General structure of correlation functions in stochastic quantization

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We investigate the general structure of stationary correlation functions in stochastic quantization. On the basis of the $(D+1)$-dimensional field-theoretical formulation (operator formalism), we prove the fluctuation dissipation theorem which establishes a link between two types of correlation functions $\langle\phi\phi\rangle$ and $\langle\phi\pi\rangle$, π being the conjugate field to ϕ. A specific structure of the self-energies to the correlation functions is clarified in $(D+1)$-dimensional momentum space, which, together with the fluctuation dissipation theorem, enables us to extract the fictitious time dependence of the correlation functions: The correlation length along the fictitious time is inversely proportional to $p^2+m^2_{\rm phys}$, $m_{\rm phys}$ being the physical (pole) mass obtained in ordinary field theory.

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I. INTRODUCTION

In the stochastic quantization (SQ) of Parisi and Wu [1], an extra degree of freedom t, called fictitious time (or simply called a time in the following if no confusion arises), is introduced in addition to the ordinary D-dimensional Euclidean coordinates x, and field variables $\phi(x)$ are regarded as random variables $\phi(X)\equiv\phi(x,t)$ subject to the Langevin equation

$$\frac{\partial}{\partial t}\phi(X)=-\kappa\frac{\delta S[\phi]}{\delta\phi(X)}+\eta(X) . \tag{1.1}$$

Here $S[\phi]$ stands for a classical action of the system. We have introduced a kernel factor $\kappa(>0)$ in the above and the statistical property of the Gaussian white noise $\eta(X)$ reads

$$\langle\eta(X)\rangle=0 , \quad \langle\eta(X)\eta(X')\rangle=2\kappa\delta^{D+1}(X-X') . \tag{1.2}$$

The quantization is supposed to be completed if we solve the Langevin Eq. (1.1) to get the solution $\phi_\eta(X)$ as a functional of η and calculate the equal-time correlation functions

$$\langle\phi_\eta(X_1)\cdots\phi_\eta(X_n)\rangle_{t_1=\cdots=t_n=t}$$

in the equilibrium limit $t\to\infty$.

It is well known that in equilibrium the correlation functions represent the corresponding Green's functions in ordinary D-dimensional field theory:

$$\begin{aligned}\lim_{t_1=\cdots=t_n=t\to\infty}&\langle\phi_\eta(X_1)\cdots\phi_\eta(X_n)\rangle\\ &=\langle T\phi(x_1)\cdots\phi(x_n)\rangle\\ &=\int\mathcal{D}\phi\,\phi(x_1)\cdots\phi(x_n)e^{-S[\phi]} .\end{aligned} \tag{1.3}$$

This equivalence is best seen in the Fokker-Planck formalism which prescribes the time development of a probability distribution $P[\phi;t]$ inherent to the stochastic process described by the above Langevin equation. The Fokker-Planck equation equivalent to (1.1) is

$$\frac{\partial}{\partial t}P[\phi;t]=H[\phi,\pi]P[\phi;t] , \tag{1.4}$$

where the Fokker-Planck Hamiltonian H is given by

$$\begin{aligned}H[\phi,\pi]&=\frac{1}{2\kappa}\int d^Dx\left[\frac{1}{2}\pi^2(x)-\kappa\pi(x)\frac{\delta S[\phi]}{\delta\phi(x)}\right]\\ &=\kappa\int d^Dx\,\frac{\delta}{\delta\phi(x)}\left[\frac{\delta}{\delta\phi(x)}+\frac{\delta S[\phi]}{\delta\phi(x)}\right] .\end{aligned} \tag{1.5}$$

Remember that the momentum operator $\pi(x)$ is represented by a differential operator $-2\kappa\delta/\delta\phi(x)$ in the ϕ-diagonal representation. We easily see that the equilibrium probability distribution $P_{\rm eq}[\phi]$ is nothing but the usual Feynman measure $e^{-S[\phi]}$ if it is normalizable. It is also to be noted that the only role of the positive kernel factor κ is to control the rate of the system's approach to equilibrium and that it has nothing to do with the equilibrium distribution itself: A different choice of κ corresponds to a different stochastic process but with the same equilibrium state.

Since its proposal, this quantization method has been applied to various problems in field theories and their results are shown to be equivalent to those obtained by the conventional quantization methods [2,3]. However, in most cases this new degree of freedom has been used just as a mathematical tool (or a computer time in numerical simulations) to generate random variables $\phi(x)$ subject to the equilibrium distribution $P_{\rm eq}[\phi]$ and then has simply been discarded by considering the equal-time correlation functions (1.3). To exhibit potential advantage of this quantization method over the conventional ones, possible dynamical roles of the fictitious time that provide us with new insight, not accessible in the conventional methods, have to be clarified.

Several years ago, an interesting observation about the dynamical role of fictitious time was made by Namiki and co-workers [4,5]. They considered the stationary two-point function $\langle\phi(X)\phi(X')\rangle$ with the nonzero time separation $\tau=t-t'$ and concentrated on the large τ behavior. They claim that the physical mass or energy gap may be obtainable from the correlation length along τ. The claim is based on the observation that in the free case the correlation length along τ is inversely proportional to the mass squared:

$$\lim_{\min(t,t')=\infty} \int d^D x \langle \phi(X)\phi(X') \rangle = \frac{1}{m^2} e^{-\kappa m^2 |\tau|} , \qquad (1.6)$$

and on a plausible argument in the interacting case. They also simulated the correlation function using a solvable quantum-mechanical model and observed a nice exponential decay with respect to τ. The measured correlation length with respect to τ showed a good agreement with the exact value of the energy gap of the system [4].

Applications of the above idea to field theories need more care and some progress has been reported. In their study of the O(N) nonlinear σ model, Okano and Schülke [6] carried out the renormalization program within the framework of SQ and obtained the β functions. They investigated the renormalization-group behavior of the correlation function with different time arguments [7] and clarified the effect of the renormalization of κ in their context [8–10]. A nice scaling behavior of the correlation length along the fictitious time consistent with the theoretical prediction was observed in their numerical simulation [7]. On the other hand, the structure of the stationary two-point functions was also investigated from the point of view of the $(D+1)$-dimensional field theoretical formulation (operator formalism) of SQ [11] and their spectral representation was derived under several assumptions [12].

The idea of extracting physical information directly from the fictitious time dynamics is itself new and very appealing: It may supply us with additional information completely independent of that obtainable in conventional ways and certain practical advantages can be expected in numerical simulations [4].

In spite of the analyses mentioned above, however, we still feel that we have not yet reached a satisfactory understanding of the fictitious time dynamics of the stationary two-point functions. It is true that the time dependence of correlation functions has intensively been investigated from the viewpoint of dynamical critical phenomena, both for real- and fictitious-time stochastic processes [13]. We want to stress here that the renormalization-group equation, on which the analyses are based, can supply us with scaling properties of correlation functions; however, the relation between the ordinary and fictitious dynamics remains unclear owing to an arbitrary function not determined in the renormalization-group analysis. The explicit time dependence of the correlation function and its relation to the physical mass have not been clarified, in general, except for a special case of the large-N limit of an O(N)-invariant model [7], because of our ignorance about the above-mentioned arbitrary function. The spectral representation derived so far is crucially dependent on rather a strong assumption [12] and one must admit that its justification remains difficult.

What is most needed is a knowledge of the dynamical structure of the stationary correlation functions (in other words, the structure of the function which is left arbitrary in the renormalization-group analysis). Only when such information as a dispersion relation between the ordinary and the fictitious momenta is provided, as in the case for the large-N limit of the O(N)-invariant model [14], can one draw definite conclusions on the above issue [7].

In this paper we investigate the dynamical structure of the stationary correlation functions in SQ in order to meet the above requirement. The investigation is based on a few general assumptions (e.g., existence of a normalizable stationary state) and no strong assumptions are needed. Though observation of the perturbative expansions of the correlation functions plays a crucial role in this study, it should be stressed that the result has a nonperturbative content in the sense that contributions from all orders of perturbation have been incorporated. For simplicity and definiteness we exclusively consider a system of self-interacting scalar field ϕ; however, the result has a model-independent nature and can easily be extended to other cases within the validity of the general assumptions.

This paper is organized as follows. In the next section, a relation between the ϕ-ϕ and the ϕ-π correlation functions is derived on the basis of the $(D+1)$-dimensional field theoretical formulation of SQ [11]. The relation, which is nothing but the fluctuation dissipation theorem, implies that all dynamics contained in $\langle \phi\phi \rangle$ can be constructed simply from those of $\langle \phi\pi \rangle$. We shall see that the relation is transformed into another one between proper self-energies to the correlation functions in SQ. From observation of the diagrammatic expression of $\langle \phi\pi \rangle$, we are able to extract a specific functional dependence of the proper self-energy on the fictitious momentum in Sec III. In Sec. IV, integration over the fictitious momentum is successfully carried out to obtain an explicit τ dependence of the stationary correlation function with a time separation τ. It is further shown that the inverse of the τ-correlation length is given by a pole position $\Omega_0(p)$ of $\langle \phi\pi \rangle$, which is proved to be proportional to $p^2+m^2_{\text{phys}}$, m_{phys} being the physical (pole) mass observed in the ordinary field theory. The last section (Sec. V) is devoted to a summary and discussion. In the Appendix, a relation which establishes a link between the proper self-energy to $\langle \phi\pi \rangle$ in SQ and the ordinary one in the usual field theory is proved by making use of the supertransformation invariance [15].

II. RELATIONSHIP BETWEEN ϕ-ϕ AND ϕ-π CORRELATION FUNCTIONS

In this section, we shall clarify the close relationship between the stationary ϕ-ϕ and ϕ-π correlation functions. The analysis is based on the $(D+1)$-dimensional field-theoretical formulation of SQ (operator formalism) [11]. The operator formalism has been constructed as a $(D+1)$-dimensional analogue of the ordinary canonical theory so that techniques developed in the ordinary field theory are also available. For example, stationary correlation functions are expressed as "vacuum-to-vacuum" expectation values of field operators. Details of the formalism are found in the original paper [11] or in the review article [3].

In the operator formalism of SQ, the canonical field operators $\phi(X)$ and $\pi(X)$, subject to the equal-time commutation relation

$$[\phi(X),\pi(X')]_0=2\kappa\delta^D(x-x') , \tag{2.1}$$

satisfy the "Heisenberg" equations

$$\frac{\partial}{\partial t}\phi(X)=[H,\phi(X)]=\pi(X)-\kappa\frac{\delta S}{\delta\phi(X)} , \tag{2.2a}$$

$$\frac{\partial}{\partial t}\pi(X)=[H,\pi(X)]=\kappa\pi(X)\frac{\partial}{\partial\phi(X)}\frac{\delta S}{\delta\phi(X)} , \tag{2.2b}$$

where H is the Fokker-Planck Hamiltonian already given in (1.5). There exist two "vacuum" states $|u_0\rangle$ and $|v_0\rangle$, i.e., the zero eigenstates of H and $H^\dagger$, respectively, and they are prescribed by the stationary conditions

$$\left[\pi(X)-2\kappa\frac{\delta S}{\delta\phi(X)}\right]|u_0\rangle=0 , \quad \langle v_0|\pi(X)=0 . \tag{2.3}$$

The stationary correlation functions are expressed as the "vacuum-to-vacuum" expectation values of the time-ordered product of operators: e.g.,

$$\begin{aligned}D(X-X')&\equiv\lim_{\substack{\min(t,t')=\infty\\ |t-t'|<\infty}}\langle\phi(X)\phi(X')\rangle\\ &=\langle v_0|T\phi(X)\phi(X')|u_0\rangle ,\end{aligned} \tag{2.4}$$

$$\begin{aligned}G(X-X')&\equiv\lim_{\substack{\min(t,t')=\infty\\ |t-t'|<\infty}}\langle\phi(X)\pi(X')\rangle\\ &=\langle v_0|T\phi(X)\pi(X')|u_0\rangle .\end{aligned} \tag{2.5}$$

The π-π correlation function is identically zero owing to the stationary condition for $\langle v_0|$ (2.3).

To derive a relation between the stationary ϕ-ϕ (D) and the ϕ-π (G) correlation functions, we differentiate $D(X-X')$ with respect to $\tau\equiv t-t'$:

$$\begin{aligned}\theta(-\tau)\frac{\partial}{\partial\tau}D(X-X')&=\theta(-\tau)\langle v_0|\phi(X')\left[\pi(X)-\kappa\frac{\delta S}{\delta\phi(X)}\right]|u_0\rangle\\ &=\tfrac{1}{2}\theta(-\tau)\langle v_0|\phi(X')\pi(X)|u_0\rangle\\ &=\tfrac{1}{2}\langle v_0|T\phi(X')\pi(X)|u_0\rangle=\tfrac{1}{2}G(X'-X) ,\end{aligned} \tag{2.6}$$

where use has been made of the Heisenberg Eq. (2.2a) and the stationary conditions (2.3). From this relation, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}D(X-X')&=\left[\theta(-\tau)\frac{\partial}{\partial\tau}-\theta(\tau)\frac{\partial}{\partial(-\tau)}\right]D(X-X')\\ &=\tfrac{1}{2}[G(X'-X)-G(X-X')] .\end{aligned} \tag{2.7}$$

It would be worthwhile to remark that the relation, which is satisfied by the full correlation functions D and G thus providing us with a generalization of a similar relation in the free case [15], is nothing but a supersymmetric Ward identity [16] and a realization of the fluctuation dissipation theorem [17,18]. The same form of the fluctuation dissipation theorem, which establishes a link between the correlation and the response functions, has been previously derived in a slightly different context [19]. Here we briefly mention the relevance to the earlier works [16–18]. The correspondence would be clear if we recall that the $(D+1)$-dimensional action integral associated with the Hamiltonian (1.5) is given by (using an overdot to denote $\partial/\partial t$)

$$\begin{aligned}I&=\frac{1}{2\kappa}\int d^{D+1}X\,\pi\dot{\phi}-\int dt\,H\\ &=\int d^{D+1}X\left[-\frac{1}{4\kappa}\pi^2+\frac{1}{2\kappa}\pi\left[\dot{\phi}+\kappa\frac{\delta S}{\delta\phi}\right]\right] .\end{aligned} \tag{2.8}$$

Introducing a source function $j(x)$ for $\phi(x)$ and adding a source term $\int d^Dx\, j(x)\phi(x)$ to the classical action S, we understand that the response function defined by

$$R(X-X')=\frac{\delta}{\delta j(X')}\langle\phi(X)\rangle_j\Big|_{j=0} \tag{2.9}$$

is nothing but ($-\frac{1}{2}$ times) the ϕ-π correlation function.

In order to study the structure of the correlation functions in more detail, we go into the $(D+1)$-dimensional momentum space (p,Ω). We define Fourier transforms as follows $[f(X)\equiv f(x,t)]$:

$$\begin{aligned}f(p,\Omega)&=\int d^{D+1}X\, f(X)e^{i\Omega t-ip\cdot x} ,\\ f(X)&=\int\frac{d^Dp}{(2\pi)^D}\frac{d\Omega}{2\pi}f(p,\Omega)e^{-i\Omega t+ip\cdot x} .\end{aligned} \tag{2.10}$$

The above fluctuation dissipation theorem (2.7) is now expressed as

$$D(p,\Omega)=\frac{1}{2i\Omega}[G(p,\Omega)-G(p,-\Omega)]=\frac{1}{\Omega}\mathrm{Im}[G(p,\Omega)] . \tag{2.11}$$

The relation implies that the dynamical structure of the ϕ-ϕ correlation function D is completely determined through that of the ϕ-π correlation function G. If we remember the fact that the perturbative expansion for G is less involved than that for D, we may expect a considerable simplification in the perturbative treatment of D. We stress that this theorem holds for any system given by a Hamiltonian of the form (1.5), as is clear from the above derivation.

Next let us introduce the proper one particle irreducible (1PI) self-energies to the correlation functions. It is well known that there appear to be two different types of self-energies $\Sigma_{\pi\phi}(p,\Omega)$ $[=\Sigma_{\phi\pi}(p,-\Omega)]$ and $\Sigma_{\pi\pi}(p,\Omega)$ in the $(D+1)$-dimensional field-theoretical treatment of SQ [5,10]. They are defined in a 2×2 matrix form by

$$\mathcal{X}(p,\Omega)=\mathcal{X}_0(p,\Omega)-\mathcal{X}_0(p,\Omega)\mathcal{Z}(p,\Omega)\mathcal{X}(p,\Omega)\ , \tag{2.12}$$

where

$$\mathcal{Z}(p,\Omega)=\begin{bmatrix} 0 & \Sigma_{\phi\pi}(p,\Omega) \\ \Sigma_{\pi\phi}(p,\Omega) & \Sigma_{\pi\pi}(p,\Omega)\end{bmatrix}\ , \tag{2.13a}$$

$$\mathcal{X}(p,\Omega)=\begin{bmatrix} D(p,\Omega) & G(p,\Omega)/2\kappa \\ G(p,-\Omega)/2\kappa & 0\end{bmatrix}\ , \tag{2.13b}$$

$$\mathcal{X}_0(p,\Omega)=\begin{bmatrix} D_0(p,\Omega) & G_0(p,\Omega)/2\kappa \\ G_0(p,-\Omega)/2\kappa & 0\end{bmatrix}\ . \tag{2.13c}$$

The free correlation functions D_0 and G_0 are explicitly written as

$$D_0(p,\Omega)=\frac{-2\kappa}{[i\Omega+\kappa(p^2+m^2)][i\Omega-\kappa(p^2+m^2)]}\ , \tag{2.14a}$$

$$G_0(p,\Omega)/2\kappa=\frac{-1}{i\Omega-\kappa(p^2+m^2)}\ . \tag{2.14b}$$

In the above we have taken into account the fact that the π-π correlation functions are identically zero both in the free and interacting cases [i.e., the (2,2) elements of $\mathcal{X}$ and $\mathcal{X}_0$] and its resulting identity $\Sigma_{\phi\phi}=0$ [i.e., the (1,1)-element of $\mathcal{Z}$]. These characteristics allow us to easily solve (2.12) for G and D.

From the (1,2) element of (2.12), we obtain

$$G(p,\Omega)/2\kappa=G_0(p,\Omega)/2\kappa-[G_0(p,\Omega)/2\kappa]\Sigma_{\pi\phi}(p,\Omega)[G(p,\Omega)/2\kappa]\ , \tag{2.15}$$

which solving for G becomes

$$G(p,\Omega)/2\kappa=\frac{-1}{i\Omega-\kappa(p^2+m^2)-\Sigma_{\pi\phi}(p,\Omega)}\ . \tag{2.16}$$

The (2,1) element gives us exactly the same form. The remaining (1,1) element seems somewhat complicated:

$$D(p,\Omega)=D_0(p,\Omega)-[G_0(p,\Omega)/2\kappa]\Sigma_{\pi\phi}(p,\Omega)D(p,\Omega)-\{D_0(p,\Omega)\Sigma_{\phi\pi}(p,\Omega)+[G_0(p,\Omega)/2\kappa]\Sigma_{\pi\pi}(p,\Omega)\}[G(p,-\Omega)/2\kappa]\ . \tag{2.17}$$

However, observing the relations [see (2.14) and (2.15)]

$$D_0(p,\Omega)=2\kappa[G_0(p,\Omega)/2\kappa][G_0(p,-\Omega)/2\kappa]\ ,$$

$$1+[G_0(p,\Omega)/2\kappa]\Sigma_{\pi\phi}(p,\Omega)=[G_0(p,\Omega)/2\kappa][G(p,\Omega)/2\kappa]^{-1}\ ,$$

and

$$G_0(p,-\Omega)/2\kappa-[G_0(p,-\Omega)/2\kappa]\Sigma_{\phi\pi}(p,\Omega)\times[G(p,-\Omega)/2\kappa]=G(p,-\Omega)/2\kappa\ ,$$

we are able to reach the relation

$$D(p,\Omega)=[G(p,\Omega)/2\kappa][2\kappa-\Sigma_{\pi\pi}(p,\Omega)]G(p,-\Omega)/2\kappa\ . \tag{2.18}$$

(For another diagrammatic derivation, see Ref. [10].) Figure 1 shows the structures of G and D diagrammatically.

We are now in a position to derive a relation between the two self-energies $\Sigma_{\pi\phi}$ and $\Sigma_{\pi\pi}$ in SQ. Because the above expression for $G/2\kappa$ (2.16) allows us to rewrite the right-hand side (RHS) of (2.11) as

$$\frac{1}{\Omega}\mathrm{Im}[G(p,\Omega)]=\frac{2\kappa}{2i\Omega}\frac{2i\Omega-[\Sigma_{\pi\phi}(p,\Omega)-\Sigma_{\pi\phi}(p,-\Omega)]}{[i\Omega-\kappa(p^2+m^2)-\Sigma_{\pi\phi}(p,\Omega)][-i\Omega-\kappa(p^2+m^2)-\Sigma_{\pi\phi}(p,-\Omega)]}$$
$$=[G(p,\Omega)/2\kappa]\left[2\kappa-\frac{2\kappa}{2i\Omega}[\Sigma_{\pi\phi}(p,\Omega)-\Sigma_{\pi\phi}(p,-\Omega)]\right][G(p,-\Omega)/2\kappa]\ ,$$

this theorem (2.11), combined with the above relation (2.18), is transformed into a relation between self-energies:

$$\Sigma_{\pi\pi}(p,\Omega)=\frac{2\kappa}{2i\Omega}[\Sigma_{\pi\phi}(p,\Omega)-\Sigma_{\pi\phi}(p,-\Omega)]=\frac{2\kappa}{\Omega}\mathrm{Im}[\Sigma_{\pi\phi}(p,\Omega)]\ . \tag{2.19}$$

The relation was first observed from the explicit forms of $\Sigma_{\pi\pi}$ and $\Sigma_{\phi\pi}$ in the one-loop calculation and was called the "optical theorem" [10]. The above derivation shows that it does hold true, in general, at any order and that it is a direct consequence of the fluctuation dissipation theorem.

We have already seen from (2.16) and (2.18) that the ϕ-ϕ correlation function is dependent on the self-energy

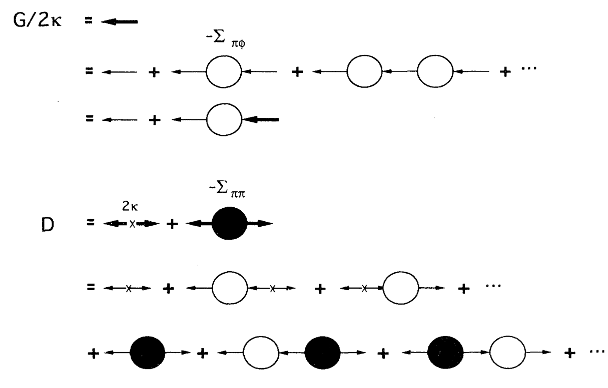

FIG. 1. Diagrammatic representations of the correlation functions $G/2\kappa$ and D. A directed thin line stands for $G_0/2\kappa$.

$\Sigma_{\pi\pi}$ as well as $\Sigma_{\pi\phi}$, while $\Sigma_{\pi\phi}$ is sufficient to determine the structure of the ϕ-π correlation function. The above relation (2.19) reveals a close link between these apparently different self-energies $\Sigma_{\pi\pi}$ and $\Sigma_{\pi\phi}$ in SQ: The latter has enough information to determine the structure of the former. Thus we are led to the conclusion that we only need to study in detail the structure of $\Sigma_{\pi\phi}(p,\Omega)$. We shall consider $\Sigma_{\pi\phi}(p,\Omega)$ exclusively in the following.

III. THE STRUCTURE OF THE SELF-ENERGY $\Sigma_{\pi\phi}(p,\Omega)$

To gain insight into the structure of $\Sigma_{\pi\phi}(p,\Omega)$, let us begin by considering a building block whose iterated use constitutes the full correlation function $G(p,\Omega)/2\kappa$. See Figs. 1 and 2. Its mathematical expression reads as $(\tau\equiv t-t')$

$$\int \frac{d^Dp}{(2\pi)^D}\frac{d\Omega}{2\pi}e^{-\Omega\tau+ip\cdot x}[G_0(p,\Omega)/2\kappa][-\Sigma_{\pi\phi}(p,\Omega)]\times[G_0(p,\Omega)/2\kappa]\ , \tag{3.1a}$$

or, in configuration space,

$$=\int d^{D+1}X_1 d^{D+1}X_2[G_0(X-X_1)/2\kappa][-\Sigma_{\pi\phi}(X_1-X_2)]\times[G_0(X_2-X')/2\kappa]\ . \tag{3.1b}$$

Here we are mainly interested in the fictitious momentum (Ω) dependence of $\Sigma_{\pi\phi}(p,\Omega)$. To extract the Ω dependence of $\Sigma_{\pi\phi}(p,\Omega)$, let us first focus our attention to the τ dependence of the above diagram Fig. 2 or (3.1).

The diagram (3.1) can be calculated perturbatively on the basis of the stochastic $[(D+1)$-dimensional] action

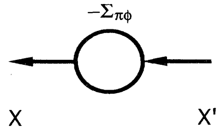

FIG. 2. A building-block diagram for the correlation function $G/2\kappa$.

(2.8). Some of its important characteristics are the following.

(i) The topological structure is the same as that of the ordinary Feynman graph.

(ii) For each internal line, connecting the ith and jth vertices and carrying momentum p_l, one of the two propagators

$$G_0(p_l;t_i-t_j)/2\kappa=\theta(t_i-t_j)e^{-K_1(p_l)(t_i-t_j)} \tag{3.2a}$$

or

$$D_0(p_l;t_i-t_j)=\frac{\kappa}{K_1(p_l)}e^{-K_1(p_l)|t_i-t_j|} \tag{3.2b}$$

is assigned, where

$$K_1(p_l)\equiv\kappa(p_l^2+m^2) \tag{3.3}$$

and t_i is the internal fictitious time attached to the ith vertex. Notice that each internal line depends exponentially on the internal time and the exponent is always negative.

(iii) We have to perform internal time integrations in addition to the ordinary internal momentum integrations.

(iv) The causal property of G_0 and the interaction form $\pi\delta S_{\text{int}}/\delta\phi$ in (2.8) make the above diagram causal, that is, $t\geq t_1\geq t_2\geq t'$ and if $t'>t$ it vanishes.

One finds it convenient to work in t-space rather than in Ω-space because the internal time integrations can easily be performed if they are split into contributions of fixed-time orderings [2,20]. Then every contribution of a fixed-time ordering is summed together. (This is the reason why the above characteristics are presented in t space.)

Suppose that a specific time ordering is fixed. Every internal time integration produces two terms, arising from the upper and the lower limits of the integration. Because of the inequality $t_1\geq t'$ [see (iv) above] and the ordering of the integration variables, after all internal integrations except for one over t_1 are performed, these internal times are set equal to either t_1 (the upper limit) or t' (the lower limit). Observe that at this stage the exponents assigned to the internal lines connecting the ith and jth vertices survive only when the two internal times t_i and t_j attached to these vertices are set differently (i.e., $t_i\neq t_j$). Otherwise (i.e., $t_i=t_j=t_1$ or $t_i=t_j=t'$) the exponents disappear.

Therefore, we can classify the terms which appear under the final t_1 integration according to their t_1 dependence. They are either (a) t_1 independent if all internal times are set equal to t_1, because all exponents of internal propagators disappear, leaving only those of external propagators

$$e^{-K_1(p)(t-t_1)}e^{-K_1(p)(t_1-t')}=e^{-K_1(p)\tau}\ ,$$

or (b) t_1 dependent if otherwise. The t_1-dependence appears in the exponential form

$$e^{-K_1(p)(t-t_1)}\exp\left[-\sum_l K_1(p_l)(t_1-t')\right],$$

where the summation over l extends over all lines whose end point times are differently set equal to t_1 and t', respectively, after the integrations. If there are n (>1) such lines $l_1, l_2, \ldots, l_n$ the summation over l is explicitly written as

$$\sum_{k=1}^{n} K_1(p_{l_k})=\kappa\sum_{k=1}^{n}(p_{l_k}^2+m^2)\equiv K_n(p_{l_k};p)\,, \tag{3.4}$$

where the line momenta p_{l_k} should satisfy $\sum_{k=1}^{n} p_{l_k}=p$ because of the momentum conservation at every vertex.

The final integration over t_1 is now easily performed. We can write down the final integration in the form

$$-\theta(\tau)\int_{t'}^{t}dt_1\left[f_0(p)e^{-K_1(p)\tau}+\sum_{n>1}\int_{p_i}\sum_{\{l_k\}}\tilde{f}_n(p_{l_k};p)\exp[-K_1(p)(t-t_1)-K_n(p_{l_k};p)(t_1-t')]\right], \tag{3.5}$$

where $\int_{p_i}$ stands for the ordinary internal momentum integrations and the summation $\sum_{\{l_k\}}$ is taken over all possible combinations $\{l_k\}$. All contributions of different time orderings have been included in the real functions f_0 and $\tilde{f}_n$. Thus after the integration over t_1 we can finally reach the general form

$$\text{Fig.2}=-\theta(\tau)\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\left[\tau f_0(p)e^{-K_1(p)\tau}+\sum_{n=1}\int_{p_i}f_n(p)e^{-K_n(p)\tau}\right]. \tag{3.6}$$

Here and hereafter, to simplify the notation, we suppress the internal momentum dependence of the functions f_n and K_n together with the summation over possible combinations of internal momenta $\sum_{\{l_k\}}$.

It is then possible to show that the real functions f_n $(n=0,1,2,\ldots)$ introduced in (3.6) are subject to the "sum" rules

$$\sum_{n=1}\int_{p_i}f_n(p)=0 \tag{3.7}$$

and

$$f_0(p)=\sum_{n=1}\int_{p_i}f_n(p)K_n(p)\,. \tag{3.8}$$

The first equality (3.7) simply reflects the fact that there is no integration interval if we set $t=t'$ in (3.5), implying that the quantity in the large parentheses in (3.6) vanishes when $\tau=t-t'=0$. It is also not difficult to see that the same structure remains even if we take the t derivative of (3.1b). In fact, it produces two terms, both of which vanish if $t=t'$ because the integration volume reduces to zero in this case also. See Fig. 3. The second sum rule (3.8) is a consequence of this property: The t derivative of (3.6) evaluated at $t=t'$ vanishes.

Now a comparison of the Fourier transform with respect to τ of (3.6) and (3.1a) yields [see (2.14b)]

$$\Sigma_{\pi\phi}(p,\Omega)=f_0(p)-[i\Omega-K_1(p)]^2\sum_{n=1}\int_{p_i}\frac{f_n(p)}{i\Omega-K_n(p)}\,,$$

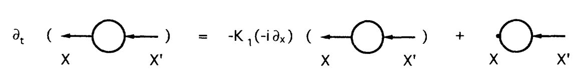

FIG. 3. Time (t) derivative of the diagram Fig. 2 or (3.1) produces two terms both of which are still causal and vanish when $\tau=t-t'=0$.

which, with the help of the sum rules (3.7) and (3.8), can be further reduced to

$$\Sigma_{\pi\phi}(p,\Omega)=-\sum_{n=1}\int_{p_i}\frac{f_n(p)[K_n(p)-K_1(p)]^2}{i\Omega-K_n(p)}\,. \tag{3.9}$$

It is worth mentioning here that $\Sigma_{\pi\phi}(p,\Omega)$ has no singularity at $\Omega=-iK_1(p)=-i\kappa(p^2+m^2)$. It has only simple poles at $\Omega=-iK_n(p)$ with $n>1$ inside the internal momentum integrations, which may be considered to represent contributions coming from higher excited states whose spectrum is continuous.

IV. FICTITIOUS TIME DYNAMICS OF CORRELATION FUNCTIONS

Since we have obtained information on the structure of the correlation functions in momentum space, especially on their fictitious momentum dependence (2.16) and (3.9), our next task is to carry out the integration over the fictitious momentum to extract their time dependence.

As was already mentioned in Sec. II, the fluctuation dissipation theorem allows us to concentrate on the ϕ-π correlation function G. The time dependence of the ϕ-ϕ correlation function D is determined through the analytic property of $G(p,\Omega)$, which is again easily understood from the fluctuation dissipation theorem (2.7):

$$D(X-X')=\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\times\int\frac{d\Omega}{2\pi}\frac{1}{2i\Omega}[G(p,\Omega)-G(p,-\Omega)]e^{-i\Omega\tau}. \tag{4.1}$$

Note that the residue at $\Omega=0$ is zero so that it has no contribution if $\tau\equiv t-t'\neq 0$.

Consider the ϕ-π correlation function G in configuration space. To make its analytical property transparent, we rewrite G in the form [see (2.16)]

$$G(X-X')=\int \frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\int \frac{d\Omega}{2\pi}\frac{-2\kappa}{i\Omega-K_1(p)-\Sigma_{\pi\phi}(p,\Omega)}e^{-i\Omega\tau}$$
$$=\int \frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\int \frac{d\Omega}{2\pi}(-2\kappa)\sum_{N=0}^{\infty}\frac{[\Sigma_{\pi\phi}(p,\Omega)]^N}{[i\Omega-K_1(p)]^{N+1}}e^{-i\Omega\tau}. \quad (4.2)$$

We realize that this expression is nothing but the direct representation of the diagrammatic expansion of G (see Fig. 1). Remember that $\Sigma_{\pi\phi}(p,\Omega)$ has only simple poles at $\Omega=-iK_n(p)$ $(n>1)$ and vanishes at $\Omega=\infty$ [see (3.9)]. Remember that the numerator $[\Sigma_{\pi\phi}(p,\Omega)]^N$ in (4.2) has only simple poles since its poles must be considered nonidentical to each other because of the presence of internal momentum integration $\int_{p_i}$ in each factor. Thus each integrand in (4.2) has only multiple poles, i.e., an $(N+1)$-fold pole at $\Omega=-iK_1(p)$ and simple poles at $\Omega=-iK_n(p)$ $(n>1)$ and is well behaved at $\Omega=\infty$. The integration over Ω is now reduced to the evaluation of residues at these poles and the summation over N.

Let us first calculate the contribution arising from the simple poles of the numerator in (4.2). From (3.9), we see that the residue of the integrand in (4.2) at one of the poles of the numerator, for example, at $\Omega=-iK_n(p)$ $(n>1)$, is given by

$$-2i\kappa\int_{p_i}f_n(p)\left[\frac{\Sigma_{\pi\phi}(p,-iK_n(p))}{K_n(p)-K_1(p)}\right]^{N-1}e^{-K_n(p)\tau}. \quad (4.3)$$

There arise N such terms for each $n>1$. Therefore, the contribution sought is

$$-2\kappa\theta(\tau)\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\sum_{N=0}^{\infty}\sum_{n>1}\int_{p_i}f_n(p)N\left[\frac{\Sigma_{\pi\phi}(p,-iK_n(p))}{K_n(p)-K_1(p)}\right]^{N-1}e^{-K_n(p)\tau}$$
$$=-2\kappa\theta(\tau)\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\sum_{n>1}\int_{p_i}f_n(p)\left[\frac{K_n(p)-K_1(p)}{K_n(p)-K_1(p)-\Sigma_{\pi\phi}(p,-iK_n(p))}\right]^2 e^{-K_n(p)\tau}. \quad (4.4)$$

We find that the ϕ-π correlation function G includes terms which decay exponentially with respect to τ with continuous exponents $K_n(p)$ $(n>1)$.

Next we turn our attention to the $(N+1)$-fold pole at $\Omega=-iK_1(p)$ of the denominator in (4.2). To calculate its residue, we have to differentiate the numerator N times with respect to Ω and then set $\Omega=-iK_1(p)$. The calculation proceeds in the following way:

$$2i\kappa\theta(\tau)\sum_{N=0}^{\infty}(-i)^{N+1}\frac{1}{N!}\sum_{n=0}^{N}\binom{N}{n}\frac{\partial^n}{\partial\Omega^n}[\Sigma_{\pi\phi}(p,\Omega)]^N\bigg|_{\Omega=-iK_1(p)}(-i\tau)^{N-n}e^{-K_1(p)\tau}$$
$$=2\kappa\theta(\tau)e^{-K_1(p)\tau}\sum_{N=0}^{\infty}\sum_{n=0}^{N}\frac{1}{N!n!}\left[-\frac{\partial^2}{\partial\tau\,\partial Q}\right]^n[-\tau\Sigma_{\pi\phi}(p,-iQ)]^N\bigg|_{Q=K_1(p)}$$
$$=2\kappa\theta(\tau)e^{-K_1(p)\tau}e^{-\partial^2/\partial\tau\partial Q}e^{-\tau\Sigma_{\pi\phi}(p,-iQ)}\bigg|_{Q=K_1(p)}, \quad (4.5)$$

where the last equality follows from the fact that the upper limit of the summation over n can be extended to infinity.

To evaluate the last factor $e^{-\partial^2/\partial\tau\partial Q}e^{-\tau\Sigma_{\pi\phi}(p,-iQ)}$ in (4.5), first calculate its Fourier transform

$$\int\frac{dQ}{2\pi}e^{i\zeta Q}e^{-\partial^2/\partial\tau\partial Q}e^{-\tau\Sigma_{\pi\phi}(p,-iQ)}=\int\frac{dQ}{2\pi}e^{i\zeta Q}e^{i\zeta\partial/\partial\tau}e^{-\tau\Sigma_{\pi\phi}(p,-iQ)}$$
$$=\int\frac{dQ}{2\pi}e^{i\zeta[Q-\Sigma_{\pi\phi}(p,-iQ)]}e^{-\tau\Sigma_{\pi\phi}(p,-iQ)}. \quad (4.6)$$

Because $\Sigma_{\pi\phi}(p,-iQ)$ is a real function of real variables p and Q [see (3.9)], the inverse Fourier transform gives us

$$e^{-\partial^2/\partial\tau\partial Q}e^{-\tau\Sigma_{\pi\phi}(p,-i\Omega)}=\int d\zeta\frac{dQ'}{2\pi}\exp\{i\zeta[Q'-Q-\Sigma_{\pi\phi}(p,-iQ')]\}e^{-\tau\Sigma_{\pi\phi}(p,-iQ')}$$
$$=\int dQ'\delta[Q'-Q-\Sigma_{\pi\phi}(p,-iQ')]e^{-\tau\Sigma_{\pi\phi}(p,-iQ')}$$
$$=\sum_{Q_0}\frac{e^{-\tau(Q_0-Q)}}{|1-(\partial/\partial Q_0)\Sigma_{\pi\phi}(p,-iQ_0)|}, \quad (4.7)$$

where $Q_0 = Q_0(p,Q)$ is a solution of the equation

$$Q = Q_0 - \Sigma_{\pi\phi}(p, -iQ_0) \ . \tag{4.8}$$

It is now clear that the contribution coming from the multipole at $\Omega = -iK_1(p)$ is expressed in a compact form as

$$2\kappa\theta(\tau)\int \frac{d^Dp}{(2\pi)^D} e^{ip\cdot(x-x')} \sum_{\Omega_0} \frac{1}{|1-(\partial/\partial Q_0)\Sigma_{\pi\phi}(p,-i\Omega_0)|} e^{-\Omega_0\tau} \ , \tag{4.9}$$

where $\Omega_0 = \Omega_0(p) \equiv Q_0(p, K_1(p))$ satisfies

$$\Omega_0 = K_1(p) + \Sigma_{\pi\phi}(p, -i\Omega_0) \ . \tag{4.10}$$

Observe that, at least perturbatively, $\Omega_0(p)$ is a real positive quantity, $\Omega_0(p) > 0$, and the denominator in (4.9) never vanishes. Thus we realize again that the correlation function decays exponentially with respect to τ. However, it should be stressed that the exponent Ω_0 is a function of p^2 and does not depend on internal momenta. We may regard the above term (4.9) as a contribution coming from low-lying one-particle states which constitute a discrete spectrum. Furthermore, we should point out that the exponent $\Omega_0(p)$ is just ($-i$ times) the pole position of $G(p,\Omega)$ [see (2.16)]. The calculation leading to (4.9) thus demonstrates explicitly an important role of the pole position of $G(p,\Omega)$ in determining the τ dependence of the correlation functions.

To summarize, we have found the following expression for the stationary ϕ-π correlation function $G(X-X')$:

$$G(X-X') = 2\kappa\theta(\tau)\int \frac{d^Dp}{(2\pi)^D} e^{ip\cdot(x-x')} \left[\sum_{\Omega_0} \frac{e^{-\Omega_0\tau}}{|1-(\partial/\partial\Omega_0)\Sigma_{\pi\phi}(p,-i\Omega_0)|} - \sum_{n>1}\int_{p_i} f_n(p) \left[\frac{K_n(p)-K_1(p)}{K_n(p)-K_1(p)-\Sigma_{\pi\phi}(p,-iK_n(p))}\right]^2 e^{-K_n(p)\tau}\right] \ . \tag{4.11}$$

Notice that this expression has been derived under no special conditions. It should be considered as a general expression for G.

In the asymptotic limit $\tau \to \infty$, the correlation function is dominated by a term which has the largest exponent (i.e., the smallest exponent in absolute value). Assume that the discrete spectrum lies below the continuous one as in the case in the ordinary field theory and that the former consists of only one state. Then the integrated correlation function behaves in the asymptotic limit like

$$\int d^Dx\, G(X-X') \underset{\tau\to\infty}{\sim} 2\kappa \frac{e^{-\Omega_0(0)\tau}}{|1-[\partial/\partial\Omega_0(0)]\Sigma_{\pi\phi}(0,-i\Omega(0))|} \ . \tag{4.12}$$

In the rest of this section we shall prove that the discrete exponent $\Omega(p)$ is proportional to $p^2 + m^2_{\rm phys}$ so that $\Omega(0) \propto m^2_{\rm phys}$, with $m_{\rm phys}$ being the physical mass defined as a pole position of the propagator in the field theory.

To prove the above statement, we shall look for a condition under which $\Omega_0(p)$ vanishes. Let p_* be the momentum such that $\Omega_0(p_*) = 0$. Then, from (4.10),

$$\begin{aligned} 0 = \Omega_0(p_*) &= K_1(p_*) + \Sigma_{\pi\phi}(p_*, -i\Omega_0(p_*)) \\ &= \kappa(p_*^2 + m^2) + \Sigma_{\pi\phi}(p_*, 0) \ . \end{aligned} \tag{4.13}$$

We can show below that the self-energy $\Sigma_{\pi\phi}(p,0)$ in SQ reduces to (κ times) the self-energy $\Sigma_{\rm FT}(p)$ to the full propagator in the ordinary field theory $\Delta'_{\rm F}(p)$:

$$\Delta'_{\rm F}(p) = \frac{1}{p^2 + m^2 + \Sigma_{\rm FT}(p)} \ . \tag{4.14}$$

This property follows from the equivalence of the equal-time ϕ-ϕ correlation function $D(X-X')|_{t=t'}$ to its counterpart $\Delta'_{\rm F}(x-x')$ in the field theory. We first write the equal-time ϕ-ϕ correlation function in momentum space:

$$\begin{aligned} \Delta'_{\rm F}(x-x') &= D(X-X')|_{t=t'} \\ &= \int \frac{d^Dp}{(2\pi)^D} e^{ip\cdot(x-x')} \int \frac{d\Omega}{2\pi} D(p,\Omega) \ . \end{aligned} \tag{4.15}$$

The last factor can be rewritten in terms of G by the fluctuation dissipation theorem (2.11):

$$\int \frac{d\Omega}{2\pi} D(p,\Omega) = \int \frac{d\Omega}{2\pi} \frac{1}{2i\Omega} [G(p,\Omega) - G(p,-\Omega)] \ . \tag{4.16}$$

Owing to the causal property of G which implies that $G(p,\Omega)$ has singularities only in the lower-half Ω plane and its asymptotic behavior $G(p,\Omega) \to 0$ as $\Omega \to \infty$ [see (2.16) and (3.9)], we can choose appropriate contours for each term in (4.16) to conclude that the only contribution comes from the residue at $\Omega = 0$ [10]:

$$\int \frac{d\Omega}{2\pi} D(p,\Omega) = \lim_{\epsilon\to 0}\int \frac{d\Omega}{2\pi} \frac{1}{2i(\Omega - i\epsilon)} [G(p,\Omega) - G(p,-\Omega)] = \tfrac{1}{2} G(p,0) \ . \tag{4.17}$$

Combining (4.14), (4.15), (4.17), and (2.16), we arrive at the desired result

$$\Sigma_{\pi\phi}(p,0)=\kappa\,\Sigma_{\rm FT}(p)\ . \tag{4.18}$$

We understand that the far RHS of (4.13) is simply $\kappa[\Delta_{\rm F}'(p_*)]^{-1}$ so that p_* is a momentum at which the full propagator $\Delta_{\rm F}'(p)$ has a pole, that is, $p_*^2+m_{\rm phys}^2=0$ from the definition of the physical (pole) mass $m_{\rm phys}$. Therefore, expanded around $p^2=p_*^2$, we see that $\Omega_0(p)$ contains a factor $p^2-p_*^2$,

$$\Omega_0(p)\propto p^2-p_*^2=p^2+m_{\rm phys}^2\ , \tag{4.19}$$

and the proportionality $\Omega_0(0)\propto m_{\rm phys}^2$ follows.

In the Appendix, another proof of the above equality (4.18), which is based on the supertransformation invariance of the stochastic diagram [15], is presented.

V. SUMMARY AND DISCUSSION

In this paper, we have investigated the τ dependence of the stationary two-point correlation functions with a finite time difference τ in SQ. The fluctuation dissipation theorem (2.7) and (2.11), which establishes a close link between the ϕ-ϕ correlation function (D) and the ϕ-π correlation function (G) was derived very easily within the framework of the $(D+1)$-dimensional field-theoretical formulation (operator formalism) of SQ. The theorem shows that the ϕ-ϕ correlation function can be constructed from the ϕ-π correlation function, thus allowing us to concentrate on the latter. Both correlation functions have essentially the same time dependence. Then we found that the ϕ-π correlation function has a simple structure (2.16) in the $(D+1)$-dimensional momentum space. A close consideration of the stochastic diagram Fig. 2 made it possible for us to extract a crucial fictitious momentum (Ω) dependence of the self-energy $\Sigma_{\pi\phi}(p\Omega)$ (3.9). From its analytic property with respect to Ω, we succeeded in performing the Ω integration to obtain the final form of the ϕ-π correlation function (4.11), which is the main result of this paper. The correlation functions are thus shown explicitly to decay exponentially with respect to τ. Furthermore, the appearance of the physical (pole) mass $m_{\rm phys}$ in their discrete exponent $\Omega_0(p)\propto p^2+m_{\rm phys}^2$ has been proved.

It should be stressed again that the analysis presented here is based on very general grounds and that no special conditions have been assumed. The result (4.11) can be considered to exhibit a general structure with respect to τ of the stationary two-point correlation functions in SQ.

General as it is, it would be worthwhile to comment explicitly on some general conditions, which have been implicitly assumed in the analysis, to clarify its range of validity and limitations. In the operator formalism, it is always assumed that there exists a stationary state $P_{\rm st}$. The existence of such a state implies that correlation functions acquire translational invariance in this state, so that the two-point correlation function $\langle\phi(X)\phi(X')\rangle$, for example, becomes a function of $X-X'$. We understand that these conditions have played a crucial role in deriving the fluctuation dissipation theorem. See (2.6) and (2.7). Of course, for any system described by a Fokker-Planck Hamiltonian of the form (1.5), we can find a stationary state $P_{\rm st}$ and, if it is normalizable, we can derive a similar fluctuation dissipation theorem. It is clear that if the stationary state $\sim e^{-S}$ is not normalizable owing to the presence of some symmetries in the classical system (e.g., gauge symmetries) or if the stochastic process is not Markovian, the present method could not be applied: The translational invariance would not be assured in the former case and a stationary state or even a stationary state condition does not seem to have been obtained explicitly in the latter case.

Next we would like to comment on the relationship between the operator formalism [11] on which the present analysis is based and the supersymmetric formulation of SQ [17,18]. The operator formalism was formulated on the basis of the so-called "Ito-related interpretation" of the Langevin equation [21,22], in which no ghost fields were required because of a trivial determinant factor. Ghost fields have been introduced to exponentiate a nontrivial determinant factor when the "Stratonovich-related interpretation" (midpoint prescription) or a similar one is adopted. Their introduction has led us to the supersymmetric formulation of SQ, which has recently been proved to be prescription independent, i.e., independent of the choice of interpretation of the Langevin equation, in the continuum limit [23]. However, their role, important as it is for the consistency of the formalism (e.g., renormalizability) [8,24], is limited to a cancellation of prescription-dependent terms which is proportional to $\theta(0)$ [25] if they appear only in internal loops. It is easily seen that every internal ghost loop and the corresponding ϕ-π loop, both of which are proportional to $\theta(0)$, have exactly the same contributions but with opposite signs. Therefore, in the present analysis where no ghost correlation functions are considered, we can safely neglect their contributions from the beginning by adopting a convention $\theta(0)=0$ [25]. Incidentally, it is a well-established fact that the Fokker-Planck equations take the same form irrespective of the choice of interpretation of the Langevin Eq. (1.1) (i.e., for the additive-noise case) [22], thus assuring interpretation-independent correlation functions for ϕ. This may be reflected by the fact that exactly the same fluctuation dissipation theorem as that obtained here has also been derived from the supersymmetric invariance of the $(D+1)$-dimensional stochastic action [16–18].

It is true that the analysis developed here largely depends on the crucial observation of the perturbative structure of the stochastic diagrams. However, the structure (3.6) found for the diagram Fig. 2 or (3.1) can be considered to hold true for any order of perturbation: All contributions from perturbative series have been taken into account in f_n. In this sense, the results (3.10), (4.11), and (4.19) are considered to possess a nonperturbative content. Of course, no essentially nonperturbative effects (e.g., instanton effect) have been incorporated here. The inclusion of such effects seems important, but is beyond the scope of the present work.

The final result (4.11) shows that the τ dependence of the stationary correlation functions appears only in exponential form. This statement, which has also been confirmed explicitly in perturbative calculations of correlation functions [26], is nothing but a realization of their "spectral decomposition" [12]. If we set up the following eigenvalue equations for the Fokker-Planck Hamiltonian H (1.5),

$$\begin{aligned} H|u_n,p\rangle &= -\lambda_n(p)|u_n,p\rangle \ , \\ H^\dagger|v_n,p\rangle &= -\lambda_n(p)|v_n,p\rangle \ , \end{aligned} \tag{5.1}$$

with

$$0\equiv\lambda_0<\lambda_1(p)\le\lambda_2(p)\le\cdots \ ,$$

we can show that the stationary two-point functions D and G can be decomposed into the spectrum [12]

$$D(X-X')=\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\sum_{n\neq0}\frac{\kappa\rho_n(p)}{\lambda_n(p)}e^{-\lambda_n(p)|\tau|} \ , \tag{5.2a}$$

$$G(X-X')=2\kappa\theta(\tau)\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\sum_{n\neq0}\rho_n(p)e^{-\lambda_n(p)\tau} \ . \tag{5.2b}$$

The summation over n should be understood to include possible internal momentum integrations, and a function $\rho_n(p)$ defined by

$$\kappa\rho_n(p)=\lambda_n(p)\langle v_0|\phi(0)|u_n,p\rangle\langle v_n,p|\phi(0)|u_0\rangle \ , \tag{5.3}$$

which is normalized and positive semidefinite,

$$\sum_{n\neq0}\rho_n(p)=1 \ , \quad \rho_n(p)\ge0 \ , \tag{5.4}$$

is an analogue to the ordinary spectral function. The results obtained in this paper are found consistent with these spectral decompositions (5.2). The present analysis, however, has made it possible to relate the eigenvalues of the Fokker-Planck Hamiltonian with dynamical quantities such as the self-energy $\Sigma_{\pi\phi}$. If we assume that the discrete spectrum always lies below the continuum one, we deduce the following correspondence: The low-lying discrete spectrum, which is assumed for simplicity to consist of only one state, is given by the pole position $\Omega_0(p)$ in the ϕ-π correlation function G,

$$\lambda_1(p)=\Omega_0(p)\propto p^2+m_{\rm phys}^2 \ , \tag{5.5}$$

and the remaining continuum spectrum corresponds to higher eigenvalues:

$$\lambda_n(p)=K_n(p)=\kappa\sum_{k=1}^{n}(p_{l_k}^2+m^2) \quad (n>1) \ . \tag{5.6}$$

Finally we shall discuss possible implications of the final result (4.11) for practical applications. The appearance of the physical mass $m_{\rm phys}$ in the exponent $\Omega_0(p)$ as a factor $p^2+m_{\rm phys}^2$ seems to offer a novel way of extracting physical information from the correlation length along the fictitious time direction. This was already expected as explained in Sec. I and was the main motivation to the present work. Our results may be considered to have partly proved the expectation: The fictitious time correlation length is inversely proportional to the physical mass squared. Incidentally, the appearance of the physical quantity (energy gap) in the exponent Ω_0 in the correlation functions or in the lowest nonzero eigenvalue λ_1 of the Fokker-Planck Hamiltonian has also been observed in second-order perturbative calculations for a simple quantum mechanical model [27,26].[1]

However, it should also be noted that to extract such physical information from the fictitious time correlation length we need to know the remaining factor of $\Omega_0(p)$ besides $p^2+m_{\rm phys}^2$. That is, if we write

$$\Omega_0(p)=h(p)(p^2+m_{\rm phys}^2) \tag{5.7}$$

the meaning of the factor $h(p)$ [or at least $h(0)$] has to be clarified. Only in the large-N limit of an O(N)-invariant model, has the exact form of $h(p)$ been derived [14]. It is generally expected that the factor $h(p)$ should be deeply connected with the renormalization of the kernel factor κ, as may be clear from the previous renormalization-group analysis [7,13]. Further study along this line of thought is now in progress.

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APPENDIX

We shall present here another proof of (4.18). The proof makes use of a (hidden) supertransformation invariance of the stochastic diagram [15] without assuming the equivalence between the equal time ϕ-ϕ correlation function and its counterpart in the field theory (4.15).

Consider the same stochastic diagram as in (3.1), but this time integrated over t':

[1]The spectrum of the Fokker-Planck Hamiltonian has been investigated in a different context both numerically and analytically. See Ref. [28].

$$J \equiv \int_{-\infty}^{\infty} dt' [\text{Fig. 2}] = -\int \frac{d^D p}{(2\pi)^D} e^{ip\cdot(x-x')} [G_0(p,0)/2\kappa] \Sigma_{\pi\phi}(p,0) [G_0(p,0)/2\kappa] , \tag{A1}$$

which may also be written as

$$J = -\int \frac{d^D p}{(2\pi)^D} \int dt_1 dt_N dt' e^{ip\cdot(x-x')} [G_0(p;t-t_1)/2\kappa] \Sigma_{\pi\phi}(p;t_1-t_N) [G_0(p;t_N-t')/2\kappa] . \tag{A2}$$

For definiteness, we consider here a self-interacting scalar theory described by an interaction Hamiltonian $g\phi^n/n!$ and suppose that there are N vertices in the above diagram J.

We follow the same technique developed in Ref. [15]. First we introduce Grassmann variables $\bar{\xi}_i \xi_i$ at each vertex i $(i=1,\ldots,N)$, which enables us to rewrite J as

$$J = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot(x-x')} \int \prod_{l=1}^{N} dt_l d\bar{\xi}_l d\xi_l dt' (-g\kappa)^N \bar{\xi}_1 \xi_1 [G_0(p;t-t_1)/2\kappa] \prod_{\{i,j\}} \mathcal{D}_{ij} [G_0(p;t_N-t')/2\kappa] . \tag{A3}$$

(A3)

Here the "super propagator" $\mathcal{D}_{ij}$ is defined[2] by

$$\mathcal{D}_{ij} \equiv D_0(t_i-t_j) + \bar{\xi}_i \xi_i G_0(t_j-t_i)/2\kappa + \bar{\xi}_j \xi_j G_0(t_i-t_j)/2\kappa \tag{A4}$$

and the product $\prod_{\{i,j\}}$ extends over all internal lines. For notational simplicity, we suppress the ordinary momentum dependence. Observe that the coupling constant has been properly scaled according to our definition of propagators [see (2.12) and (2.13)].

Now let t_2 be the time attached to one of the vertices directly connected to the vertex V_1 whose time variable is t_1 and define

$$\mathcal{I}(t_1,\bar{\xi}_1\xi_1;t_2,\bar{\xi}_2\xi_2) = \int \prod_{l=3}^{N} dt_l d\bar{\xi}_l d\xi_l dt' \prod_{\{i,j\}} \mathcal{D}_{ij} [G_0(t_N-t')/2\kappa] . \tag{A5}$$

Obviously

$$J = (-g\kappa)^N \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot(x-x')} \int dt_1 d\bar{\xi}_1 d\xi_1 dt_2 d\bar{\xi}_2 d\xi_2 [G_0(t-t_1)/2\kappa] \bar{\xi}_1 \xi_1 \mathcal{I} . \tag{A6}$$

Then we consider the supertransformation

$$\delta t_2 = \theta(t_1-t_2)\Delta_2 , \quad \delta\xi_2 = -\kappa\epsilon a , \quad \delta\bar{\xi}_2 = -\kappa\epsilon\bar{a} , \tag{A7}$$

where ϵ is an infinitesimal parameter, a and $\bar{a}$ Grassmann numbers and $\Delta_2 = \epsilon(\bar{a}\xi_2 + \bar{\xi}_2 a)$. Remark that the only invariant combination of t_2, ξ_2, and $\bar{\xi}_2$ which is linear in t_2 and bilinear in ξ_2 and $\bar{\xi}_2$ is $t_2 + \theta(t_1-t_2)\bar{\xi}_2\xi_2/\kappa$. The variation of the superpropagator $\mathcal{D}_{2i}$ under this transformation is easily seen to be

$$\begin{aligned} \delta\mathcal{D}_{2i} &= \theta(t_1-t_2)\Delta_2 \frac{\partial}{\partial t_2} D_0(t_2-t_i) - \kappa\Delta_2 G_0(t_i-t_2)/2\kappa \\ &= -\theta(t_1-t_2)\Delta_2 \left[\bar{\xi}_2\xi_2 \frac{\partial}{\partial t_i} G_0(t_2-t_i)/2\kappa + \kappa G_0(t_2-t_i)/2\kappa \right] , \end{aligned} \tag{A8}$$

where use has been made of (2.7)

We shall show that the above $\mathcal{I}$ is invariant under the transformation (A7). Following the same line of thought expounded in the appendix of Ref. [15], we can arrive at the following expression for $\delta\mathcal{I}$:

$$\begin{aligned} \delta\mathcal{I} = -\theta(t_1-t_2)\Delta_2 \int \prod_{l=3}^{N} dt_l d\bar{\xi}_l d\xi_l dt' \sum_{V_2-V_p} \sum_{V_p-V_q} \cdots \sum_{V_r-V_N} \bar{\xi}_p\xi_p\bar{\xi}_q\xi_q \cdots \bar{\xi}_r\xi_r\bar{\xi}_N\xi_N \\ \times [G_0(t_2-t_p)/2\kappa][G_0(t_p-t_q)/2\kappa] \cdots [G_0(t_r-t_N)/2\kappa] \frac{\partial}{\partial t'} [G_0(t_N-t')/2\kappa] \prod_{\{ij\}}{}' \mathcal{D}_{ij} , \end{aligned} \tag{A9}$$

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[2] A similar but slightly different superpropagator appears in the superspace formulation of SQ [16,17,24]. Either superpropagator can be used in the proof for the difference disappears under the Grassmann integrations.

where the summation $\sum_{V_i - V_j}$ extends over all vertices V_j directly connected with the vertex V_i and the last factor is a product of the remaining propagators not included in the product of $(G_0/2\kappa)$'s. Note that the integrand is written as a total derivative with respect to t'. The integration over t' is trivially performed to give us the boundary $(t'=\pm\infty)$ values of the integrand, which are both zero thanks to the causal property and the exponential time dependence of G_0. Thus we have proved the invariance of $\mathcal{I}$ under the supertransformation (A7).

This invariance implies that $\mathcal{I}$ is dependent on t_2, ξ_2, and $\bar{\xi}_2$ only through the invariant combination $t_2+\theta(t_1-t_2)\bar{\xi}_2\xi_2/\kappa$, which enables us to perform the integrations in (A6). Since

$$\mathcal{I}(t_1,\bar{\xi}_1\xi_1;t_2,\bar{\xi}_2\xi_2)=\mathcal{I}(t_1,\bar{\xi}_1\xi_1;t_2+\theta(t_1-t_2)\bar{\xi}_2\xi_2/\kappa,0)$$
$$=\mathcal{I}(t_1,\bar{\xi}_1\xi_1;t_2,0)+\theta(t_1-t_2)\frac{\partial}{\partial t_2}\mathcal{I}(t_1,\bar{\xi}_1\xi_1;t_2,0)\bar{\xi}_2\xi_2/\kappa ,$$

we have

$$J=(-g\kappa)^N\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\int dt_1 d\bar{\xi}_1 d\xi_1[G_0(t-t_1)/2\kappa]\bar{\xi}_1\xi_1\mathcal{I}(t_1,\bar{\xi}_1\xi_1;t_1,0)/\kappa . \tag{A10}$$

Observe that the net effect of the integration over t_2, ξ_2, and $\bar{\xi}_2$ is summarized as a replacement $t_2\to t_1$, $\bar{\xi}_2\xi_2\to 0$ and a multiplication by a factor $1/\kappa$.

We are able to apply this technique repeatedly to J, the consequence of which amounts to replacements $t_i\to t_1$, $\bar{\xi}_i\xi_i\to 0$ for $i=2,\ldots,N$ and the multiplication by a factor $(1/\kappa)^{N-1}$: All of the super propagators $\mathcal{D}_{ij}$ are to be replaced by $D_0(t_1-t_1)$ after the integrations. The final form of J is

$$J=\kappa(-g)^N\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot(x-x')}\int dt_1 dt'[G_0(t-t_1)/2\kappa]\prod_{\text{int. lines}}D_0(0)[G_0(t_1-t')/2\kappa] . \tag{A11}$$

Since the topological structure of the stochastic diagram (3.1) or (A1) is the same as that of the corresponding Feynman diagram in the field theory, comparison between (A1) and (A11) immediately yields the conclusion that the self-energy $\Sigma_{\pi\phi}(p,0)$ is nothing but (κ times) the ordinary proper self-energy $\Sigma_{\text{FT}}(p)$ in field theory:

$$\Sigma_{\pi\phi}(p,0)=-\kappa(-g)^N\prod_{\text{int. lines}}D(q_l,0)=-\kappa(-g)^N\prod_{\text{int. lines}}\Delta_{\text{F}}(q_l)=\kappa\Sigma_{\text{FT}}(p) , \tag{A12}$$

where integrations over internal line momenta q_l are implicit as before.

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