

Role of zero modes in the canonical quantization of heavy-fermion QED in light-cone coordinates

Robert W. Brown, Jin Woo Jun,* Shmaryu M. Shvartsman, and Cyrus C. Taylor

Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106

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Four-dimensional heavy-fermion QED is studied in light-cone coordinates with (anti)periodic field boundary conditions. We carry out a consistent light-cone canonical quantization of this model using the Dirac algorithm for a system with first- and second-class constraints. To examine the role of the zero modes, we consider the quantization procedure in the zero mode and the nonzero-mode sectors separately. In both sectors we obtain the physical variables and their canonical commutation relations. The physical Hamiltonian is constructed via a step-by-step exclusion of the unphysical degrees of freedom. An example using this Hamiltonian in which the zero modes play a role is the verification of the correct Coulomb potential between two heavy fermions.

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I. INTRODUCTION

There has been recent progress [1–4] toward a framework for describing decay processes involving heavy quarks. Its basis has been the study of QCD in which some of the quark masses are taken to infinity. In this limit, the quark spin degrees of freedom decouple from each other and the couplings of the heavy quarks to the gluon degrees of freedom become mass independent and are described by a Wilson line.

The remarkable consequence of this independence is that all matrix elements of vector and axial-vector currents bilinear in either or both of a pair of heavy quarks taken between either vector or pseudoscalar mesons in initial and final states at arbitrary momentum transfer are determined in terms of a single normalized function. The question arises as to how one calculates this function.

One of the more promising approaches to problems in QCD is light-cone quantization [5, 6]. Light-cone quantization has turned out to be a useful tool for the perturbative treatment of field theories [7, 8]. In its extension [9–13] to the nonperturbative domain, we have come to realize that careful attention must be paid to the nontrivial vacuum structure of light-cone quantum field theory. Some mathematical aspects of the question about the existence of such vacuum states were considered in [14, 15] (see also other references therein). For instance, the light-cone vacuum in the massless Schwinger model can be understood only by a careful study of the zero modes of the constraints imposed by the light-cone frame [16, 17]. Indeed it has been conjectured that the dynamics of zero modes in QCD in light-cone quantization provide the mechanism for confinement [5, 6].

In the present paper we consider a four-dimensional

heavy-fermion QED, as an initial step toward the study of heavy-quark QCD in light-cone quantization. We carry out a consistent canonical quantization of this gauge invariant model in a light-cone domain restricted in its “spatial” directions. It is well known that in such a restricted region one has problems with zero modes [18, 17]. The canonical quantization of the massless Schwinger model on a lightlike hyperplane, taking into account zero-mode contributions, was considered in [17]. Neglecting such contributions, Tang *et al.* [19] carried out the discretized light-cone quantization of four-dimensional QED and Mustaki [20] developed the canonical quantization of two-dimensional QED on the null plane. We note that these questions have also been investigated by first quantizing and then taking the heavy-quark limit, in two-dimensional QCD [21].

In studying the quantization of gauge field theories one is confronted by first-class constraints and their corresponding gauge conditions. A consistent canonical quantization formalism for such problems was proposed long ago by Dirac [22] and Bergmann [23], and its generalization to fermionic (Grassmann-odd) constraints by Casalbuoni [24]. [There are in fact books on this subject [25, 26]. We will not discuss here other approaches involving path integration, Becchi-Rouet-Stora-Tyutin (BRST) or Batalin-Fradkin-Vilkovisky (BFV) methods.] In the Dirac approach some problems arise. They involve the determination of the dynamical (physical) variables and the construction of the physical Hamiltonian. One needs to prove, also, that in the physical sector the S matrix does not depend on the form of the gauge conditions.

The specific gauge theory we address in this paper is in terms of light-cone variables where, as we shall see, the quantization comes with some important constraints involving zero-frequency variables which require special attention. To examine the role of the zero modes explicitly we consider the quantization procedure in the separated zeroless-mode and zero-mode sectors. In such sectors we choose special gauge conditions and obtain the physical variables and their canonical commutation relations. The physical Hamiltonian is constructed by systematic elim-

*Permanent address: Department of Physics, Inje University, Kimhae, 621-749, Republic of Korea.

ination of the nondynamical variables.

In Secs. II–V the canonical quantization of heavy-fermion QED is addressed by way of the Dirac-Bergmann algorithm. In Sec. VI we calculate the interaction potential between a heavy fermion and a heavy antifermion using old-fashioned perturbation theory. Conclusions and a brief discussion follow in Sec. VII.

II. CONSTRAINED DYNAMICS OF HEAVY-FERMION QED

We begin with the Lagrangian of heavy-fermion QED obtained [1] by a generalized Foldy-Wouthuysen transformation that removes the terms mixing fermion and antifermion fields in the action. We have

$$\mathcal{L} = i\bar{\Psi}\not{U}U^\mu D_\mu\Psi - M\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (2.1)$$

where the Minkowski metric is $\text{diag } g_{\mu\nu} = (1, -1, -1, -1)$,

$$D_\mu = \partial_\mu + ieA_\mu \quad (2.2)$$

is the covariant derivative,

$$\not{U}U^\mu = \gamma \cdot UU^\mu, \quad (2.3)$$

and U^μ is the given four-velocity of the heavy fermion satisfying the condition $U^2 = 1$. The field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.4)$$

and we use the system of units where $\hbar = c = 1$. The heavy-fermion limit means that MU^μ is greater than any other momentum in the problem under consideration.

The light-cone coordinates in four dimensions are $x^\mu = (x^+, x^-, x^j)$, $j = 1, 2$, where

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3). \quad (2.5)$$

The variable x^+ plays the role of the “time” variable. In terms of such coordinates the Lagrangian \mathcal{L} becomes

$$\begin{aligned} \mathcal{L} = & i\bar{\Psi}\not{U}(U_+\partial_- + U_-\partial_+ - U_j\partial_j)\Psi - M\bar{\Psi}\Psi \\ & + \frac{1}{2}F_{+-}^2 + F_{+j}F_{-j} - \frac{1}{4}F_{jk}F_{jk} \\ & - e\bar{\Psi}\not{U}(U_+A_- + U_-A_+ - U_jA_j)\Psi, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} F_{jk} &= \partial_j A_k - \partial_k A_j, \\ F_{+-} &= \partial_+ A_- - \partial_- A_+, \\ F_{\pm j} &= \partial_\pm A_j - \partial_j A_\pm, \\ A_\mp &= \frac{1}{\sqrt{2}}(A^0 \pm A^3), \\ U_\mp &= \frac{1}{\sqrt{2}}(U^0 \pm U^3), \\ \partial_\pm &= \partial/\partial x^\pm. \end{aligned} \quad (2.7)$$

The fermion part of the Lagrangian (2.1) differs from ordinary QED by the change $\gamma^\mu \rightarrow \not{U}U^\mu$. Let us consider the equations of motion for the fermion field Ψ . They are

$$i\not{U}U^\mu D_\mu\Psi = M\Psi. \quad (2.8)$$

As in ordinary QED, let us introduce, in light-cone coordinates, two fermion fields [27–30]

$$\begin{aligned} \Psi_{(\pm)} &= P_{(\pm)}\Psi, \\ P_{(\pm)} &= \frac{1}{2}(1 \pm \gamma_0\gamma_3). \end{aligned} \quad (2.9)$$

Since $U^2 = 1$, Eqs. (2.8) may be rewritten in the form

$$iU^\mu D_\mu\Psi_{(\pm)} = M(\sqrt{2}U_\pm\gamma_0\Psi_{(\mp)} - U_j\gamma_j\Psi_{(\pm)}). \quad (2.10)$$

From these equations it follows that both fields $\Psi_{(\pm)}$ are dynamical fields because both of them contain the derivative with respect to the “time” coordinate x^+ . This distinguishes heavy-fermion QED from ordinary QED where the fields $\Psi_{(\pm)}$ are not independent [27–30] in light-cone quantization: The field $\Psi_{(+)}$ can be related to the dynamical field $\Psi_{(-)}$ in ordinary QED. (See, however, the discussion of zero modes in [31].)

The Lagrangian (2.1) is gauge invariant. This means that the classical theory contains “first-class” constraints and we need a quantization prescription for systems with constraints such as that provided by Dirac [22].

According to this procedure “primary constraints” arise when one cannot relate a velocity to its corresponding canonical momentum. The consistency condition of a primary constraint, which means that the constraint must be conserved in time, is used either as a condition which defines a Lagrange multiplier function or as a “secondary” constraint. There could even be “tertiary constraints,” and so on. This procedure terminates when no new constraints appear.

All constraints are divided further into two groups. Constraints whose Poisson brackets with all other constraints vanish on the constraint surface are called “first-class” constraints. Otherwise, they are called “second-class” constraints. If a constraint is first class, a subsidiary condition (gauge condition) must be imposed in order to determine the corresponding Lagrange multiplier function. When all constraints finally become second class, we can invert the matrix of Poisson brackets between the constraints and replace the Poisson brackets by the Dirac brackets. The quantization procedure consists of replacement of the Dirac brackets by a commutator for bosons or an anticommutator for fermions.

The infrared problems in the quantization step can be regularized by considering the system to be contained in a finite volume. Let us consider the quantization of the theory in the restricted region $-L \leq x^- \leq L$, $-L \leq x^j \leq L$ and impose periodic boundary condition for the bosonic variables and antiperiodic boundary condition for the fermionic ones. We choose antiperiodic boundary conditions for the fermions following Refs. [6, 32, 33, 17]. Any boundary condition, however, implies that there is an additional constraint to be satisfied, in view of the field dependence in the Schwinger string factor. In this paper we ignore such constraints. On the other hand, as we take the box length to infinity, the fields vanish and no such constraint arises.

In such a restricted region, the fact that the classical equations are first order in the “time” x^+ leads to

the well-known problem of zero modes. To separate the zero-mode contributions, it is useful to generalize the orthogonal projection operators found in [17]:

$$\begin{aligned} \mathcal{P}_{(a)} * \phi(\underline{x}) &= \int_{-L}^L \mathcal{P}_{(a)}(\underline{x}, \underline{y}) \phi(\underline{y}) d\underline{y}, \\ \mathcal{P}_{(a)} &= (\mathcal{P}, \mathcal{P}_j, \mathcal{Q}, \mathcal{Q}_j), \quad j = 1, 2, \\ \mathcal{Q} &= \mathcal{I} - \mathcal{P}, \quad \mathcal{Q}_j = \mathcal{I} - \mathcal{P}_j, \\ \mathcal{P}(\underline{x}, \underline{y}) &= \frac{1}{2L} \delta(x_\perp - y_\perp), \\ \mathcal{P}_j(\underline{x}, \underline{y}) &= \frac{1}{2L} \epsilon_{jk} \delta(x^k - y^k) \delta(x^- - y^-), \end{aligned} \quad (2.11)$$

where $\underline{x} = (x^-, x^1, x^2) = (x^-, x_\perp)$, antidiag $\epsilon_{jk} = (1, 1)$, and \mathcal{I} is a unit operator, $\mathcal{I} * \phi(\underline{x}) = \phi(\underline{x})$. Here $\mathcal{P}, \mathcal{P}_j$ are the projection operators into the zero-mode sectors of x^-, x^j , respectively, and $\mathcal{Q}, \mathcal{Q}_j$ are the projection operators which eliminate the zero modes in the sectors of x^-, x^j , respectively. We shall speak of \mathcal{P} as the projector onto the \mathcal{P} sector, and \mathcal{Q} as the projector onto the \mathcal{Q} sector.

In the same way it is possible to define the product of two projectors:

$$\begin{aligned} \mathcal{P}_{(a)} \mathcal{P}_{(b)} * \phi(\underline{x}) &= \mathcal{P}_{(a)} * (\mathcal{P}_{(b)} * \phi)(\underline{x}) \\ &= \mathcal{P}_{(b)} \mathcal{P}_{(a)} * \phi(\underline{x}). \end{aligned} \quad (2.12)$$

Because of this definition, the operator $\mathcal{P}_1 \mathcal{P}_2$ is orthogonal to $\mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{Q}_1 \mathcal{Q}_2$. One can verify that the operators $\mathcal{P}_1 \mathcal{P}_2$ and $\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2$ are projection operators. Further, $\mathcal{P}_1 \mathcal{P}_2 + \mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2 = 1$, so that the two subspaces defined by these projection operators span the space.

We define the canonical momenta Π_ϕ as

$$\begin{aligned} \Pi_\phi &= \frac{\partial_r \mathcal{L}}{\partial \dot{\phi}}, \\ \dot{\phi} &= \partial_+ \phi, \end{aligned} \quad (2.13)$$

where the label r denotes the right derivative. Then

$$\begin{aligned} \Pi_+ (\equiv \Pi_{A_+}) &= 0, \\ \Pi_- (\equiv \Pi_{A_-}) &= \dot{A}_- - \partial_- A_+, \\ \Pi_j (\equiv \Pi_{A_j}) &= \partial_- A_j - \partial_j A_-, \\ \Pi_\Psi &= i \bar{\Psi} \psi U_-, \\ \Pi_{\bar{\Psi}} &= 0. \end{aligned} \quad (2.14)$$

The velocity \dot{A}_- can be expressed through the momentum Π_- and we have five primary constraints $\Phi^{(1)} \approx 0$, where

$$\Phi^{(1)} = \begin{cases} \phi_+^{(1)} = \Pi_+, \\ \phi_j^{(1)} = \Pi_j - \partial_- A_j + \partial_j A_-, \\ \phi_\Psi^{(1)} = \Pi_\Psi - i \bar{\Psi} \psi U_-, \\ \phi_{\bar{\Psi}}^{(1)} = \Pi_{\bar{\Psi}}. \end{cases} \quad (2.15)$$

The canonical Hamiltonian density on the constraint surface (2.15) is

$$\begin{aligned} \mathcal{H}_c &= -\Pi_\Psi U_-^{-1} (U_+ \partial_- - U_j \partial_j) \Psi - i M \Pi_\Psi \psi U_-^{-1} \Psi \\ &\quad - i e \Pi_\Psi U_-^{-1} U^\mu A_\mu \Psi + \frac{1}{2} \Pi_-^2 \\ &\quad - (\partial_- \Pi_- + \partial_j \Pi_j) A_+ + \frac{1}{4} F_{jk} F_{jk}. \end{aligned} \quad (2.16)$$

Consider the Poisson brackets between the constraints $\phi_j^{(1)}$ from (2.15):

$$\left\{ \phi_j^{(1)}(\underline{x}), \phi_k^{(1)}(\underline{y}) \right\} = -2 \delta_{jk} \partial_- \delta(\underline{x} - \underline{y}). \quad (2.17)$$

From this expression we see that the constraints $\phi_j^{(1)}$ commute only in the \mathcal{P} sector. Hence in the \mathcal{P} sector the consistency conditions for the constraints $\phi_j^{(1)}$ may lead to secondary constraints. Thus let us consider the decomposition of the primary constraints $\phi_+^{(1)}$ and $\phi_j^{(1)}$ into the \mathcal{Q} and \mathcal{P} sector constraints:

$$\begin{aligned} \phi_+^{(1, \mathcal{Q})} &= \mathcal{Q} * \Pi_+, \\ \phi_j^{(1, \mathcal{Q})} &= \mathcal{Q} * (\Pi_j - \partial_- A_j + \partial_j A_-), \end{aligned} \quad (2.18)$$

$$\begin{aligned} \phi_+^{(1, \mathcal{P})} &= \mathcal{P} * \Pi_+, \\ \phi_j^{(1, \mathcal{P})} &= \mathcal{P} * (\Pi_j + \partial_j A_-). \end{aligned} \quad (2.19)$$

Following the Dirac prescription we construct the primary Hamiltonian density

$$\begin{aligned} \mathcal{H}^{(1)} &= \mathcal{H}_c + \lambda_\Psi \phi_\Psi^{(1)} + \lambda_{\bar{\Psi}} \phi_{\bar{\Psi}}^{(1)} + \lambda_+^{(\mathcal{Q})} \phi_+^{(1, \mathcal{Q})} \\ &\quad + \lambda_j^{(\mathcal{Q})} \phi_j^{(1, \mathcal{Q})} + \lambda_+^{(\mathcal{P})} \phi_+^{(1, \mathcal{P})} + \lambda_j^{(\mathcal{P})} \phi_j^{(1, \mathcal{P})}, \end{aligned} \quad (2.20)$$

where $\lambda_+^{(\mathcal{Q}, \mathcal{P})}, \lambda_j^{(\mathcal{Q}, \mathcal{P})}$ are Grassmann-even Lagrange multipliers for the constraints (2.18), (2.19), and $\lambda_\Psi, \lambda_{\bar{\Psi}}$ are Grassmann-odd Lagrange multipliers for the fermionic constraints $\phi_\Psi^{(1)}, \phi_{\bar{\Psi}}^{(1)}$, respectively. The consistency conditions for the constraints $\phi_\Psi^{(1)}, \phi_{\bar{\Psi}}^{(1)}$ do not give new constraints, and hence lead to the determination of their Lagrange multipliers. We will consider the \mathcal{Q} and \mathcal{P} sectors separately.

III. \mathcal{Q} SECTOR

In this section we are going to consider the quantization procedure in the \mathcal{Q} sector. In this sector the primary Hamiltonian density is

$$\mathcal{H}^{(1, \mathcal{Q})} = \mathcal{H}_c + \lambda_+^{(\mathcal{Q})} \phi_+^{(1, \mathcal{Q})} + \lambda_j^{(\mathcal{Q})} \phi_j^{(1, \mathcal{Q})}. \quad (3.1)$$

The consistency conditions for the constraints $\phi_j^{(1, \mathcal{Q})}$ determine their Lagrange multipliers, and for the constraint $\phi_+^{(1, \mathcal{Q})}$ gives a new constraint

$$\phi_+^{(2, \mathcal{Q})} = \mathcal{Q} * (\partial_- \Pi_- + \partial_j \Pi_j + i e \Pi_\Psi \Psi). \quad (3.2)$$

This constraint does not commute with the fermionic constraints. Thus let us consider a new ‘‘Gaussian law’’ constraint which is equivalent to (3.2) in the presence of (2.15)

$$\phi_+^{(2, \mathcal{Q})} = \mathcal{Q} * (\partial_- \Pi_- + \partial_j \Pi_j + i e \Pi_\Psi \Psi + i e \bar{\Psi} \Pi_{\bar{\Psi}}), \quad (3.3)$$

and, on the constraint surface, commutes with all constraints. The fermionic constraints are second-class constraints and commute now with all nonfermionic ones. Therefore we will omit them in future consideration.

The consistency condition of the constraint (3.3) gives neither a new constraint nor determines any Lagrange multiplier. The constraints $\phi_j^{(1,\mathcal{Q})}, \phi_{j+4}^{(\mathcal{Q})}, \phi_\Psi^{(1)}, \phi_{\bar{\Psi}}^{(1)}$ are sec-

ond class, and the $\phi_+^{(1,\mathcal{Q})}, \phi_+^{(2,\mathcal{Q})}$ are first class. We need two gauge conditions. We choose them in the form

$$\begin{aligned}\phi_G^{(1,\mathcal{Q})} &= \mathcal{Q} * A_+, \\ \phi_G^{(2,\mathcal{Q})} &= \mathcal{Q} * \partial_- A_-.\end{aligned}\quad (3.4)$$

So in the \mathcal{Q} sector the constraints $\Phi^{(\mathcal{Q})} \approx 0$ are

$$\Phi^{(\mathcal{Q})} = \begin{cases} \phi_+^{(1,\mathcal{Q})} \equiv \phi_1^{(\mathcal{Q})}, \\ \phi_G^{(1,\mathcal{Q})} \equiv \phi_2^{(\mathcal{Q})}, \\ -\mathcal{Q} * \Delta_-^{-1} [\partial_- \Pi_- + \partial_k \Pi_k + ie(\Pi_\Psi \Psi + \bar{\Psi} \Pi_{\bar{\Psi}})] \equiv \phi_3^{(\mathcal{Q})}, \\ \phi_G^{(2,\mathcal{Q})} \equiv \phi_4^{(\mathcal{Q})}, \\ \phi_j^{(1,\mathcal{Q})} \equiv \phi_{j+4}^{(\mathcal{Q})}, \end{cases}\quad (3.5)$$

where $\Delta_- = \partial_-^2$ and Δ_-^{-1} is a nondegenerate operator whose matrix elements in the \mathcal{Q} sector are

$$\begin{aligned}(\Delta_-^{-1})_{\underline{x}, \underline{y}} &= H^{\mathcal{Q}}(x^- - y^-) \delta(x_\perp - y_\perp), \\ H^{\mathcal{Q}}(x - y) &= \frac{|x - y|}{2} - \frac{(x - y)^2}{4L}.\end{aligned}\quad (3.6)$$

The constraints $\Phi^{(\mathcal{Q})}$ are second class. The matrix of Poisson brackets between these constraints

$$A_{a,b}(\underline{x}, \underline{y}) = \{\phi_a^{(\mathcal{Q})}(\underline{x}), \phi_b^{(\mathcal{Q})}(\underline{y})\}, \quad a, b = 1, \dots, 6, \quad (3.7)$$

is quasidiagonal (block diagonal) and its nonvanishing elements are found to be

$$\begin{aligned}A_{1,2}(\underline{x}, \underline{y}) &= -A_{2,1}(\underline{x}, \underline{y}) = -\mathcal{Q}(\underline{x}, \underline{y}), \\ A_{3,4}(\underline{x}, \underline{y}) &= -A_{4,3}(\underline{x}, \underline{y}) = -\mathcal{Q}(\underline{x}, \underline{y}), \\ A_{j+4, j'+4}(\underline{x}, \underline{y}) &= -2\delta_{jj'} \partial_- \mathcal{Q}(\underline{x}, \underline{y}).\end{aligned}\quad (3.8)$$

The inverse matrix $A_{a,b}^{-1}(\underline{y}, \underline{z})$,

$$\int d\underline{y} A_{c,a}(\underline{x}, \underline{y}) A_{a,b}^{-1}(\underline{y}, \underline{z}) = \delta_{cb} \delta^{(\mathcal{Q})}(\underline{x} - \underline{z}), \quad (3.9)$$

$$\begin{aligned}\delta^{(\mathcal{Q})}(\underline{x} - \underline{z}) &= D^{\mathcal{Q}}(x^- - z^-) \delta(x_\perp - z_\perp), \\ D^{\mathcal{Q}}(x - z) &= \delta(x - z) - \frac{1}{2L},\end{aligned}\quad (3.10)$$

has nonvanishing elements

$$\begin{aligned}A_{1,2}^{-1}(\underline{x}, \underline{y}) &= -A_{2,1}^{-1}(\underline{x}, \underline{y}) = -\mathcal{Q}(\underline{x}, \underline{y}), \\ A_{3,4}^{-1}(\underline{x}, \underline{y}) &= -A_{4,3}^{-1}(\underline{x}, \underline{y}) = -\mathcal{Q}(\underline{x}, \underline{y}), \\ A_{j+4, j'+4}^{-1}(\underline{x}, \underline{y}) &= -\frac{1}{2} \delta_{jj'} \delta(x_\perp - z_\perp) G^{\mathcal{Q}}(x^- - z^-),\end{aligned}\quad (3.11)$$

$$G^{\mathcal{Q}}(x - z) = \frac{\epsilon(x - z)}{2} - \frac{x - z}{2L}.\quad (3.12)$$

The functions $D^{\mathcal{Q}}(x - z)$ and $G^{\mathcal{Q}}(x - z)$ are the δ function and the matrix element of the operator ∂_-^{-1} , respectively, in the \mathcal{Q} sector.

Consider two variables

$$\omega_j^{(\mathcal{Q})} = \mathcal{Q} * A_j.\quad (3.13)$$

It is easy to show that the Dirac brackets between these variables are

$$\begin{aligned}\{\omega_j^{(\mathcal{Q})}(\underline{x}), \omega_{j'}^{(\mathcal{Q})}(\underline{y})\}_D \\ = -\frac{1}{2} \delta_{jj'} \delta(x^\perp - y^\perp) G^{\mathcal{Q}}(x^- - y^-).\end{aligned}\quad (3.14)$$

Thus we can consider the variables $\omega_j^{(\mathcal{Q})}(\underline{x})$ as the physical variables in the \mathcal{Q} sector.

IV. \mathcal{P} SECTOR

In this section we will consider the canonical quantization procedure in the \mathcal{P} sector. The primary Hamiltonian density in this sector is

$$\mathcal{H}^{(1,\mathcal{P})} = \mathcal{H}_c + \lambda_+^{(\mathcal{P})} \phi_+^{(1,\mathcal{P})} + \lambda_j^{(\mathcal{P})} \phi_j^{(1,\mathcal{P})}.\quad (4.1)$$

The \mathcal{P} sector is a space $H^{(\mathcal{P})}$ of functions which depend on the variables x^\perp only. On the other hand the \mathcal{P} sector is the sector of zero modes in the x^- direction. To take into account the zero-mode contributions in the directions x^1, x^2 one can decompose the space $H^{(\mathcal{P})}$ into a direct sum of two orthogonal subspaces

$$H^{(\mathcal{P})} = H^{(\mathcal{P}\mathcal{P}_1\mathcal{P}_2)} \oplus H^{[\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2)]},\quad (4.2)$$

where $H^{(\mathcal{P}\mathcal{P}_1\mathcal{P}_2)}$ is the space of zero modes in all directions and is defined by the projector $\mathcal{P}\mathcal{P}_1\mathcal{P}_2$ and the space $H^{[\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2)]}$ is defined by the projector $\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2)$. Such a decomposition leads to the corresponding decomposition of the primary constraints $\phi_+^{(1,\mathcal{P})}, \phi_j^{(1,\mathcal{P})}$ (2.19), and the primary Hamiltonian density $\mathcal{H}^{(1,\mathcal{P})}$ (4.1):

$$\begin{aligned}\phi_+^{(1,\mathcal{P})} &= \left(\varphi_+^{(1)}, \psi_+^{(1)} \right), \\ \phi_j^{(1,\mathcal{P})} &= \left(\varphi_j^{(1)}, \psi_j^{(1)} \right),\end{aligned}\quad (4.3)$$

$$\begin{aligned} \mathcal{H}^{(1,\mathcal{P})} = & \mathcal{H}_c + \lambda_+^{(1)} \varphi_+^{(1)} + \Lambda_+^{(1)} \psi_+^{(1)} \\ & + \lambda_j^{(1)} \varphi_j^{(1)} + \Lambda_j^{(1)} \psi_j^{(1)}. \end{aligned} \quad (4.4)$$

Here $\lambda_+^{(1)}, \Lambda_+^{(1)}, \lambda_j^{(1)}, \Lambda_j^{(1)}$ are the Lagrange multipliers; the primary constraints in the $\mathcal{PP}_1\mathcal{P}_2$ -subsector are

$$\begin{aligned} \varphi_+^{(1)} &= \mathcal{PP}_1\mathcal{P}_2 * \Pi_+, \\ \varphi_j^{(1)} &= \mathcal{PP}_1\mathcal{P}_2 * \Pi_j, \end{aligned} \quad (4.5)$$

and

$$\chi_+^{(1)} = \mathcal{PP}_1\mathcal{P}_2 * (ie\Pi_\Psi\Psi), \quad (4.7)$$

$$\chi_j^{(1)} = \mathcal{PP}_1\mathcal{P}_2 * (-ie\Pi_\Psi U_-^{-1} U_j \Psi), \quad (4.8)$$

$$\chi_+^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * (\partial_j \Pi_j + ie\Pi_\Psi\Psi), \quad (4.9)$$

$$\chi_j^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * (\partial_j \Pi_- + \partial_k F_{kj} - ie\Pi_\Psi U_-^{-1} U_j \Psi). \quad (4.10)$$

As it happened in the \mathcal{Q} sector the constraints (4.7)–(4.10) do not commute with the fermionic constraints. Thus let us consider new constraints which are equivalent to (4.7)–(4.10) in the presence of (2.15):

$$\chi_+^{(1)} = \mathcal{PP}_1\mathcal{P}_2 * \rho, \quad (4.11)$$

$$\chi_j^{(1)} = \mathcal{PP}_1\mathcal{P}_2 * \rho_j, \quad (4.12)$$

$$\chi_+^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * (\partial_j \Pi_j + \rho), \quad (4.13)$$

$$\chi_j^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * (\partial_j \Pi_- + \partial_k F_{kj} - \rho_j), \quad (4.14)$$

where

$$\rho = ie(\Pi_\Psi\Psi + \overline{\Psi}\Pi_\Psi), \quad (4.15)$$

and

$$\rho_j = \frac{U_j}{U_-} \rho, \quad (4.16)$$

which is the consequence of the heavy mass limit.

Let us consider the quantization procedure where the strong constraint

$$\mathcal{PP}_1\mathcal{P}_2 * \rho|\psi\rangle = 0, \quad (4.17)$$

holds, where $|\psi\rangle$ is the state vector. The quantity $\mathcal{PP}_1\mathcal{P}_2 * \rho$ is the total light-cone electric charge of the fermions per unit volume. This is proportional to the usual electric charge, which must vanish in a compact system as a consequence of Gauss's law.

The consistency condition for the constraints (4.13) and (4.14) does not give any new constraints. So in the \mathcal{P} sector we have 14 constraints: $\varphi_+^{(1)}, \varphi_j^{(1)}, \chi_+^{(1)}, \chi_j^{(1)}, \psi_+^{(1)}, \psi_j^{(1)}, \chi_+^{(2)}, \chi_j^{(2)}, \phi_\Psi^{(1)}, \phi_{\overline{\Psi}}^{(1)}$. The constraints $\varphi_+^{(1)}, \varphi_j^{(1)}, \psi_+^{(1)}$ and $\chi_+^{(2)}$ are first class and the rest are second class. We need four gauge conditions.

We consider the gauge conditions for the first-class constraints $\varphi_+^{(1)}$ and $\psi_+^{(1)}$ in the form

$$\psi_+^{(1)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * \Pi_+,$$

$$\psi_j^{(1)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * (\Pi_j + \partial_j A_-), \quad (4.6)$$

are the primary constraints in the $\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2)$ subsector.

The consistency condition for the constraints $\varphi_+^{(1)}, \varphi_j^{(1)}$ and $\psi_+^{(1)}, \psi_j^{(1)}$ do not determine their Lagrange multipliers, and instead give new constraints in the form (we denote them as $\chi_+^{(1)}, \chi_j^{(1)}, \chi_+^{(2)}, \chi_j^{(2)}$, correspondingly)

$$\varphi_G^{(1)} = \mathcal{PP}_1\mathcal{P}_2 * A_+,$$

$$\psi_G^{(1)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * A_+. \quad (4.18)$$

Let us discuss the gauge conditions for the first-class constraint $\chi_+^{(2)}$ in the $\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2)$ subsector. Note that in this subsector the operator

$$\Delta = \partial_k \partial_k \quad (4.19)$$

is an invertible operator. Let us choose the gauge condition for the constraints $\chi_+^{(2)}$ in the form

$$\chi_G^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * \Delta^{-1} \partial_k A_k. \quad (4.20)$$

The constraints $\chi_j^{(2)}$ do not commute with the primary constraints $\psi_j^{(1)}$ but do commute with the rest. On the other hand, the primary constraints $\psi_j^{(1)}$ do not commute with the gauge condition $\chi_G^{(2)}$. Thus let us consider a linear combination of the constraints $\psi_j^{(1)}$ and $\chi_+^{(2)}$,

$$\psi_j'^{(1)} = \psi_j^{(1)} + \alpha_j \chi_+^{(2)}, \quad (4.21)$$

in such a way that $\psi_j'^{(1)}$ will commute with the gauge condition $\chi_G^{(2)}$. Here α_j are some functions to be found. We obtain

$$\psi_j'^{(1)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * \left(\Pi_j^\perp - \frac{\partial_j}{\Delta} \rho + \partial_j A_- \right), \quad (4.22)$$

$$\Pi_j^\perp = \left(\delta_{jk} - \frac{\partial_j \partial_k}{\Delta} \right) \Pi_k. \quad (4.23)$$

Here Π_j^\perp are two-dimensional transverse momenta. Let us introduce longitudinal momenta and an analogous decomposition for the potentials A_k :

$$\Pi_j = \Pi_j^\perp + \Pi_j^\parallel, \quad \Pi_j^\parallel = \frac{\partial_j \partial_k}{\Delta} \Pi_k, \quad (4.24)$$

$$A_j = A_j^\perp + A_j^\parallel, \\ A_j^\perp = \left(\delta_{jk} - \frac{\partial_j \partial_k}{\Delta} \right) A_k, \quad A_j^\parallel = \frac{\partial_j \partial_k}{\Delta} A_k. \quad (4.25)$$

With this decomposition the constraints depend only on transverse or longitudinal components, but not both:

$$\chi_+^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2) * (\partial_j \Pi_j^\parallel + \rho), \quad (4.26)$$

$$\chi_G^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2) * \Delta^{-1} \partial_k A_k^\parallel, \quad (4.27)$$

$$\chi_j^{(2)} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2) * (\partial_j \Pi_- + \Delta A_j^\perp - \rho_j). \quad (4.28)$$

Consider the primary first-class constraints $\varphi_j^{(1)}$. Naively one might try to choose the gauge condition as $\mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_j \approx 0$, which means that the zero modes of A_j are eliminated. This contradicts, however, the fact that the integral over a closed loop,

$$\oint A_j dx_j, \quad (4.29)$$

is gauge invariant [34], and need not vanish if taken over a noncontractible loop. Instead, we choose the gauge condition $\varphi_{jG}^{(1)} \approx 0$ in the form

$$\varphi_{jG}^{(1)} = \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_j - f_j(x^+), \quad (4.30)$$

where $f_j(x^+)$ is any function of the ‘‘time’’ x^+ . The constraints $\varphi_{jG}^{(1)}$ depend on ‘‘time’’ explicitly. It is well known that the consistency condition for this type of constraint can be written in the form

$$\partial_+ \varphi_{jG}^{(1)} + \int \{ \varphi_{jG}^{(1)}, \mathcal{H}_c(x) \} d\mathbf{x} \approx 0. \quad (4.31)$$

This relation determines the Lagrange multiplier $\lambda_j^{(1)}$ in terms of the derivative $\partial_+ f_j$. In the next section we will show that the physical Hamiltonian does not depend on the function $f_j(x^+)$.

So the constraints $\Phi^{(\mathcal{P})} \approx 0$ in the \mathcal{P} sector are (we omit here the fermionic constraints)

$$\Phi^{(\mathcal{P})} = \left\{ \begin{array}{l} \varphi_+^{(1)} \equiv \phi_1^{(\mathcal{P})} \\ \varphi_G^{(1)} \equiv \phi_2^{(\mathcal{P})} \\ \varphi_j^{(1)} \equiv \phi_{j+2}^{(\mathcal{P})} \\ \varphi_{jG}^{(1)} \equiv \phi_{j+4}^{(\mathcal{P})} \end{array} \right\} \mathcal{P}\mathcal{P}_1\mathcal{P}_2 \text{ subsector} \\ \left\{ \begin{array}{l} \psi_+^{(1)} \equiv \phi_7^{(\mathcal{P})} \\ \psi_G^{(1)} \equiv \phi_8^{(\mathcal{P})} \\ \chi_+^{(2)} \equiv \phi_9^{(\mathcal{P})} \\ \chi_G^{(2)} \equiv \phi_{10}^{(\mathcal{P})} \\ \psi_j^{(1)} \equiv \phi_{j+10}^{(\mathcal{P})} \\ \chi_j^{(2)} \equiv \phi_{j+12}^{(\mathcal{P})} \end{array} \right\} \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2) \text{ subsector} \quad (4.32)$$

The constraints (4.32) are second-class constraints and the matrix of their Poisson brackets is block diagonal.

The physical variables are among the constraints

$\phi_{j+10}^{(\mathcal{P})}, \phi_{j+12}^{(\mathcal{P})}$. The matrix of Poisson brackets between these constraints is antidiagonal and its nonzero elements are

$$\{ \phi_{j+10}^{(\mathcal{P})}(\underline{x}), \phi_{j'+12}^{(\mathcal{P})}(\underline{y}) \} \\ = A_{j+10, j'+12}(\underline{x}, \underline{y}) \\ = -\delta_{jj'} \Delta [\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2)](\underline{x}, \underline{y}), \quad (4.33)$$

$$A_{j+12, j'+10}(\underline{x}, \underline{y}) = \delta_{jj'} \Delta [\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2)](\underline{x}, \underline{y}). \quad (4.34)$$

The inverse matrix is antidiagonal too, and is easily found to be

$$A_{j+12, j'+10}^{-1}(\underline{x}, \underline{y}) = -\delta_{jj'} \frac{1}{\Delta} [\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2)](\underline{x}, \underline{y}),$$

$$A_{j+10, j'+12}^{-1}(\underline{x}, \underline{y}) = \delta_{jj'} \frac{1}{\Delta} [\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2)](\underline{x}, \underline{y}). \quad (4.35)$$

Consider two pairs of conjugated variables:

$$\omega_1^{(\mathcal{P})} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2) * A_-, \\ \omega_2^{(\mathcal{P})} = \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2) * \Pi_-, \quad (4.36)$$

$$\omega_3^{(\mathcal{P})} = \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_-, \\ \omega_4^{(\mathcal{P})} = \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * \Pi_-. \quad (4.37)$$

The nonzero Dirac brackets between these variables are

$$\{ \omega_1^{(\mathcal{P})}(\underline{x}), \omega_2^{(\mathcal{P})}(\underline{y}) \}_D = 2[\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2)](\underline{x}, \underline{y}) \\ = \frac{1}{L} \left(\delta(x_\perp - y_\perp) - \frac{1}{4L^2} \right), \quad (4.38)$$

$$\{ \omega_3^{(\mathcal{P})}(\underline{x}), \omega_4^{(\mathcal{P})}(\underline{y}) \}_D = (\mathcal{P}\mathcal{P}_1\mathcal{P}_2)(\underline{x}, \underline{y}) = \frac{1}{8L^3}. \quad (4.39)$$

Thus one can consider these variables as physical variables in the \mathcal{P} sector.

V. PHYSICAL HAMILTONIAN

We have come to the step where we can find the ‘‘physical’’ Hamiltonian. Such a Hamiltonian is defined as

$$H^{\text{phys}} = \int \mathcal{H}_c^{\text{phys}} d\mathbf{x}, \\ \mathcal{H}_c^{\text{phys}} = \mathcal{H}_c|_{\Phi \approx 0}, \quad (5.1)$$

where

$$\Phi = (\Phi^{(\mathcal{Q})}, \Phi^{(\mathcal{P})}, \phi_\Psi^{(1)}, \phi_\Psi^{(1)}), \quad (5.2)$$

are all constraints in the problem under consideration.

From Eq. (5.1) it follows that to construct the Hamiltonian density $\mathcal{H}_c^{\text{phys}}$ we have to express all variables

through the physical variables using the constraints $\Phi \approx 0$. For this purpose let us return to expression (2.16) for the canonical Hamiltonian and rewrite it in the form

$$H_c = \int \mathcal{H}_c^{(F)} d\mathbf{x} + \int \mathcal{H}'_c d\mathbf{x}. \quad (5.3)$$

$$\begin{aligned} \mathcal{H}'_c = & \frac{1}{2}(\mathcal{Q} * \Pi_-)^2 + \frac{1}{2}[\mathcal{P}\mathcal{P}_1\mathcal{P}_2 * \Pi_- + \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * \Pi_-]^2 \\ & + (\mathcal{Q} * \Pi_-)[\mathcal{P}\mathcal{P}_1\mathcal{P}_2 * \Pi_- + \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * \Pi_-] \\ & - (\partial_- \Pi_- + \partial_j \Pi_j)[\mathcal{Q} * A_+ + \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_+ + \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * A_+] \\ & + \frac{1}{4}(\mathcal{Q} * F_{jk})^2 + \frac{1}{4}[\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * F_{jk}]^2 + \frac{1}{2}(\mathcal{Q} * F_{jk})[\mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * F_{jk}] \\ & + e\bar{\Psi}U^\mu\Psi[\mathcal{Q} * A_\mu + \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_\mu + \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * A_\mu], \end{aligned} \quad (5.5)$$

is the Hamiltonian density of the electromagnetic field and its interaction with the fermions. Then the physical Hamiltonian becomes

$$\begin{aligned} H^{\text{phys}} &= \int \mathcal{H}_c^F d\mathbf{x} + \int \mathcal{H}'_c{}^{\text{phys}} d\mathbf{x}, \\ \mathcal{H}'_c{}^{\text{phys}} &= \mathcal{H}'_c|_{\Phi \approx 0}. \end{aligned} \quad (5.6)$$

Because of the property

$$e\bar{\Psi}U^\mu\Psi = U^\mu\rho, \quad (5.7)$$

and the strong condition (4.17) one can neglect the term $e\bar{\Psi}U^\mu\Psi\mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_\mu$ in the Hamiltonian density $\mathcal{H}'_c{}^{\text{phys}}$ (5.6). This is because the physical Hamiltonian H'^{phys} , which corresponds to the density $\mathcal{H}'_c{}^{\text{phys}}$, by itself is proportional to the zero mode of $\mathcal{H}'_c{}^{\text{phys}}$:

$$H'^{\text{phys}} = 8L^3\mathcal{P}\mathcal{P}_1\mathcal{P}_2 * \mathcal{H}'_c{}^{\text{phys}}. \quad (5.8)$$

Thus the physical Hamiltonian does not depend on the functions $f_j(x^+)$ involved in the gauge condition (4.30).

Using the constraints $\Phi \approx 0$ we are able to write

$$\begin{aligned} \mathcal{Q} * \Pi_-|_{\Phi \approx 0} &= -(\partial_j \omega_j^{(\mathcal{Q})} + e\mathcal{Q} * \partial_-^{-1} \bar{\Psi}U_- \Psi), \\ \mathcal{Q} * A_\pm|_{\Phi \approx 0} &= 0, \\ \mathcal{Q} * F_{jk}|_{\Phi \approx 0} &= \partial_j \omega_k^{(\mathcal{Q})} - \partial_k \omega_j^{(\mathcal{Q})}, \\ \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_+|_{\Phi \approx 0} &= 0, \\ \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * A_+|_{\Phi \approx 0} &= 0, \\ \mathcal{P}\mathcal{P}_1\mathcal{P}_2 * A_j|_{\Phi \approx 0} &= f_j(x^+), \\ \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * A_k^\parallel|_{\Phi \approx 0} &= 0, \\ \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * A_k^\perp|_{\Phi \approx 0} &= -\frac{\partial_k \omega_2^{(\mathcal{P})}}{\Delta} + \frac{U_k \omega_1^{(\mathcal{P})}}{U_-}, \\ \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * F_{jk}|_{\Phi \approx 0} & \\ &= \mathcal{P}(\mathcal{Q}_1 + \mathcal{Q}_2 - \mathcal{Q}_1\mathcal{Q}_2) * F_{jk}^\perp|_{\Phi \approx 0} \\ &= U_-^{-1}(U_k \partial_j - U_j \partial_k) \omega_1^{(\mathcal{P})}. \end{aligned} \quad (5.9)$$

After substitution of Eqs. (5.9) into Eq. (5.6) we get

Here

$$\mathcal{H}_c^{(F)} = \bar{\Psi}U(-iU_+ \partial_- + iU_k \partial_k)\Psi + M\bar{\Psi}\Psi, \quad (5.4)$$

is the Hamiltonian density of fermions. The expression \mathcal{H}'_c ,

$$\begin{aligned} \mathcal{H}'_c{}^{\text{phys}} = & \frac{1}{2}(\partial_j \omega_j^{(\mathcal{Q})} + e\mathcal{Q} * \partial_-^{-1} \bar{\Psi}U_- \Psi - \omega_2^{(\mathcal{P})} - \omega_4^{(\mathcal{P})})^2 \\ & + \frac{1}{4}(\mathcal{F}_{jk}^{(\mathcal{Q})} + \mathcal{F}_{jk}^{(\mathcal{P})})(\mathcal{F}_{jk}^{(\mathcal{Q})} + \mathcal{F}_{jk}^{(\mathcal{P})}) \\ & + e\bar{\Psi}U^\mu\Psi \left(-U_j \omega_j^{(\mathcal{Q})} + U_+ \omega_3^{(\mathcal{P})} + \frac{U_j \partial_j \omega_2^{(\mathcal{P})}}{\Delta} \right. \\ & \left. + \frac{1 - U_+ U_-}{U_-} \omega_1^{(\mathcal{P})} \right), \end{aligned} \quad (5.10)$$

where

$$\begin{aligned} \mathcal{F}_{jk}^{(\mathcal{Q})} &= \partial_j \omega_k^{(\mathcal{Q})} - \partial_k \omega_j^{(\mathcal{Q})}, \\ \mathcal{F}_{jk}^{(\mathcal{P})} &= U_-^{-1}(U_k \partial_j - U_j \partial_k) \omega_1^{(\mathcal{P})}. \end{aligned} \quad (5.11)$$

Having the physical Hamiltonian and the Poisson brackets between all physical variables one can now quantize heavy-fermion QED. According to the Dirac prescription the quantization procedure consists of replacing the Dirac brackets of bosons with bosons (or bosons with fermions) by a commutator and that of fermions with fermions by an anticommutator

$$\begin{aligned} \{A, B\}_D &\rightarrow -i[\hat{A}, \hat{B}] \text{ for bosons,} \\ \{A, B\}_D &\rightarrow -i[\hat{A}, \hat{B}]_+ \text{ for fermions,} \end{aligned} \quad (5.12)$$

where the caret means that the classical object A becomes an operator \hat{A} . Then we come to the equal "time" commutation relations

$$[\hat{\omega}_j^{(\mathcal{Q})}(\mathbf{x}), \hat{\omega}_k^{(\mathcal{Q})}(\mathbf{y})] = -\frac{i}{2} \delta_{jk} \delta(\mathbf{x}_\perp - \mathbf{y}_\perp) G^{(\mathcal{Q})}(x^- - y^-), \quad (5.13)$$

$$[\hat{\omega}_1^{(\mathcal{P})}(x_\perp), \hat{\omega}_2^{(\mathcal{P})}(y_\perp)] = \frac{i}{L} \left(\delta(x_\perp - y_\perp) - \frac{1}{4L^2} \right), \quad (5.14)$$

$$[\hat{\omega}_3^{(\mathcal{P})}, \hat{\omega}_4^{(\mathcal{P})}] = \frac{i}{8L^3}, \quad (5.15)$$

$$[\hat{\Psi}(\mathbf{x}), \hat{\Psi}(\mathbf{y})]_+ = \frac{\Psi}{U_-} \delta(\mathbf{x} - \mathbf{y}). \quad (5.16)$$

The physical Hamiltonian \hat{H}^{phys} can be obtained from (5.10) by the change $\omega_a \rightarrow \hat{\omega}_a$, where $\hat{\omega}_a$ are all operators satisfying the relations (5.13)–(5.16).

We wish to find a realization of the commutation relations (5.13)–(5.16). Using the periodic boundary conditions for $\omega_j^{(\mathcal{Q})}(\mathbf{x})$ and Eq. (5.13) one might write (from now on we will omit the caret)

$$\omega_j^{(\mathcal{Q})}(\underline{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\underline{P}_n} \frac{1}{\sqrt{2P_n^+}} [a_j(\underline{P}_n) \exp(-i\underline{P}_n \cdot \underline{x}) + a_j^\dagger(\underline{P}_n) \exp(i\underline{P}_n \cdot \underline{x})], \quad (5.17)$$

where

$$\underline{P}_n = \frac{\pi}{L}(n^+, n^j), \quad n^+ = 1, 2, \dots, \quad n^j = 0, \pm 1, \pm 2, \dots, \quad \Omega = 8L^3, \quad (5.18)$$

and for any vectors $\underline{a} \cdot \underline{b} = a^\dagger b^- - a_j b_j$. The operators $a_j(\underline{P}_n)$ and $a_j^\dagger(\underline{P}_n)$ satisfy the nonzero commutation relations

$$[a_j(\underline{P}_n), a_{j'}^\dagger(\underline{P}_m)] = i\delta_{jj'}\delta_{\underline{P}_n \underline{P}_m}. \quad (5.19)$$

Consider the operators $\omega_{1,2}^{(\mathcal{P})}(x_\perp)$. Their decomposition into a Fourier series can be written in the form

$$\begin{aligned} \omega_1^{(\mathcal{P})}(x_\perp) &= \frac{1}{\sqrt{\Omega}} \sum_{\underline{P}_{\perp n}} [a(\underline{P}_{\perp n}) \exp(-i\underline{P}_{\perp n} \cdot x_\perp) + a^\dagger(\underline{P}_{\perp n}) \exp(i\underline{P}_{\perp n} \cdot x_\perp)] + \frac{i}{\sqrt{\Omega}}(a - a^\dagger), \\ \omega_2^{(\mathcal{P})}(x_\perp) &= \frac{i}{\sqrt{\Omega}} \sum_{\underline{P}_{\perp n}} [a(\underline{P}_{\perp n}) \exp(-i\underline{P}_{\perp n} \cdot x_\perp) - a^\dagger(\underline{P}_{\perp n}) \exp(i\underline{P}_{\perp n} \cdot x_\perp)] - \frac{1}{\sqrt{\Omega}}(a + a^\dagger), \end{aligned} \quad (5.20)$$

where

$$\underline{P}_{\perp n} = \frac{\pi}{L}n^j, \quad n^j = 0, \pm 1, \pm 2, \dots, \quad a_\perp \cdot b_\perp = a_j b_j, \quad (5.21)$$

and the operators $a(\underline{P}_{\perp n}), a^\dagger(\underline{P}_{\perp n}), a, a^\dagger$ satisfy the nonzero commutation relations

$$[a(\underline{P}_{\perp n}), a^\dagger(\underline{P}_{\perp m})] = \delta_{\underline{P}_{\perp n} \underline{P}_{\perp m}}, \quad [a, a^\dagger] = 1. \quad (5.22)$$

The realization for the zero-mode operators $\omega_{3,4}^{(\mathcal{P})}$ is

$$\omega_3^{(\mathcal{P})} = \frac{1}{\sqrt{\Omega}}(c + c^\dagger), \quad \omega_4^{(\mathcal{P})} = -\frac{i}{\sqrt{\Omega}}(c - c^\dagger), \quad [c, c^\dagger] = 1. \quad (5.23)$$

The operators $a_j(\underline{P}_n), a(\underline{P}_{\perp n}), a, c$ and $a_j^\dagger(\underline{P}_n), a^\dagger(\underline{P}_{\perp n}), a^\dagger, c^\dagger$ can be interpreted as annihilation and creation operators.

The same decomposition in terms of creation and annihilation operators can be made for the fermion operators $\Psi(\underline{x}), \bar{\Psi}(\underline{x})$ satisfying the antiperiodic boundary conditions

$$\Psi(\underline{x}) = \frac{1}{\sqrt{\Omega U_-}} \sum_{\beta, \underline{Q}_n} [b_\beta(\underline{Q}_n) u_+^\beta \exp[-i\underline{Q}_n \cdot \underline{x}] + d_\beta^\dagger(\underline{Q}_n) u_-^\beta \exp(i\underline{Q}_n \cdot \underline{x})], \quad (5.24)$$

where

$$\begin{aligned} \underline{Q}_n &= (Q_n^+, Q_n^j) = \frac{\pi}{L}(n^+ + 1/2, n^j + 1/2), \\ n^+ &= 0, 1, 2, \dots, \quad n^j = 0, \pm 1, \pm 2, \dots, \\ \Lambda_\pm u_\pm^\beta &= u_\pm^\beta, \quad \Lambda_\pm = \frac{1}{2}(1 \pm \psi), \\ \bar{u}_\pm^\beta u_\pm^\alpha &= \pm \delta_{\alpha\beta}, \quad \bar{u}_\pm^\beta u_\mp^\alpha = 0, \end{aligned} \quad (5.25)$$

and the operators $b_\beta(\underline{Q}_n), d_\beta^\dagger(\underline{Q}_n)$ satisfy the nonzero anticommutation relations

$$\begin{aligned} [b_\beta(\underline{Q}_n), b_\alpha^\dagger(\underline{Q}_m)]_+ \\ = [d_\beta(\underline{Q}_n), d_\alpha^\dagger(\underline{Q}_m)]_+ = \delta_{\alpha\beta} \delta_{\underline{Q}_n \underline{Q}_m}. \end{aligned} \quad (5.26)$$

VI. CALCULATION OF HEAVY FERMION POTENTIAL

We wish to see the physical Hamiltonian H^{phys} found in the previous section in action. The example we choose

is the static heavy fermion potential. (This calculation is the analogue in QED of the calculation of the quark-antiquark potential in QCD.) For this purpose we consider old-fashioned perturbation theory up to second order:

$$E \approx E_{(0)} + eE_{(1)} + e^2E_{(2)}. \quad (6.1)$$

The physical Hamiltonian H^{phys} can be expressed in terms of the normal product of the creation and annihilation operators introduced in Sec. V:

$$H^{\text{phys}}(\text{operator}) \equiv: (H_{(0)} + eH_{(1)} + e^2H_{(2)}) : , \quad (6.2)$$

where the symbol $: :$ is the normal ordering one.

The construction of the normalized eigenstate which describes the two static heavy fermions \underline{r} apart yields

$$|f\bar{f}\rangle = \left(\sum_{\underline{Q}_n}^M 1 \right)^{-1/2} \sum_{\underline{Q}_n}^M \exp[-i(\underline{Q}_n - M\underline{U}) \cdot \underline{r}] b_\alpha^\dagger(\sqrt{2}M - Q_n^+, -Q_{\perp n}) d_\beta^\dagger(Q_n^-) |0\rangle, \quad (6.3)$$

where $\alpha \neq \beta$ and we choose

$$U^+ = U^- = \frac{1}{\sqrt{2}}, \quad U^1 = U^2 = 0.$$

In Eq. (6.3) we have the effective summation over Q_n which has modulus less than M . It can be shown that the corresponding eigenvalue ($E_{(0)}$) of $H_{(0)}$ turns out to be $\sqrt{2}M$ and the mean value of the operator $H_{(1)}$ in the state (6.3) is zero:

$$E_{(1)} = \langle \bar{f}f | H_{(1)} | f\bar{f} \rangle = 0. \quad (6.4)$$

The last term in Eq. (6.1) is

$$E_{(2)} = \langle \bar{f}f | H_{(2)} | f\bar{f} \rangle + \sum_m' \frac{|\langle \bar{f}f | H_{(1)} | m \rangle|^2}{E_{(0)} - E_{(0)}^m}, \quad (6.5)$$

where $|m\rangle$ is an eigenstate of $H_{(0)}$ and $E_{(0)}^m$ is the corresponding eigenvalue and the summation does not cover the state $|f\bar{f}\rangle$. After some straightforward calculation we get

$$\langle \bar{f}f | H_{(2)} | f\bar{f} \rangle = -\frac{1}{2\Omega} \sum_{\underline{P}_n} \frac{1}{(P_n^+)^2} [1 + \cos(\underline{P}_n \cdot \underline{r})], \quad (6.6)$$

and

$$\begin{aligned} & \sum_m' \frac{|\langle \bar{f}f | H_{(1)} | m \rangle|^2}{E_{(0)} - E_{(0)}^m} \\ &= \frac{1}{2\Omega} \sum_{\underline{P}_n} \frac{1}{2P_n^{+2} + P_{\perp n}^2} \frac{P_{\perp n}^2}{P_n^{+2}} [1 + \cos(\underline{P}_n \cdot \underline{r})]. \end{aligned} \quad (6.7)$$

If we remember that $\underline{r} = (-\frac{1}{\sqrt{2}}r^3, r^1, r^2)$ and make use of the new vectors

$$\mathbf{P}_n \equiv (P_n^1, P_n^2, \sqrt{2}P_n^+), \quad \mathbf{r} \equiv (r^1, r^2, r_c^3), \quad (6.8)$$

then (in the limit $L \rightarrow \infty$),

$$\begin{aligned} e^2 E_{(2)} &= -\frac{e^2}{\Omega} \sum_{\mathbf{P}_n} \frac{1}{\mathbf{P}_n^2} [1 + \cos(\mathbf{P}_n \cdot \mathbf{r})] \\ &= -\frac{e^2}{4\sqrt{2}\pi r} + \text{const}, \end{aligned} \quad (6.9)$$

where r_c^3 is the Cartesian z component of the vector \mathbf{r} . The additional factor $1/\sqrt{2}$ in the expression (6.9) comes from the fact that the energy $e^2 E_{(2)}$ corresponds to the

Hamiltonian P^- in the light-cone coordinates. In the reference frame where $\mathbf{P} = 0$ we have

$$P^- = \frac{1}{\sqrt{2}} P^0.$$

This gives the potential in the Cartesian coordinates,

$$V(r) = -\frac{e^2}{4\pi r}, \quad (6.10)$$

and clearly shows the Coulomb potential together with an irrelevant infinite constant in Eq. (6.9) (which is proportional to M) arising from the fermion self-energy.

VII. CONCLUSIONS

We have considered the light-cone canonical quantization of the heavy-fermion QED taking into account the zero-mode contributions explicitly. We have imposed periodic boundary conditions for bosonic fields and antiperiodic ones for fermionic fields. As in ordinary QED, this model is gauge invariant, which means that there are unphysical degrees of freedom. To quantize the theory, we have used the Dirac algorithm for a system with first- and second-class constraints and the corresponding gauge conditions. In order to make explicit the role of the zero modes, we have considered gauge fixing and quantization procedures in the zero-mode and nonzero-mode sectors, separately. In all sectors we obtained the physical variables and their canonical (anti)commutation relations. The physical Hamiltonian was constructed by excluding unphysical degrees of freedom. We have considered the role of all physical fields in the calculation of the potential between static heavy fermions.

We suggest that this approach can be used for the case of finite-mass QED or, perhaps, one might first address heavy-quark QCD. An important goal is the calculation of the QCD quark-antiquark potential.

Note added in proof. The role of zero modes in light-cone QED using a Lagrangian approach has been studied very recently by Kalloniatis and Pauli [35].

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