors of the separable potential:

$$I(\mathcal{E},P)=\int \frac{dk}{2\pi}\bar{v}(k)u(k-P)\bar{v}(k-P)u(k) \times \left[\frac{1}{\mathcal{E}-E(k-P,k)+i\epsilon} - \frac{1}{\mathcal{E}+E(k-P,k)} \right] . \tag{4.8}$$

In ordinary coordinates, the calculation can be carried through as follows: Evaluating the integrand of Eq. (4.8), one obtains [10]

$$I(\mathcal{E},P)=\frac{1}{2}\int \frac{dk}{2\pi}\frac{4m^2+P^2-E^2(k-P,k)}{\mathcal{E}^2-E^2(k-P,k)+i\epsilon} \times \left[\frac{1}{E(k)}+\frac{1}{E(k-P)} \right] . \tag{4.9}$$

This integral is logarithmically divergent and can be split into divergent and convergent parts:

$$I(\mathcal{E},P)=I_{\rm div}+I_{\rm conv}(\mathcal{E},P) \tag{4.10}$$

with

$$I_{\rm div}=\frac{1}{2}\int \frac{dk}{2\pi}\left[\frac{1}{E(k)}+\frac{1}{E(k-P)} \right] \tag{4.11}$$

and

$$I_{\rm conv}(\mathcal{E},P)=\frac{1}{2}\int \frac{dk}{2\pi}\frac{4m^2+P^2-\mathcal{E}^2}{\mathcal{E}^2-E^2(k-P,k)+i\epsilon} \times \left[\frac{1}{E(k)}+\frac{1}{E(k-P} \right] . \tag{4.12}$$

The divergent constant $I_{\rm div}$ is exactly canceled by the 1 in the denominator of Eq. (4.7), if we invoke the Hartree condition (4.5). Therefore, the net result for $\tau(\mathcal{E},P)$ is finite and independent of the bare coupling constant, as it should be:

$$\tau(\mathcal{E},P)=-\frac{1}{I_{\rm conv}(\mathcal{E},P)} . \tag{4.13}$$

$I_{\rm conv}$ can be evaluated analytically, although the calculation is much more tedious than the corresponding light-cone calculation, and the result agrees with the one obtained above [22]. However, this is not our principal goal here. We shall only consider the integral $I(\mathcal{E},P)$ in the limit of very large momenta P and demonstrate how it can be used to rederive the light-cone expressions. First, let us split the k axis for the fermions or antifermions into two complementary regions of momenta:

$$\begin{aligned} &\text{region 1:}\;\; \Delta < k < P-\Delta \;; \\ &\text{region 2:}\;\; k<\Delta \;\;\text{or}\;\; k>P-\Delta \;. \end{aligned} \tag{4.14}$$

Here, Δ is a positive momentum, and we shall assume the hierarchy

$$P \gg \Delta \gg m \;, \tag{4.15}$$

for reasons which will soon become clear. Our aim is to eliminate region 2 and set up an effective theory defined exclusively in region 1. In the limit $P\to\infty$ (infinite momentum frame), this effective theory will be shown to become identical to the light-cone formulation.

In the $q\bar{q}$ scattering problem, the imaginary part of $I(\mathcal{E},P)$ arises from the kinematical point where the fermion and antifermion in the intermediate state have the same momenta as in the initial state (up to a possible exchange). If we restrict ourselves to initial momenta in region 1, we can trivially restrict ourselves to region 1 when evaluating ImI. Because of Lorentz invariance, this assumption is perfectly harmless. As far as the real part is concerned, let us first consider the convergent part $I_{\rm conv}$. In region 1, the square roots in the single-particle energies (4.3) can be expanded [this is, of course, the motivation for requiring the inequalities (4.15)]. Rescaling the integration variable via $k=Px$ and introducing $\delta=\Delta/P$, we find

$$I^{(1)}_{\rm conv}(s)=\frac{1}{2}(4m^2-s)\int_{\delta}^{1-\delta}\frac{dx}{2\pi}\frac{1}{sx(1-x)-m^2+i\epsilon} \;, \tag{4.16}$$

where s is again the invariant mass of the fermion-antifermion state:

$$s=\mathcal{E}^2-P^2 \;. \tag{4.17}$$

The contribution to $I_{\rm conv}$ coming from region 2 can be neglected in the limit $P\to\infty$. To see this, we first note that it is sufficient to consider the region $k>P-\Delta$, owing to the symmetry of the integrand under exchange of k and $P-k$. Since the integral $I_{\rm conv}$ is convergent, it is evident that the contribution from $k>P-\Delta$ will be suppressed for large P (a more detailed analysis reveals that it is of order $1/P$). The divergent part on the other hand, $I_{\rm div}$, is independent of P and receives comparable contributions from regions 1 and 2. If we nevertheless want to project out region 2, we can do this by introducing an effective coupling constant. First, we write (4.7) in the limit $P\to\infty$ more explicitly as

$$\tau(s)=\frac{Ng^2}{1-Ng^2[I^{(1)}_{\rm div}+I^{(2)}_{\rm div}+I^{(1)}_{\rm conv}(s)]} \;. \tag{4.18}$$

Defining

$$g^2_{\rm eff}=\frac{g^2}{1-Ng^2I^{(2)}_{\rm div}} \;, \tag{4.19}$$

we obtain

$$\tau(s)=\frac{Ng^2_{\rm eff}}{1-Ng^2_{\rm eff}[I^{(1)}_{\rm div}+I^{(1)}_{\rm conv}(s)]} \;. \tag{4.20}$$

The renormalization condition for $g^2_{\rm eff}$ becomes

$$1=\frac{Ng^2_{\rm eff}}{2\pi}\int_{\Delta}^{P-\Delta}\frac{dk}{E(k)} \;, \tag{4.21}$$

where we have used the Hartree condition (4.5) and Eq. (4.19). Since, in view of condition (4.15), we can approximate $E(k)$ by k in region 1, this yields

$$1=\frac{Ng^2_{\rm eff}}{2\pi}\int_{\delta}^{1-\delta}\frac{dx}{x}=Ng^2_{\rm eff}I^{(1)}_{\rm div} \;. \tag{4.22}$$

Substituting Eq. (4.22) into Eq. (4.20), we finally get

$$\tau(s)=-\frac{1}{I^{(1)}_{\rm conv}(s)} \;. \tag{4.23}$$

In the limit $P\to\infty$, if Δ is kept constant, δ goes to zero and Eqs. (4.16) and (4.23) agree with the light-cone result, Eq. (3.15) of Sec. III, up to an overall factor. This different normalization is due to the fact that the form factors of the separable potential were defined differently in light-cone and ordinary coordinates. Using the explicit form of the spinors from Ref. [10], one finds, for region 1 [as defined in (4.15)],

$$\bar{v}(k-P)u(k)\bar{v}(k')u(k'-P) = -\frac{m^2}{4}\left[\frac{1}{k}-\frac{1}{P-k}\right]\left[\frac{1}{k'}-\frac{1}{P-k'}\right] . \tag{4.24}$$

Inserting this result into Eq. (4.4) for V and comparing with Eqs. (3.5)–(3.7), the difference in normalization can be accounted for. Similarly, the Hartree condition (4.22) and the definition of the effective coupling constant (4.19) coincide with Eqs. (3.26) and (3.27) above.

This derivation shows clearly that in intermediate states, the fermion or antifermion momenta are not confined to $0<k<P$. However, if we restrict ourselves to $q\bar{q}$ pairs with very large center-of-mass momentum, the contribution from outside this interval becomes trivial in the sense that it does not depend on the dynamics (i.e., the variable s). Therefore, it can be taken into account by means of an effective coupling constant. The status of the vacuum from this point of view can be summarized as follows: In region 1 which is used to define the effective theory, the vacuum indeed looks trivial, since one can neglect the fermion mass in the expression for the single particle energy. Outside this region, the fermion mass plays a role and the vacuum is definitely different from the free massless vacuum. However, provided we look only at fast moving $q\bar{q}$ pairs, this dynamics is hidden in the coupling constant and modified renormalization condition of the effective theory.

V. SUMMARY AND CONCLUSIONS

In this paper, we have applied continuum light-cone quantization to the Gross-Neveu model "family." The motivation for this study was the desire to understand in a simple, but nonperturbative case the implications of renormalizability on light-cone dynamics. On the technical side, the main tool was the $1/N$ expansion in the form of the equations of motion approach. This method can be taken over almost literally from equal time quantization, the only difference being that certain dynamical equations become equations of constraint and have to be used in order to eliminate (nondynamical) left-handed fermion fields.

To zeroth order or equivalently at the Hartree level, the situation in light-cone and ordinary coordinates is very similar: Dynamical fermion mass generation occurs. However, the value of the mass (or condensate) cannot be predicted, but has to be put in by hand via the renormalization condition. This is different from the two-dimensional gauge theories where the coupling constant provides a mass scale which determines the condensate. In the (massless) Gross-Neveu models, the dimensionless coupling constant can be eliminated in favor of the physical fermion mass in all physical observables, an example of dimensional transmutation. In the present formalism, this is achieved by using consistently the Hartree equation, either in light-cone or in normal coordinates.

To order $1/\sqrt{N}$, fluctuations are considered and $q\bar{q}$ scattering and boundstates come into focus. In equal-time quantization, the Bethe-Salpeter equation has the standard RPA form, describing forward and backward moving pairs. In light-cone quantization, we have derived the analogue of the 't Hooft equation with the expected simpler TDA structure by systematically eliminating the left-handed fermion fields. This light-cone Tamm-Dancoff equation has two conspicuous features: First, the interaction is of separable type, and secondly, the coupling constant gets automatically renormalized in the course of eliminating the left-handed fermions. The first feature is shared by the RPA equation in ordinary coordinates and is at the origin of the fact that these models can be solved analytically. The second feature, described by the constant Z, is specific for the light cone, and could be understood in terms of projecting out a certain region of fermion momenta. We have shown that it is possible to write the Tamm-Dancoff equation in the usual light-cone form with the variable $x\in[0,1]$, although with an effective coupling constant and a corresponding modified renormalization condition. Applications of this formalism to the nonchiral, the chirally invariant and the massive Gross-Neveu models allowed us to derive scattering and bound-state properties of the fermion-antifermion system in a concise way, much more elegantly than in ordinary coordinates.

Finally, we have reproduced the main results independently by working in normal coordinates, in the infinite momentum limit. Here, the light-cone theory could be understood as an effective theory defined for a certain range of momenta. Apart from the infinite momentum frame, we had to use one other "trick," namely, prevent fermion or antifermion momenta from coming close to the region $k\sim m$. This was achieved by introducing the momentum Δ in defining the effective theory and enabled us to work with a "trivial vacuum," thus playing a similar role as the small interval in Ref. [18]. In this way, the appearance of the effective coupling constant, which is somewhat miraculous in the light-cone derivation, is made very compelling. Of course, this infinite momentum calculation is not meant as a substitute for the light-cone calculation. After all, there is little gain from the light-cone if one first has to go through the whole derivation in normal coordinates, with all its spinorial complications. The idea was to have an independent confirmation of the first derivation and try to clarify the structure and physics content of the light-cone TDA equation.

This study shows that there are cases where UV renormalization and light-cone techniques can be reconciled without destroying the well-known advantages of light-cone quantization. As far as the problem of symmetry breaking is concerned, the situation here is perhaps less mysterious than in other cases, since renormalization has forced us into keeping some residual, nontrivial vacuum structure. To verify the consistency of our procedure, one should evaluate matrix elements of currents and verify the corresponding Ward identities. This has not been done yet, but since we have via the infinite momentum frame approach a way to relate the light-cone to the ordinary coordinate formulation, we believe that possible difficulties can be resolved.

It will be interesting to try to apply similar methods to other, more complicated field theories. As far as the Gross-Neveu models are concerned, there is one problem which we have avoided so far: the description of baryons. As discussed in Refs. [1,10], baryons can be calculated by means of the Hartree approximation in the large-N limit, provided the effects of the Dirac sea are included self-consistently. In order to be able to take advantage of the light-cone quantization, one needs to understand clearly how to boost such objects. Although this can be done in principle, there are still some technical problems in practice which are presently under investigation. The solution of these problems may possibly give a handle on the more general issue of how to describe and quantize solitons on the light cone.

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